Bulletin T.CXXII de l'Académie Serbe des Sciences et des Arts - 2001
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 26

A CLASS OF EXPONENTIALLY BOUNDED DISTRIBUTION SEMIGROUPS

M. MIJATOVIĆ, S. PILIPOVIĆ

(Presented at the 9th Meeting, held on December 22, 2000)

A b s t r a c t. A structural theorem for a vector valued exponentially bounded distribution is used for introducing and studyng of a class distribution semigroups. An infinitesimal generator of such a semigroup is not necessarily densely defined, but if it is the case, then it corresponds to a distribution semigroup introduced by Lions. This result is obtained by Wang and Kunstmann for a class of exponentially bounded quasi-distribution semigroups. In fact we show that our class of distribution semigroup is identical to Wang-Kunstmann's one. Our approach is completely different and gives new characterizations. Applications to equations $\dfrac{\partial u}{\partial t} = Au + f$, *where* A *is not necessarily densely defined and* f *is an exponential vector valued distribution supported by* $[0, \infty)$, *are given.*

This paper is written much before the publishing of Wang's and Kunstmann's paper but because of various reasons it is published with a very long delay. Here it is given in the primary version as an original approach although some parts are consequences of published results of Wang and Kunstmann.

AMS Mathematics Subject Classification (2000): 47D03
Key Words: distribution semigroups, integrated semigroups

0. *Introduction*

One-time integrated and n-times integrated exponentially bounded semigroups (n-t.i.e.b.s., in short), $n \in \mathbb{N}$, of operators on a Banach space introduced by Arendt, were developed in [1-2], [13-16], [19], [22], [29-30] and applied to abstract Cauchy problems with operators which do not generate C_0-semigroups. More generally, α-times integrated semigroups $\alpha \in \mathbb{R}^+ \cup \{0\} = [0, \infty)$ are introduced and analyzed in [13-15] and in [21] in connection with certain classes of partial differential and pseudodifferential operators on L^p spaces. Distribution semigroups were introduced and analyzed by Lions (cf. [18] and after that in [5-11], [32] and many other papers. By results of Sova in [25], Arendt had proved in [2] that every exponentially bounded distribution semigroup is an n-th distributional derivative of an n-t.i.e.b.s. with densely defined infinitesimal generator, where n is sufficiently large. The corresponding result for a distribution semigroup which is not exponentially bounded and which infinitesimal generator is not densely defined is proved in [35] and [17]. Local n-times integrated semigroups are introduced and analyzed in [3], [20], [23] and [28], where the relations with distribution semigroups were given.

Wang and Kunstmann have introduced and analyzed in [35] and [17] a quasi-distribution semigroup, QDSG, in short; specially in the case when it is exponentially bounded, EQDSG. A $\mathcal{G} \in \mathcal{D}'_+(L(E))$ (the notation is given in the next section) is a QDSG if and only if $\mathcal{G}$ is a distributional derivative of order n of an n.t.i.e.b.s. We call it a 0-distribution semigroup; in the case when it is exponentially bounded, we denote it by 0-EDSG. C_0-semigroup is among them.

We will investigate 0-EDSG having not densely defined infinitesimal generators. Lions has studied in [18] distribution semigroups with densely defined generators using the structural properties and advantages of the space of tempered distributions. Wang has developed his own approach by constructing a space of generalized functions. The natural frame for such investigations is the space of exponentially bounded distributions $\mathcal{K}'_1$ ([12]). We analyze 0-EDSG using this space.

Let us present the results of the paper. An infinitesimal generator A of an n-t.i.e.b.s. is the generator of a 0-EDSG and conversely, where n is sufficiently large. If A is densely defined, then a 0-EDSG is an exponentially bounded distribution semigroup, EDSG in short (cf. [18], Definition 6.1).

The composition law for a 0-EDSG is given by

$$\langle S(t+s,x), \varphi(t,s)\rangle = \langle S(t, S(s,x)), \varphi(t,s)\rangle,$$

$$\varphi \in \mathcal{K}_1(\mathbb{R}^2),\ \text{supp}\varphi \subset [0,\infty)\times[0,\infty).$$

If $S = G'$ where G is strongly continuous and supported by $[0,\infty)$, then the above condition is sufficient for S being an 0-EDSG. Relations between a 0-EDSG and its infinitesimal generator are determined. They are not the same as in the case of an n-t.i.e.b.s.

It is known that when an operator A generates an n-t.i.e.b.s. on a Banach space E, then there exists a Banach space $E_1 \subset D(A^n)$ continuously imbedded in E such that the part of A in E_1 is the generator of a strongly continuous semigroup (cf. [4]). In this paper a Banach space $E_0 \subset E$ is constructed such that a 0-EDSG in E has the restriction on E_0 forming an EDSG.

In example 1 the results are applied to equation $u' = Au + f$, where $E = C_b(\mathbb{R})$ or $E = L^\infty(\mathbb{R})$,

$$A = \sum_{j=0}^{k} a_j\Big(\frac{d}{dx}\Big)^j, \quad Re\sum_{j=0}^{k} a_j(ix)^j < \infty,\ \ f \in \mathcal{K}_1'(E_0)$$

and E_0 is a suitable subspace of E. Note, A is not densely defined. In example 2 is solved

$$\frac{\partial u}{\partial t} - iH_m(\partial)u = f,\ f \in \mathcal{K}_1'(L^p(\mathbb{R}^n)),\ \text{supp}f \subset [a,\infty),$$

where $H_m(\partial)$ is a pseudodifferential operator $i(\Delta)^{\frac{m}{2}},\ m > 0$. Another approach to this equation in the framework of L^p spaces is given in [5-6] for $m = 2$ and [15]. The assumption $f \in \mathcal{K}_1'(L^p(\mathbb{R}^n))$ justifies our approach.

1. *Preliminaries from the theory of distributions*

Denote by E a Banach space with a norm $\|\cdot\|$; $L(E) = L(E,E)$ is the space of bounded linear operators from E into E and $C(\mathbb{R}, L(E))$ is the space continuous mappings from $\mathbb{R}$ into $L(E)$. We refer to [26-27] and [31] for the definitions of spaces $\mathcal{D}(\mathbb{R}), \mathcal{E}(\mathbb{R}), \mathcal{S}(\mathbb{R})$, their strong duals and $\mathcal{S}'(E) = L(\mathcal{S}(\mathbb{R}), E)$. Moreover, we refer to [33] for the space $\mathcal{S}_+ = \{\varphi; |t^k\varphi^{(v)}(t)| < C_{k,v},\ t \in [0,\infty),\ k, v \in \mathbb{N}_0\}$ ($\mathbb{N}_0 = \mathbb{N}\cup\{0\}$) and its dual $\mathcal{S}_+'$, which consists of tempered distributions supported by $[0,\infty)$. Recall ([12]), the space of exponentially decreasing test functions on the real line $\mathbb{R}$ is defined by $\mathcal{K}_1(\mathbb{R}) = \{\varphi; |e^{k|t|}\varphi^{(v)}(t)| < C_{k,v},\ t \in \mathbb{R},\ k, v \in \mathbb{N}_0\}$. The space $\mathcal{K}_1(\mathbb{R}^2)$ is defined in an appropriate way. This space has the same topological properties as $\mathcal{S}(\mathbb{R})$. Note,

$$f \in \mathcal{K}_1'(\mathbb{R}) \text{ if and only if } e^{-r|x|}f \in \mathcal{S}'(\mathbb{R}) \text{ for some } r \in \mathbb{R}. \tag{1}$$

The strong dual of $\mathcal{K}_1(\mathbb{R})$, $\mathcal{K}_1'(\mathbb{R})$ is the space of exponential distributions. The space $\mathcal{K}_{1+}' \subset \mathcal{K}_1'(\mathbb{R})$ consists of distributions which are supported by $[0,\infty)$.

We will also use the space $\mathcal{K}_{1+} = \{\varphi; |e^{k|t|}\varphi^{(v)}(t)| < C_{k,v},\ t \in [0,\infty),$ $k, v \in \mathbb{N}_0\}$ which has the same topological properties as $\mathcal{S}_+$. Note, its dual space is $\mathcal{K}_{1+}'$.

Essential role will play spaces $\mathcal{K}_{1_0}$ and $\mathcal{D}_0$, subspaces of $\mathcal{K}_1$ and $\mathcal{D}$, respectively, which elements are supporte by $[0,\infty)$. In the sequal, we will use the family of distributions

$$f_n(t) = \begin{cases} \dfrac{H(t)t^{n-1}}{(n-1)!}, & n \in \mathbb{N} \\ f_{n+n_1}^{(n_1)}(t), & -n \in \mathbb{N}_0, n_1 \in \mathbb{N}, n+n_1 > 0, t \in \mathbb{R}, \end{cases}$$

where H is Heaviside's function. Note $f_{-1} = \delta'$.

Let $\mathcal{K}_1'(E) = L(\mathcal{K}_1, E)$ denote the space of continuous linear functions $\mathcal{K}_1 \to E$ with respect to the topology of uniform convergence on bounded sets of $\mathcal{K}_1$. Denote $\mathcal{K}_{1+}'(E) = L(\mathcal{K}_{1+}, E)$. It is a subspace of $\mathcal{K}_1'(E)$ with elements supported by $[0,\infty)$. There holds $\mathcal{K}_1'(E) = \mathcal{K}_1'(\mathbb{R})\widehat{\otimes}E = L(\mathcal{K}_1, E)$ where $\widehat{\otimes}$ denotes the completion of tensor product with respect to the $\mathcal{E}$-topology which is equivalent with the π-topology, since $\mathcal{K}_1'(\mathbb{R})$ is nuclear (cf. [31]). Also we have $\mathcal{K}_{1+}'(E) = \mathcal{K}_{1+}'\widehat{\otimes}E$.

The convolution of $f \in \mathcal{K}_{1+}'(E)$ and $g \in \mathcal{K}_{1+}'$ is defined by $\langle f * g, \varphi\rangle = \langle f, \check{g} * \varphi\rangle$, $\varphi \in \mathcal{K}_1(\mathbb{R})$, $(\check{g}(t) = g(-t))$. One can prove easily that $f * g = g * f \in \mathcal{K}_{1+}'(E)$.

Let $T : [0,\infty) \to L(E)$ be strongly continuous. Then, it is exponentially bounded at infinity if there exist $M \geq 0$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0, \tag{2}$$

In this case $\varphi \to \int\limits_0^\infty T(t)\varphi(t)dt$, $\varphi \in \mathcal{K}_1(\mathbb{R})$, defines an element of $\mathcal{K}_{1+}'(E)$.

We need the following representation for elements of $\mathcal{K}_{1+}'(L(E))$.

Theorem 1. *Let* $S \in \mathcal{K}_{1+}'(L(E))$.

a) There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist a strongly continuous function $F_n : \mathbb{R} \to L(E)$, $\operatorname{supp} F_n \subset [0, \infty)$ and positive constants m_n and C_n, such that

$\|F_n(t)\| \leq C_n e^{m_n t},\ t \geq 0,\ S = F_n^{(n)}$ *($^{(n)}$ is the distributional n-th derivative).*

b) Let $\psi, \varphi \in \mathcal{K}_1(\mathbb{R})$. Then

$$\langle S(t, \langle S(s,x), \psi(s)\rangle), \varphi(t)\rangle = \int S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0)}(t)ds\,dt. \quad (3)$$

c) Let $\varphi(t,s) \in \mathcal{K}_1(\mathbb{R}^2)$ and $\varphi_v(t)$, $\psi_v(s)$ be sequences in $\mathcal{D}(\mathbb{R})$ such that the product sequence $\varphi_v(t) \cdot \psi_v(s)$ converge to $\varphi(t,s)$ in $\mathcal{K}_1(\mathbb{R}^2)$ as $v \to \infty$. Then, the limit

$$\lim_{v\to\infty} \langle S(t, \langle S(s,x), \psi_v(s)\rangle), \varphi_v(t)\rangle$$

defined by the left hand side of (3) exists and defines an element of $\mathcal{K}_1'(\mathbb{R}^2)$ which we denote by $S(t, S(s,x))$ i.e.

$$\langle S(t, S(s,x)), \varphi(t,s)\rangle = \lim_{v\to\infty} \int S_{n_0}(t, S_{n_0}(s,x))\psi_v^{(n_0)}(s)\varphi_v^{(n_0)}(t)ds\,dt, \quad (4)$$

where $\varphi \in \mathcal{K}_1(\mathbb{R}^2)$.

d) Also, we have for $\varphi \in \mathcal{K}_1(\mathbb{R}^2)$ and $r, p \in \mathbb{N}$,

$$(i) \quad \Big\langle \frac{\partial^r}{\partial t^r} S(t, S(s,x)), \varphi(t,s)\Big\rangle = (-1)^r \langle S(t, S(s,x)),\ \frac{\partial^r}{\partial t^r}\varphi(t,s)\rangle;$$

$$(ii) \quad \Big\langle \frac{\partial^r}{\partial s^p} S(t, S(s,x)), \varphi(t,s)\Big\rangle = \Big\langle S(t, \frac{\partial^p}{\partial s^p} S(s,x)), \varphi(t,s)\Big\rangle$$

$$= (-1)^p \langle S(t, S(s,x)),\ \frac{\partial^p}{\partial s^p}\varphi(t,s)\rangle.$$

P r o o f. a) Let $k \in \mathbb{N}$ and $\mathcal{K}_{1,k}(\mathbb{R})$ be the spaces of functions $\varphi \in C^k(\mathbb{R})$, such that $\lim\limits_{|x|\to\infty} \sup\limits_{i\leq k}\{e^{k|x|}|\varphi^{(i)}(x)|\} = 0$. Then, $\mathcal{K}_{1,k}(\mathbb{R}),\ k \in \mathbb{N}$, are Banach spaces with the norms $\|\varphi\| = \sup\{e^{k|x|}|\varphi^{(i)}(x)|;\ x \in \mathbb{R},\ i \leq k\}$ and

$$\mathcal{K}_1(\mathbb{R}) = \operatorname{proj}\lim_{k\to\infty} \mathcal{K}_{1,k}(\mathbb{R}), \quad \mathcal{K}_1'(L(E)) = \operatorname{ind}\lim_{k\to\infty} \mathcal{K}_{1,k}'(L(E))$$

in the sense of strong topologies because the inclusion mappings $\mathcal{K}_{1,k+1}(\mathbb{R}) \to \mathcal{K}_{1,k}(\mathbb{R})$ are compact, $k \in \mathbb{N}$. Thus, there exists $n_1 \in \mathbb{N}$ such that $S \in$

$\mathcal{K}'_{1,n_1}(L(E))$. Also, there exists n_0 such that for every $t \in \mathbb{R}$, $f_{n_0}(t-\cdot)\theta(\cdot) \in \mathcal{K}_{1,n_1}(\mathbb{R})$, where $\theta \in C^\infty(\mathbb{R})$, $\theta(x) = 0$, for $x \leq -1$ and $\theta(x) = 1$, for $x \geq -\dfrac{1}{2}$. Then

$$(S * f_{n_0})(t) = \Big\langle S(u), \frac{(t-u)^{n_0-1}}{\Gamma(n_0)} H(t-u)\theta(u)\Big\rangle, \quad t \in \mathbb{R},$$

it is continuous, supported by $[0,\infty)$, of exponential growth and $(S*f_{n_0})^{(n_0)} = S$. This implies the proof of assertion for every $n \geq n_0$.

b) Assume that φ and ψ are supported by $[\alpha, \beta]$.

$$\langle S(s,x), \psi(x)\rangle = (-1)^{n_0} \int_\alpha^\beta S_{n_0}(s,x)\psi^{(n_0)}(s)ds$$

$$= (-1)^{n_0} \lim_{v\to\infty} \sum_{i=0}^{v} S_{n_0}(s_i, x)\psi^{(n_0)}(s_i)\Delta s_i.$$

The continuity of S_{n_0} implies

$$(-1)^{n_0} \int_\alpha^\beta S_{n_0}(t, \langle S_{n_0}(s,x), \psi^{(n_0)}(s)\rangle)\varphi^{(n_0)}(t)dt$$

$$= \lim_{v\to\infty} \sum_{i=0}^{v} \int_\alpha^\beta S_{n_0}(t, S_{n_0}(s_i,x))\psi^{(n_0)}(s_i)\varphi^{(n_0)}(t)\Delta s_i\, dt$$

$$= \int_\alpha^\beta S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0)}(t)ds\, dt.$$

Note $\|S_{n_0}(t, S_{n_0}(s,x))\| \leq e^{k|t|}e^{k|s|}\|x\|$, $x \in E$. Thus, both sides in (3) exist for $\varphi, \psi \in \mathcal{K}_1(\mathbb{R})$. By taking sequences φ_v and ψ_v of test functions in $\mathcal{D}(\mathbb{R})$ which converge in $\mathcal{K}_1(\mathbb{R})$ to φ and ψ, respectively, we obtain (3).

c) This assertion directly follows from a) by using the integral form given in (3).

d) (i) Assertion b) and (4) imply

$$\Big\langle \frac{\partial^r}{\partial t^r} S(t, S(s,x)), \psi(s)\varphi(t)\Big\rangle$$

$$= (-1)^r \int S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0+r)}(t)ds\, dt, \quad \varphi, \psi \in \mathcal{K}_1(\mathbb{R})$$

and this gives the assertion.

(ii) The same arguments imply

$$\Big\langle \frac{\partial^p}{\partial s^p} S(t, S(s,x)), \psi(s)\varphi(t)\Big\rangle$$

$$= (-1)^p \int S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0+p)}(s)\varphi^{(n_0)}(t)ds\,dt$$

$$= \langle S_{n_0}(t, \langle S_{n_0}(s,x), \psi^{(n_0+p)}(s)\rangle), \varphi^{(n_0)}(t)\rangle, \quad \varphi, \psi \in \mathcal{K}_1(\mathbb{R}). \qquad \square$$

By using (1) one can prove easily

$$f \in \mathcal{K}'_{1+}(L(E)) \text{ if and only if } e^{-r|x|}f \in \mathcal{S}'_{+}(L(E)) \text{ for some } r \geq 0. \tag{5}$$

Let f satisfy (5). Then the Laplace transformation of f is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) = \langle f(t), e^{-\lambda t}\eta(t)\rangle, \quad Re\lambda > r,$$

where $\eta \in \mathbb{C}^\infty(\mathbb{R})$, $\operatorname{supp}\eta = [-\epsilon, \infty)$, $\epsilon > 0$ and $\eta \equiv 1$ on $[0,\infty)$. As in the case of tempered distributions, one can easily show that this definition does not depend on η (cf. [27]). If $f \in L^1([0,\infty), E)$ (which means $\|\int f(t)dt\|_E < \infty$), then

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt = \langle f(t), e^{-\lambda t}\rangle, \quad Re\lambda > 0,$$

where integral is taken in Bochner's sense.

2. *0-exponentially bounded distribution semigroup*

Let $T : (0,\infty) \to L(E)$ be strongly continuous, integrable in a neighborhood of 0, i.e., integrable on $(0,\epsilon)$ for some $\epsilon > 0$ and exponentially bounded at infinity i.e. satisfies (2) for some $M \geq 0$ and $\omega \in \mathbb{R}$. The Laplace transformation of T is defined by $\mathcal{L}(T)(\lambda) = R(\lambda) = \int_0^\infty e^{-\lambda t}T(t)dt$, $Re\lambda > \omega$, where the integral is understood in Bochner's sense.

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. It is well-known that A is the infinitesimal generator of this semigroup if and only if there exists $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ and $R : \{\lambda \in \mathbb{C}; Re\lambda > \omega\} \to L(E)$, defined by $R(\lambda) = (\lambda I - A)^{-1}$, $Re\lambda > \omega$, is the Laplace transformation of $(T(t))_{t\geq 0}$.

Arendt ([2]) defined an n-t.i.e.b.s. as follows.

Let $(S(t))_{t\geq 0}$ be a strongly continuous (on $[0,\infty)$) exponentially bounded family in $L(E)$ and $n\in\mathbb{N}$. Then, it is called n-t.i.e.b.s. if $S(0,x)=0$ and

$$S(t,S(s,x))=\tfrac{1}{(n-1)!}\Big[\int\limits_t^{t+s}(t+s-r)^{n-1}S(r,x)dr \\ -\int\limits_0^s(t+s-r)^{n-1}S(r,x)dr\Big],\ t,s\geq 0,\ x\in E. \tag{6}$$

Let $S:(0,\infty)\to L(E)$ be strongly continuous, exponentially bounded, integrable in a neighbourhood of 0, satisfy (2) for some $M>0$, $\omega\in\mathbb{R}$ and

$$R(\lambda)=\lambda^n\int\limits_0^\infty e^{-\lambda t}S(t)dt,\quad Re\lambda>\omega,\ \ n\in\mathbb{N}. \tag{7}$$

Then, Arendt proved that $(R(\lambda))_{Re\lambda>\omega}$ is a pseudoresolvent if and only if (6) holds.

Theorem 2. *Let $S\in\mathcal{K}'_{1+}(L(E))$ and $R(\lambda)=\mathcal{L}(S)(\lambda)$, $Re\lambda>\omega$.*

a) Then, $(R(\lambda))_{Re\lambda\geq\omega}$ is a pseudoresolvent if and only if there exists $n_0\in\mathbb{N}$ such that $S_{n_0}(t)=(S*f_{n_0})(t)$, $t\in\mathbb{R}$, is continuous, $S_{n_0}(0)=0$ and satisfies

$$\langle S(t,S(s,x)),\varphi(t)\psi(s)\rangle=\langle (S_{n_0}(t,S_{n_0}(s,x)))^{(n_0,n_0)},\varphi(t)\psi(s)\rangle$$

$$\Big\langle\tfrac{1}{(n_0-1)!}\Big(\int\limits_t^{t+s}(t+s-r)^{n_0-1}S_{n_0}(r,x)dr \\ -\int\limits_0^s(t+s-r)^{n_0-1}S_{n_0}(r,x)dr\Big)^{(n_0,n_0)},\ \varphi(t)\psi(s)\Big\rangle, \tag{8}$$

for every $\varphi,\psi\in\mathcal{K}_{1_0}(\mathbb{R})$ and $x\in E$.

b) If $(R(\lambda))_{Re\lambda>\omega}$ is a pseudoresolvent then

$$\langle S(t,S(s,x)),\varphi(t,s)\rangle=\langle S(t+s,x),\varphi(t,s)\rangle,\ \ \varphi\in\mathcal{K}_{1_0}(\mathbb{R}^2),\ x\in E. \tag{9}$$

If (9) holds and $S=G'$, where G is strongly continuous, $G:\mathbb{R}\to L(E)$ and supp$G\subset[0,\infty)$, *then $(R(\lambda))_{Re\lambda>\omega}$ is a pseudoresolvent.*

Remark. If (8) holds, then it holds for every $n\geq n_0$ with $S_n=S*f_n$ because $S_{n_0}=S_{n_0+(n-n_0)}^{(n-n_0)}$.

P r o o f.

a) Since the necessity simply follows, we will prove the sufficiency of (8). We have $S = S_{n_0}^{(n_0)}$. Let $x \in E$. Then, (8) implies

$$(S_{n_0}(t, S_{n_0}(s,x)))^{(n_0,n_0)} = (F(t,s,x))^{(n_0,n_0)} \text{ (in the sense of distributions)},$$

in open sets not intersecting lines $x = 0$ and $y = 0$, where

$$F(t,s,x) = \frac{1}{(n_0-1)!}\Big[\int_t^{t+s} (t+s-r)^{n_0-1} S_{n_0}(r,x)dr$$

$$-\int_0^s (t+s-r)^{n_0-1} S_{n_0}(r,x)dr\Big],\ t,s \geq 0,$$

$$F(t,s,x) = 0, \quad t \leq 0 \quad \text{or} \quad s \leq 0.$$

Since both sides are supported by $[0,\infty)$ and $S_{n_0}(t, S_{n_0}(s,x))$ and $F(t,s,x), t,s \in \mathbb{R}$ are continuous, it follows that

$$S_{n_0}(t, S_{n_0}(s,x)) = F(t,s,x), \ \ t,s \geq 0.$$

Thus, $R(\lambda) = \lambda^{n_0}\mathcal{L}(S_{n_0})(\lambda),\ Re\lambda > \omega$, is a pseudoresolvent.

b) Let $(R(\lambda))_{Re\lambda>\omega}$ be a pseudoresolvent. By assertion a) and (4)

$$\langle S(t,S(s,x)), \varphi(t,s)\rangle = \langle S_{n_0}(t, S_{n_0}(s,x)), \varphi^{(n_0,n_0)}(t,s)\rangle$$

$$= \Big\langle D_t^{n_0} D_s^{n_0}\Big(\frac{1}{(n_0-1)!}\Big[\int_0^{t+s}(t+s-r)^{n_0-1}S_{n_0}(r,x)dr - \int_0^t (t+s-r)^{n_0-1}S_{n_0}(r,x)dr$$

$$-\int_0^s (t+s-r)^{n_0-1}S_{n_0}(r,x)dr\Big]\Big), \varphi(t,s)\Big\rangle,\ \varphi \in \mathcal{K}_{1_0}(\mathbb{R}^2),$$

where $D_t^{n_0} = \dfrac{\partial^{n_0}}{\partial t^{n_0}}$. Then,

$$\langle S(t,S(s,x)), \varphi(t,s)\rangle$$

$$= \Big\langle D_t^{n_0} D_s^{n_0}\Big(\frac{1}{(n_0-1)!}\Big[\int_0^{t+s}(t+s-r)^{n_0-1}S_{n_0}(r,x)dr - \int_0^t (t+s-r)^{n_0-1}S_{n_0}(r,x)dr$$

$$-\int_0^s (t+s-r)^{n_0-1} S_{n_0}(r,x)dr\Big]\Big),\ \varphi(t,s)\Big\rangle$$

$$= \Big\langle D_t^{n_0}\Big(S_{n_0}(t+s-r) - S_{n_0}(s,x) - \sum_{j=1}^{n_0-2} \frac{t^j}{j!} S^{(j+1)}(s,x)\Big), \varphi(t,s)\Big\rangle$$

$$= \langle S(t+s,x),\ \varphi(t,s)\rangle.$$

This implies

$$\langle S(t,S(s,x)), \varphi(t,s)\rangle = \langle S(t+s,x), \varphi(t,s)\rangle,\quad \varphi \in \mathcal{K}_{1_0}(\mathbb{R}^2).$$

Now we prove the second part of b). By (9) we have

$$S(\varphi * \psi, x) = S(\varphi, S(\psi, x)),\quad \varphi, \psi \in \mathcal{K}_{1_0},\ x \in E. \tag{10}$$

Let $\varphi, \psi \in \mathcal{K}_{1_0}$ and $x \in E$. Then by using

$$\mathcal{G}(\varphi, \mathcal{G}(\psi, x)) = \langle S(t, S(s,x)), \varphi'(t)\psi'(s)\rangle,$$

$$\mathcal{G}(\varphi * \psi, x) = -\langle S(t+s,x)), \varphi'(t)\psi(s)\rangle,$$

we obtain

$$\int\int S(t,S(s,x))\varphi'(t)\psi'(s)dt\,ds = -\int\int S(t+s,x))\varphi'(t)\psi(s)dt\,ds$$

$$= \int\int \Big(\int_t^{t+s} S(t+s-r,x)dr - \int_0^s S(r,x)dr\Big)\varphi'(t)\psi'(s)dt\,ds.$$

Put $P(t,s,x) = 0$ if $t \le 0$ or $s \le 0$ and

$$P(t,s,x) = \int_t^{t+s} S(t+s-r,x)dr - \int_0^s S(r,x)dr - S(t,S(s,x)),\quad t>0,\ s>0.$$

This is a continuous function on $\mathbb{R}^2$ and $P(t,0,x) = 0,\ t \in \mathbb{R},\ P(0,s,x) = 0,\ s \in \mathbb{R}$. We have

$$\Big\langle P(t,s,x), \varphi'(t)\psi'(s)\Big\rangle = 0,\quad \varphi, \psi \in \mathcal{K}_{1_0}(\mathbb{R}). \tag{11}$$

Put, for fixed φ,

$$p(s,x) = \int P(t,s,x)\varphi'(t)dt,\quad s \in \mathbb{R}.$$

It is continuous on $\mathbb{R}$ and $p(0,x)=0$.

Now by (11) we have

$$\langle p(s,x), \psi'(s)\rangle = 0, \quad \psi \in \mathcal{K}_{1_0}(\mathbb{R}).$$

The continuity argument implies $p(s,x)=c$.

Thus, we have $\langle P(t,s,x), \varphi'(t)\rangle = 0$, $\forall \varphi \in \mathcal{K}_{1_0}$. This implies $P(t,s,x)=C(s),\ t>0,\ s>0$, where C is a continuous function. With the same procedure, but first applying $P(t,s,x)$ on $\psi'(s)$ we obtain

$$P(t,s,x)=C_1(t), \quad t>0, \quad s>0,$$

where C_1 is a continuous function.

This implies $C_1(t)=C(s),\ t>0,\ s>0$. Thus $C_1(t)=C(s)$ and $P(t,s,x)=0,\ t>0,\ s>0$. □

Definition 1. *Let $S \in \mathcal{K}'_{1+}(L(E))$. Then, S is called a 0-exponentially bounded distribution semigroup, a 0-EDSG, in short, if there exists $n_0 \in \mathbb{N}$, such that $S_{n_0} = S * f_{n_0}$ is continuous on $\mathbb{R}$, supported by $[0,\infty)$, exponentially bounded and satisfies (8). It is called non-degenerate if $\langle S(t,x), \varphi(t)\rangle = 0$ for all $\varphi \in \mathcal{K}_{1_0}$, implies $x=0$. We will also use the notation $(S(t))_{t\geq 0}$ for an 0-EDSG.*

Clearly, C_0-semigroup is a 0-EDSG.

Let $(S(t))_{t\geq 0}$ be a 0-EDSG, and $R(\lambda)=\mathcal{L}(S)(\lambda)$, where $Re\lambda > \omega$. Then, by the resolvent equation, $kerR(\lambda)$ is independent of $Re\lambda > \omega$. Hence, by the uniqueness theorem $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate. In that case there exists a unique operator $A: D(A)\to E,\ (D(A)\subset E)$ satisfying $(\omega,\infty)\subset \rho(A)$ such that $R(\lambda)=(\lambda I - A)^{-1},\ Re\lambda > \omega$. This operator is called the generator of $(S(t))_{t\geq 0}$. We put this in the following definition.

Definition 2. *A closed operator A is the generator of a 0-EDSG $(S(t))_{t\geq 0}$ if $(a,\infty)\subset\rho(A)$ for some $a\in\mathbb{R}$ and the function $\lambda \mapsto (\lambda I - A)^{-1} = \mathcal{L}(S)(\lambda),\ Re\lambda > a$, and it is injective, where the Laplace transformation is understood in the sense of distribution theory.*

Theorem 2 and the above definition directly imply the next Proposition.

Proposition 1. *Let $(S_n(t))_{t\geq 0},\ n\in\mathbb{N}$ be an n-t.i.e.b.s. Then $S_n * f_{-n}$ is a 0-EDSG. If $(S(t))_{t\geq 0}$ is a 0-EDSG, then there is $n_0\in\mathbb{N}$ such that $S * f_n$ is an n-t.i.e.b.s. for every $n \geq n_0$.*

Also, A is the generator of an n-t.i.e.b.s. $(S(_n(t))_{t\geq 0}$ *if and only if A is the generator of a 0-EDSG,* $S_n * f_{-n}$.

The relations of a 0-EDSG $(S(t))_{t\geq 0}$ and its infinitesimal generator A are slightly different then in the case of an n-t.i.e.b.s. However, in the proofs we have to use results for n-t.i.e.b.s. and after that to apply the n-th distributional derivative.

Theorem 3. *Let A be a generator of a 0-EDSG* $(S(t))_{t\geq 0}$. *Then, for all* $\varphi \in \mathcal{K}_1$, *we have*

a) $A\langle S(t,x), \varphi(t)\rangle = \langle S(t,Ax),\ \varphi(t)\rangle,\ \ x \in D(A)$

b) For every $x \in E$, $\langle S(t,x), \varphi(t)\rangle \in D(A)$.

c) $\langle S(t,x), \varphi(t)\rangle = \langle f_1(t,x), \varphi(t)\rangle + \langle (f_1 * S)(t, Ax), \varphi(t)\rangle,\ x \in D(A)$ *and*

$$A\langle (f_1 * S)(t,x), \varphi(t)\rangle = \langle S(t,x), \varphi(t)\rangle - \langle f_1(t,x), \varphi(t)\rangle,\ \ x \in E.$$

In particular

$$A\langle S(t,x), \varphi(t)\rangle = -\langle S(t,x), \varphi'(t)\rangle - \varphi(0)x,\ \ x \in E.$$

Remark. We will use also the notation $A\langle S(t,x), \varphi(t)\rangle = \langle AS(t,x), \varphi(t)\rangle$.

P r o o f.

a) Let $n_0 \in \mathbb{N}$ such that $S_{n_0} = S * f_{n_0}$ is an n_0-t.i.e.b.s. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $x \in D(A)$. Then,

$$\langle S(t,x), \varphi(t)\rangle = (-1)^{n_0}\langle S_{n_0}(t,x), \varphi^{(n_0)}(t)\rangle,\ \ n_0 \in \mathbb{N}.$$

Proposition 3.3 in [2] implies $S_{n_0}(t,x) \in D(A)$ and $AS_{n_0}(t,x) = S_{n_0}(t,Ax)$.

This and the continuity of A imply

$$A\langle S(t,x), \varphi(t)\rangle = (-1)^{n_0} A\langle S_{n_0}(t,x), \varphi^{(n_0)}(t)\rangle$$

$$= (-1)^{n_0} A \int S_{n_0}(t,x)\varphi^{(n_0)}(t)dt = (-1)^{n_0} A \lim_{v\to\infty} \sum_{i=1}^{v} S_{n_0}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j$$

$$= (-1)^{n_0} \lim_{v\to\infty} \sum_{j=1}^{v} AS_{n_0}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j$$

$$= (-1)^{n_0}\langle AS_{n_0}(t,x), \varphi^{(n_0)}(t)\rangle = \langle S(t,Ax), \varphi(t)\rangle,\ x \in E,\ \varphi \in \mathcal{D}$$

where $\Big(\sum\limits_{j=1}^{v} S_{n_0}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j\Big),\ v\in\mathbb{N}$ is a sequence of integral sums for $\int S_{n_0}(t,x)\varphi^{(n_0)}(t)dt$.

Let $\varphi\in\mathcal{K}_1$ and φ_v be a sequence in $\mathcal{D}(\mathbb{R})$ which converges to φ in $\mathcal{K}_1$. Then,

$$A\langle S(t,x),\varphi(t)\rangle=\lim_{v\to\infty}\langle S(t,Ax),\varphi_v(t)\rangle=\langle S(t,Ax),\varphi(t)\rangle.$$

This implies the assertion.

b) By Proposition 3.3 in [2], we have $\int\limits_0^t S_{n_0}(s,x)ds\in D(A)$ for every $x\in E$. Thus, $\Big\langle\int\limits_0^t S_{n_0}(s,x)ds,\varphi(t)\Big\rangle\in D(A)$ for every $\varphi\in\mathcal{K}_1$ and $x\in E$. By putting $\varphi^{(n_0+1)}$ instead of φ, we obtain $\langle S(\cdot,x),\varphi\rangle\in D(A)$ for every $\varphi\in\mathcal{K}_1$.

c) Note, for $x\in D(A),\ t\geq 0,\ S_{n_0}(t,x)\in D(A),\ AS_{n_0}(t,x)=S_{n_0}(t,Ax)$, $S_{n_0}(t,x)=\dfrac{t^{n_0}}{n_0!}x+\displaystyle\int\limits_0^t S_{n_0}(s,Ax)ds$ (Proposition 3.3. in [2]). This implies

$$\langle S(t,x),\varphi(t)\rangle=(-1)^{n_0}\langle S_{n_0}(t,x),\varphi^{(n_0)}(t)\rangle$$

$$(-1)^{n_0}\langle f_{n_0+1}(t,x),\varphi^{(n_0)}(t)\rangle+(-1)^{n_0}\langle f_1*S_{n_0})(t,Ax),\varphi^{(n_0)}(t)\rangle$$

$$=\langle f_1(t,x),\varphi(t)\rangle+\langle(f_1*S_{n_0}^{(n_0)})(t,Ax),\varphi(t)\rangle$$

$$=\langle f_1(t,x),\varphi(t)\rangle+\langle(f_1*S)(t,Ax),\varphi(t)\rangle,\ \ x\in D(A),\ \varphi\in\mathcal{K}_1,$$

which gives the first assertion.

Again by using the quoted Proposition 3.3 in [2], it follows

$$A\langle(f_1*S)(t,x),\varphi(t)\rangle=(-1)^{n_0}\langle A(f_1*S_{n_0})(t,x),\varphi^{(n_0)}(t)\rangle$$

$$=(-1)^{n_0}\langle S_{n_0}(t,x),\varphi^{(n_0)}(t)\rangle-(-1)^{n_0}\langle f_{n_0+1}(t,x),\varphi^{(n_0)}(t)\rangle$$

$$=\langle S_{n_0}^{(n_0)}(t,x),\varphi(t)\rangle-\langle f_1(t,x),\varphi(t)\rangle=\langle S(t,x),\varphi(t)\rangle-\langle f_1(t,x),\varphi(t)\rangle,$$

which gives the second assertion. The particular case follows by putting φ' in the second relation in assertion c). □

3. *Relations with distribution semigroup of Lions*

We follow Definition 6.1 in [18] of a distribution semigroup. Note, instead of $\mathcal{S}(\mathbb{R})$ which is used as the basic space in [18], we use in this paper the space $\mathcal{K}_1(\mathbb{R})$.

Let us recall conditions for a $\mathcal{G} \in \mathcal{D}'_+(L(E))$ to be a *distribution semigroup.*

(D.1) $\mathcal{G}(\varphi * \psi, \cdot) = \mathcal{G}(\varphi, \mathcal{G}(\psi, \cdot)), \quad \varphi, \psi \in \mathcal{D}_0$;

(D.2) $\bigcap_{\varphi \in \mathcal{D}_0} N(\mathcal{G}(\varphi, \cdot)) = \{0\}$;

(D.3) The linear hull $\mathcal{R}$ of $\bigcup_{\varphi \in \mathcal{D}_0} R(\mathcal{G}(\varphi, \cdot))$ is dense in E;

(D.4) for every $x \in \mathcal{R}$ there exists a function $u : \mathbb{R} \to E$ such that $\operatorname{supp} u \subset [0, \infty)$, $u(0) = x$ and u is continuous for $t \geq 0$ and $\mathcal{G}(\varphi, x) = \int_0^\infty \varphi(t)u(t)dt$ for any $\varphi \in \mathcal{D}_0$.

If, in addition, there exists $\xi_0 \in \mathbb{R}$ such that

(D.5) $e^{-\xi t}\mathcal{G} \in \mathcal{S}'_+(L(E))$, for $\xi > \xi_0$, then it is called *an exponentially bounded distribution semigroup,* EDSG *in short.*

With another convolution $\varphi \otimes \psi = \int_0^t \varphi(x-t)\psi(t)dt$, $\varphi, \psi \in \mathcal{D}$, Wang [35] and Kunstmann [17] have introduced a quasi-distribution semigroup, QDSG in short, as an element of $\mathcal{D}'(L(E))$ satisfying

(Q.D.1) $\mathcal{G}(\varphi \otimes \psi, \cdot) = \mathcal{G}(\varphi, \mathcal{G}(\psi, \cdot))$, $\varphi, \psi \in \mathcal{D}$,

(Q.D.2) = (D.2).

One can simply show that (Q.D.1) implies $\mathcal{G} \in \mathcal{D}'_+(L(E))$. If, moreover, (D.5) holds, than it is called exponentially bounded quasi-distribution semigroup, EQDSG in short. In fact in [35] is used another definition of exponential boundedness.

Let $(S_0(t))_{t \geq 0}$ be a 0-EDSG with an infinitesimal generator which is not densely defined. We recall:

$$S_0(\varphi, x) = 0 \quad \text{for every} \;\; \varphi \in \mathcal{D}_0 \Rightarrow x = 0. \tag{12}$$

As in [18], we extend $(S_0(t))_{t \geq 0}$ on $T \in \mathcal{E}'(\mathbb{R}), \operatorname{supp} T \subset [0, \infty)$ using δ sequences $\{\rho_v\}$ in $\mathcal{D}_0$, $(\rho_v \to \delta)$. We denote by $D(S_0(T))$ the set of $x \in E$ such that:

(i) $S_0(\rho_v, x) \to x$, $\;v \to \infty$,

(ii) $\lim_{v \to \infty} S_0(T * \rho_v, x)$ exists, and does not depend on ρ_v for which (i) holds.

This means that if $\tilde{\rho}_v$ is another δ sequence in $\mathcal{D}_0$ for which (i) holds, then $S_0(T * \tilde{\rho}_v, x) - S_0(T * \rho_v, x) \to 0$ as $v \to \infty$. This limit defines $S_0(T, x)$.

Because of (12), we can define the closure of $S_0(T, \cdot)$ which will be denoted by $\overline{S_0(T, \cdot)}$.

In the next theorem we summerize basic relations between C_0-semigroups and various distribution exponentially bounded semigroups.

Theorem 4. *a) Let $(T(t))_{t \geq 0}$ be a C_0-semigroup and A its infinitesimal generator. Then by*

$$\mathcal{J}(\varphi, x) = (T * \check{\varphi})(0)(x) = \langle T(t, x), \varphi(t) \rangle, \ \ \varphi \in \mathcal{K}_{1^+}, \ x \in E$$

is defined an EDSG, $Ax = \overline{\mathcal{J}(-\delta', x)}$, $x \in D(A)$ and

$$A\mathcal{J}(\varphi, x) = -\int_0^\infty T(t, x)\varphi'(t)dt - \langle \delta(t, x), \varphi(t) \rangle =$$

$$= -\int_0^\infty T(t, x)\varphi'(t)dt - \varphi(0)x, \ x \in D(A), \varphi \in \mathcal{K}_{1^+}.$$

b) Let $(S_0(t))_{t \geq 0}$ be a 0-EDSG with a densely defined infinitesimal generator A. Then, by

$$\mathcal{J}_0 : \varphi \to \langle S_0(t, \cdot), \varphi(t) \rangle, \quad \varphi \in \mathcal{K}_{1^+},$$

is defined an EDSG. Moreover, $A = \mathcal{J}_0(-\delta', \cdot)$.

If A is not densely defined, then by $\mathcal{J}_0$ is defined an element of $\mathcal{K}'_{1^+}(L(E))$ which satisfies all the properties for an EDSG except that $\{\mathcal{J}_0(\varphi, x);\ \varphi \in \mathcal{D}_0,\ x \in E\}$ is dense in E.

c) Let $(S_n(t))_{t \geq 0}$, $n \in \mathbb{N}_0$ be an n-t.i.e.b.s. Assume that its infinitesimal generator A is densely defined. Then,

$$\mathcal{J}_n(\varphi, x) = (S_n * \check{\varphi})(0)(x), \ \ \varphi \in \mathcal{K}_{1^+}, \tag{13}$$

defines an element of $\mathcal{K}'_{1^+}(L(E))$ which is an EDSG if and only if $n = 0$.

d) Let $S \in \mathcal{D}'_+(L(E))$. It is a 0-EDSG if and only if it is an EQDSG.

P r o o f.

a) Clearly $\mathcal{J} \in \mathcal{K}'_{1^+}(L(E))$. Then,

$$(T * (\varphi \check{*} \psi))(0)(x) = ((T * \check{\psi}) * \check{\varphi})(0)(x), \ \varphi, \psi \in \mathcal{D}_0(\mathbb{R})$$

and this implies $\mathcal{J}(\varphi * \psi, x) = \mathcal{J}(\varphi, \mathcal{J}(\psi, x)),\ x \in E,\ \varphi, \psi \in \mathcal{D}_0(\mathbb{R})$.

Put $y = \mathcal{J}(\psi, x)$, where $\psi \in \mathcal{D}_0(\mathbb{R})$ and $x \in E$ are fixed. Then, a distribution $\varphi \mapsto \mathcal{J}(\varphi, y),\ \varphi \in \mathcal{D}_0(\mathbb{R})$ is determined by a continuous function $u : [0, \infty) \to E,\ u(0) = y$. In fact, we have $u(t) = (T * \check{\psi})(t)$.

Let $x \in E$ be fixed. If $\mathcal{J}(\varphi, x) = 0$ for every $\varphi \in \mathcal{D}_0$, then it follows $x = 0$ or $x \neq 0$ and $T = \sum\limits_{\alpha=0}^{m} a_\alpha \delta^{(\alpha)} \otimes I$, where I is the identity operator and $a_\alpha \in \mathbb{C},\ \alpha = 0, 1, ..., m$. But since T could not be of this form, we have $x = 0$. Let us prove that $\{\langle T(t, x), \varphi(t)\rangle;\ \varphi \in \mathcal{D}_0,\ x \in E\}$ is dense in E. For every $t_0 \in [0, \infty)$ and every $x \in E$,

$$\langle T(t, x), \delta_n(t - t_0)\rangle \text{ converges in } E \text{ to } T(t_0, x) \text{ as } n \to \infty,$$

where $\delta_n(t - t_0)$ is a δ - sequence. For example, we can take $\delta_n(t - t_0) = n\varphi(n(t - t_0))$, where $\varphi \in \mathcal{D}_0(\mathbb{R}),\ \int \varphi = 1$. Since the set $\{T(t_0, x); t_0 \in [0, \infty),\ x \in E\}$ is dense in E, the assertions follows. Thus, we verify all the conditions which $\mathcal{J}$ has to satisfy to be an EDSG.

We have to prove that $Ax = \overline{\mathcal{J}(-\delta', x)}, x \in D(A)$; this implies that A is the infinitesimal generator of semigroup $\mathcal{J}$.

By noting that for $x \in D(A),\ T(\cdot, x)$ is differentiable, we have

$$\overline{\mathcal{J}(-\delta', x)} = (T * \delta')(0)(x) = T'(0)(x) = Ax.$$

Let $x \in D(A)$, then we have

$$A\mathcal{J}(\varphi, x) = \mathcal{J}(-\delta' * \varphi, x) = \mathcal{J}(-\varphi', x) = \int\limits_0^\infty T'(t, x)\varphi(t)dt, \quad \varphi \in \mathcal{K}_{1^+}.$$

Now, by partial integration, we obtain

$$A\mathcal{J}(\varphi, x) = T(t, x)\varphi(t)|_0^\infty - \int\limits_0^\infty T(t, x)\varphi'(t)dt = -\int\limits_0^\infty T(t, x)\varphi(t)dt - \varphi(0)x.$$

b) Let $(S_0(t))_{t \geq 0}$ be a 0-EDSG with a densely defined infinitesimal generator A. Let $S_0(\varphi, x),\ \varphi \in \mathcal{K}_{1^+},\ x \in E$, be defined by

$$S_0(\varphi, x) = (S_0(\cdot, x) * \check{\varphi})(0) = \langle S_0(t, x), \varphi(t)\rangle. \tag{14}$$

Let $\varphi, \psi \in \mathcal{D}_0$. Then,

$$(S_0(\cdot, x) * (\varphi \check{*} \psi))(0) = \langle S_0(t, x), (\varphi * \psi)(t)\rangle$$

$$= \langle S_0(t+s,x),\varphi(t)\psi(s)\rangle = \langle S_0(t,\langle S_0(s,x),\psi(s)\rangle),\varphi(t)\rangle.$$

This implies

$$S_0(\varphi*\psi,x)=S_0(\varphi,S_0(\psi,x)),\ x\in E,\ \varphi,\psi\in\mathcal{D}_0.$$

Clearly, for fixed $\psi\in\mathcal{D}_0$ and $x\in E$, $\mathcal{D}_0\ni\varphi\mapsto S_0(\varphi,S_0(\psi,x))$ is determined by $u(t)=(S_0(\cdot,x)*\check{\psi})(t)$ which is continuous and supported by $[0,\infty)$.

Since S_0 is a non-degenerate one and not of the form $\sum\limits_{\alpha=0}^{m} a_\alpha\delta^{(\alpha)}\otimes I$, it follows

$$(S_0(\varphi,x)=0 \text{ for every } \varphi\in\mathcal{K}_{1_0})\Rightarrow(x=0).$$

Similary as in [18], pp. 152-153, we will prove that $\{S_0(\varphi,x);\varphi\in\mathcal{D}_0,\ x\in E\}$ is dense in E. This will imply that (13) defines an EDSG.

The duals of $D(A)$ and E, $D(A)'$ and E' are equal, since $D(A)$ is dense in E. Thus,

$\mathcal{H}(\varphi,\cdot)={}^t\langle S_0(t,\cdot),\varphi(t)\rangle\in D(A)',\ \varphi\in\mathcal{K}_{1+}$, where superscript t denotes a transposed operator.

Moreover, $\mathcal{H}\in\mathcal{K}_1'(L(D(A)',E)),\ \mathrm{supp}\mathcal{H}\subset[0,\infty)$. Theorem 3, a) and c) implies

$$-\mathcal{H}(\varphi)\,{}^tA-\mathcal{H}(\varphi')=\varphi(0)I_{E'}$$

$$-{}^tA\mathcal{H}(\varphi)-\mathcal{H}(\varphi')=\varphi(0)I_{D(A)'}.$$

Let $x'\in E'$ and

$$\langle\langle S_0(t,x),\varphi(t)\rangle,x'\rangle=0,\quad\text{for every}\ \ \varphi\in\mathcal{D}_0,\ x\in E. \tag{15}$$

We have to prove that $x'=0$ which means, by Hahn-Banach theorem, that $\{\langle S_0(t,x),\varphi(t)\rangle;\ \varphi\in\mathcal{D}_0,\ x\in E\}$ is dense in E. Now (15) implies $\langle\mathcal{H}(\varphi,\cdot),x'\rangle=0$ which implies $x'=0$.

We define $A_0x=\overline{S_0(-\delta',x)}$. Then, $A_0=A$ because the resolvent for both operators are determined by $\mathcal{L}(S_0)(\lambda), Re\lambda>\omega$, (cf. [18]).

Let $\overline{D(A)}\neq E$. Then, except of density of $\{S_0(\varphi,x);\ \varphi\in\mathcal{D}_0,\ x\in E\}$, one can simply prove that S_0 verifies all the other properties which should be satisfied by an EDSG.

c) Let $\mathcal{J}$ be an EDSG. As it is remarked by Arendt, Theorem 4.3 in [2] and Theorem 3.2 in [25] imply that there exists an n-t.i.e.b.s., $(S_n)(t))_{t\geq0}$, $n\in\mathbb{N}$, such that

$$\mathcal{J}(\varphi,x)=\langle S_n^{(n)}(t,x),\varphi(t)\rangle=(S_n^{(n)}(\cdot,x)*\check{\varphi})(0),\ \varphi\in\mathcal{D},\ x\in E.$$

This implies $S_n^{(n)} = S_n * f_{-n} = S_0 = \mathcal{J}$, where S_0 satisfies Theorem 2.

If for some $n \in \mathbb{N}$, (13) defines an EDSG then $(S_n(t))_{t\geq 0}$ determine exponentially bounded distribution semigroups which is impossible by the uniqueness of an EDSG with the given infinitesimal generator A.

d) The properties of a $0-EDSG$ given in Theorems 1,2, Proposition 1 and in [35], Theorem 3.8, as well as arguments in [35], Section 4, imply the assertion. □

The last part of Theorem 4 shows that our investigations give another approach and insight into the theory of EQDSG.

With the asumptions as in the second part of Theorem 4 b), we have the following theorem.

Theorem 5. *Let $(S_0(t)_{t\geq 0}$ be a $0-EDSG$ with an infinitesimal generator A such that $\overline{D(A)} \neq E$. Denote by E_0 the closure in E of the set $\tilde{E} = \{S_0(\varphi, x); x \in E, \varphi \in \mathcal{D}_0\}$. Then the set $\{S_0(\varphi, x); x \in E_0, \varphi \in \mathcal{D}_0\}$ is dense in E_0 and $S_{0|E_0\times\mathcal{K}_{1+}}$ defines an EDSG with the infinitesimal generator $A = A_{|D(A)\cap E_0}$.*

P r o o f. Let $y \in E_0$ and $y_n = S_0(\varphi_n, x_n) \in \tilde{E} \subset E_0$ be a sequence which converges to y in E_0. We have $S_0(\varphi_n, x_n) = \lim_{m\to\infty} S_0(\varphi_n * \rho_m, x_n)$, where ρ_m is a δ sequence in $\mathcal{D}_0$. Since

$$S_0(\varphi_n * \rho_m, x_n) = S_0(\rho_m, (S_0(\varphi_n, x_n)) = S_0(\rho_m, y_n), m, n \in \mathbb{N},$$

we obtain that y is an limit point for $\{S_0(\varphi, x); x \in E_0, \varphi \in \mathcal{D}_0\}$ and $y_n \in \tilde{E}$.

Since $\{S_0(\varphi, x), x \in E_0, \varphi \in \mathcal{D}_0\} \subset D(A)$, we have that $D(A) \cap E_0$ is dense in E_0. This completes the proof. □

4. *Applications*

Let A be an operator on E, and $T \in \mathcal{K}_1'$ (E), supp$T \subset [a, \infty), a \geq 0$. Then, $u \in \mathcal{K}_1'$ (E) is a solution to equation

$$u' = Au + T, \text{ in } \mathcal{K}_1'(E) \tag{16}$$

if $\langle u(t), \varphi(t)\rangle \in D(A)$ for every $\varphi \in \mathcal{K}_{1+}'(\mathbb{R})$ and (16) holds.

Let A be an infinitesimal generator of a $0-EDSG(S_0(t))_{t\geq 0.}$ We assume that $D(A)$ is supplied with the graph norm $\|x\| + \|Ax\|, x \in D(A)$.

Let $U \in \mathcal{K}_1'(L(E, D(A))), V \in \mathcal{K}_1'(L(D(A), E))$ and supp$U \subset [a, \infty)$, supp $V \subset [b, \infty), a, b \geq 0$. Then, $U * V$ and $V * U$ are elements of $\mathcal{K}_1'(L(D(A)))$

and $\mathcal{K}'_1(L(E))$, respectively and their supports are contained in $[a+b,\infty)$ (cf. [28]).

Theorem 6. *Let A be an infinitesimal generator of a $0-EDSG$ $(S_0(t))_{t\geq 0}$. Then $(S_0(t))_{t\geq 0}$ is an $EDSG$ with an infinitesimal generator A on $E_0\times\mathcal{K}'_{1+}$, where $E=\overline{\{S_0(\varphi,x);\varphi\in\mathcal{D}_0,x\in E\}}$ and*

$$\Big(-A+\frac{\partial}{\partial t}\Big)*S_0=\delta\otimes I_{E_0}, S_0*\Big(-A+\frac{\partial}{\partial t}\Big)=\delta\otimes I_{D(A)\cap E_0}.$$

Let $T\in\mathcal{K}'_1(E_0)$ and $\mathrm{supp}T\subset[a,\infty)$ for some $a\geq 0$. Then, $u=S_0*T$ is a unique solution to

$$u'=Au+T \text{ in } \mathcal{K}'_1(L(E_0)), \mathrm{supp}u\subset[a,\infty).$$

The proof of Theorem 6 is the same as the proof of Theorem 4.1 in [18].

Example 1. Let $A=\sum_{j=0}^{k}a_j\Big(\frac{d}{dx}\Big)^j, p(x)=\sum_{j=0}^{k}a_j(ix)^j(i^2=-1)$, $Rep(x)<\infty$, $a_j\in\mathbb{C}$, $j=0,1,...,k$, $E=C_b(\mathbb{R})$ or $E=L^\infty(\mathbb{R})$, $Af=\sum_{j=0}^{k}a_j\Big(\frac{d}{dx}\Big)^jf$, where

$$D(A)=\{f\in E;\sum_{j=0}^{k}a_j\Big(\frac{d}{dx}\Big)^jf\in E \text{ distributionally }\}.$$

It is proved in [16] that

$$S_tf=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(\phi_t)*f, t\geq 0, f\in E,$$

where $(\phi_t(\xi)=f_1(t)*e^{p(\xi)t})_{t\geq 0}$, is an 1-t.i.e.b.s. with the generator A such that $\overline{D(A)}\neq E$. Here $\mathcal{F}$ denotes the Fourier transformation and $\mathcal{F}^{-1}$ denotes its inverse; $\mathcal{F}(f)(\lambda)=\tilde{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\lambda t}f(t)dt, \lambda\in\mathbb{R}$. We have

$$\langle f_{-1}(t)*\phi_t(\xi),\alpha(t))\rangle=\langle e^{p(\xi)t},\alpha(t)\rangle, \alpha\in\mathcal{K}'_{1+}(\mathbb{R}).$$

Thus $\mathcal{F}^{-1}(e^{p(\xi)t})=\frac{d}{dt}\Big(\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(\phi_t)\Big)$, in the sense of distributions, defines a $0-EDSG$ which is not an EDSG. If $T\in\mathcal{K}'_1(E_0), \mathrm{supp}T\subset[a,\infty), a\geq 0$, then the solution to equation $u'=Au+T$ is given by $\mathcal{F}^{-1}(e^{p(\xi)t})*T(t)$ (cf. Theorem 6).

Example 2. Let $m>0$. Consider

$$\frac{\partial}{\partial t}u(x,t)-i\Delta^{\frac{m}{2}}u(x,t)=f(x,t), \quad f\in\mathcal{K}'_1(L^p(\mathbb{R}^n)), \mathrm{supp}f\subset[a,\infty), a\geq 0.$$

With $m = 2$ and $f \in \mathcal{T}'(L^p(\mathbb{R}^n))$ this equation is analyzed in [5] and [6].

We recall the definition of the space M_p of Fourier multipliers on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ (cf. [25]): $a \in M_p$ if and only if a is a tempered distribution and $\|a\|_{M_p} = \sup_{\|f\|_{L^p}=1} \|\mathcal{F}^{-1}(a\tilde{f})\|_{L^p} < \infty$.

Let $m > 0, k > 0$ and $H_m(\xi) = |\xi|^m, \xi \in \mathbb{R}^n$. Then, the following asertion is proved in [25]:

$$a_{H_m,k}(\xi,t) = \frac{k}{t^k}\int_0^t (t-s)^{k-1} e^{-isH_m(\xi)} ds \in M_p, \quad t > 0,\ \xi \in \mathbb{R}^n,$$

if:

$$\text{for } m \neq 1,\ k \geq n\Big|\frac{1}{p} - \frac{1}{2}\Big|, \ \text{ for } \ m = 1,\ k \geq (n-1)\Big|\frac{1}{p} - \frac{1}{2}\Big|. \tag{17}$$

Moreover, $\|a_{H_m,k}(\xi,t)\|_{M_p}$ does not depend on t.

Define, for $\varphi \in \mathcal{K}_1(\mathbb{R})$, $u \in \mathcal{S}(\mathbb{R}^n)$,

$$G_0(\varphi,u)(x) = \mathcal{F}^{-1}\Big(\int_{\mathbb{R}} e^{-itH_m(\xi)}\varphi(t)dt\tilde{u}(\xi))(x), x \in \mathbb{R}^n.$$

Let k be an integer which satisfies (17). Then,

$$G_0(\varphi,u)(x) = \frac{(-1)^k}{k!}\int_{\mathbb{R}} t^k\varphi^{(k)}(t)\mathcal{F}^{-1}(a_{H_m,k}(\xi,t)\tilde{u}(\xi))dt(x), x \in \mathbb{R}^n.$$

It implies that for given $\varphi \in \mathcal{K}_1(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^n)$ there exists $C_\varphi > 0$ such that

$$\|G_0(\varphi,u)\|_{L^p},$$

which shows that it can be extended on $L^p(\mathbb{R}^n)$ as a continuous linear operator.

Moreover, G_0 is an EDSG with the infinitesimal generator

$$A = iH_m(\partial) = i\Big(\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial_n^2}\Big)^{\frac{m}{2}}$$

which is defined on the closure in L^p of the set $\{u \in \mathcal{S}; \mathcal{F}^{-1}(|\xi|^m\tilde{u}) \in L^p\}$.

Thus we have that $u = G_0 * f$ is the solution to equation

$$\frac{\partial}{\partial t}u - iH_m(\partial)u = f, f \in \mathcal{K}_1'(L^p), \mathrm{supp} f \subset [a,\infty).$$

REFERENCES

[1] W. A r e n d t, *Resolvent positive operators and integrated semigroups,* Proc. London Math. Soc., (3) 54 (1987), 321–349.

[2] W. A r e n d t, *Vector valued Laplace transforms and Cauchy problems,* Israel J. Math., 59 (1987), 327–352.

[3] W. A r e n d t, O. E l - M e n n a o u, V. K e y a n t u o, *Local integrated semigroups,* J. Math. Anal. Appl., 186 (1994), 572–595.

[4] W. A r e n d t, F. N e u b r a n d e r, U. S c h l o t t e r b e c k, *Interpolation of Semigroups and Integrated semigroups,* Semigroup From, 45 (1992), 26–37.

[5] M. B a l a b a n e, H. A. E m a m i r a d, *Smooth distribution group and Schrödinger equation in $L^p(\mathbb{R}^n)$,* J. Math. Anal. Appl., 70 (1979), 61–71.

[6] M. B a l a b a n e, H. A. E m a m i r a d, *L^p estimates for Schrödinger evolution equations,* Trans. Amer. Math. Soc., 291 (1985), 357–373.

[7] J. C h a z a r i a n, *Problémes de Cauchy abstraits et applications aux quelques problemes mixtes,* J. Func. Anal., 7(1971), 387–446.

[8] G. D a P r a t o, U. M o s c o, *Regolarizzazione dei semigruppi analitici,* Ann. Scuola Norm. Sup. Pisa, 19(1965), 563–576.

[9] G. D a P r a t o, U. M o s c o, *Semigruppi distribuzioni analitici,* Ann. Scuola Norm. Sup. Pisa, 19(1965), 367–396.

[10] C. F o i a s, *Remarques sur les semi-groupes distributions d' operateurs normaux,* Portugal. Math., 19(1960), 227–242.

[11] D. F u j i w a r a, *A characterization of exponential distribution semigroups,* J. Math. Soc., 18(1966), 267–274.

[12] M. H a s u m i, *Note on the $n-$ dimensional tempered ultradistributions,* Tôhoku Math. J., 13(1961), 94–104.

[13] M. H i e b e r, *Integrated semigroups and differential operators on L^p spaces,* Math. Ann., 291(1991), 1–16.

[14] M. H i e b e r, *Integrated semigroups and the Cauchy problem for systems in L^p - spaces,* J. Math. Anal. Appl., to appear

[15] M. H i e b e r, *L^p spectra of pseudodifferential operators generating integrated semigroups,* Trans. of the Amer. Math. Soc., 347(1995), 4023–4035.

[16] H. K e l l e r m a n n, M. H i e b e r, *Integrated semigroups,* J. Func. Anal., 84(1989), 160–180.

[17] P. C. K u n s t m a n n, *Distribution semigroups and abstract Cauchy problems,* Trans. Amer. Math. Soc., 351(1999), 837–856.

[18] J. L. L i o n s, *Semi-groupes distributions,* Portugal. Math., 19 (1960), 141–164.

[19] G. L u m e r, *Evolution equations. Solutions for irregular evolution problems via generalized initial values. Applications to periodic shocks models,* Ann. Univ. Saraviensis, 5 Saarbrücken, 1994.

[20] I. V. Melnikova, M. A. Alshansky, *Well-posedness of Cauchy problem in a Banach space: regular and degenerate cases,* Itogi nauki i tehn., Series sovr. matem. i ee prilog. Analiz-9/VINITI, 27 (1995), 5–64.

[21] M. M i j a t o v i ć, S. P i l i p o v i ć, F. V a j z o v i ć, α - *times integrated semigroup* ($\alpha \in \mathbb{R}^+$), J. Math. Anal. Appl., 210 (1997), 790–803.

[22] F. N e u b r a n d e r, *Integrated semigroups and their applications to the obstract Cauchy problem,* Pac. J. Math., 135 (1988), 111–155.

[23] S. O h a r u, *Semigroups of linear operators in Banach spaces,* Publ. Res. Inst. Math. Sci., 204 (1973),189–198.

[24] S. S j ö s t r a n d, *On the Riesz means of solutions of Schrödinger equation,* Ann. Scuola Norm. Sup. Pisa, 24 (1970), 331–348.

[25] M. S o v a, *Problémes de Cauchy paraboliques abstraits de classes supérieurs et les semigroupes distributions,* Ricerche Mat. 18 (1969), 215–238.

[26] L. S c h w a r t z, *Theórie des distributions,* 2 vols., Hermann, Paris, (1950-1951).

[27] L. S c h w a r t z, *Theórie des distributions à valeurs vectorielles,* Annales Inst. Fourier, $1^{ére}$ partie: 7 (1957), 1-141; $2^{éme}$ partie: 8 (1958), 1–207.

[28] N. T a n a k a, N. O k a z a w a, *Local C-semigroups and local integrated semigroups,* Proc. London Math. Soc., (1990), 63–90.

[29] H. R. T h i e m e, *Integrated semigroups and integrated solutions to abstract Cauchy problems,* J. Math. Anal. Appl., 152 (1990), 416–447.

[30] H. R. T h i e m e, *Semiflows generated by Lipschitz perturbations of non-densely defined operators,* Diff. Integr. Equations, 6 (1990), 1035–1066.

[31] F. T r e v e s, *Topological Vector Spaces, Distributions and Kernels,* Acad. Press, New York, (1967)

[32] T. U s h i j a m a, *Some properties of regular distribution semigroups,* Proc. Japan Acad., 45 (1969), 224–227.

[33] V. S. V l a d i m i r o v, *Generalized Functions in Mathematical Physics,* Mir, Moscow, (1979).

[34] V. S. V l a d i m i r o v, Y. N. D r o ž ž i n o v, B. I. Z a v i a l o v, *Multidimensional Tauberian Theorems for Generalized Functions,* Nauk, Moscow, (1986) (In Russian).

[35] S. W a n g, *Quasi-Distribution Semigroups and Integrated Semigroups,* J. Func. Anal., 146 (1997), 352–381.

Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad
Yugoslavia