Bulletin T.CXXII de l'Académie Serbe des Sciences et des Arts - 2001
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 26

A NEW QUANTITATIVE ANALYSIS OF SOME BASIC PRINCIPLES OF THE THEORY OF FUNCTIONS OF A REAL VARIABLE

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(Presented at the 4th Meeting, held on May 25, 2001)

A b s t r a c t. *We define and study a new regularity requirement that one can put on sets of reals where a given function is to be continuous. The requirement is expressed in terms of number systems used in representing reals as strings of digits. We compare it with the classical requirement expressed in terms of the Lebesgue measure.*

AMS Mathematics Subject Classification (2000): 28A05, 28A10, 26A21, 05D10.
Key Words: measurable, continuous

1. *Introduction*

Recall the following three basic principles of the theory of functions of a real variable sometimes known as 'Littlewood's three principles' [1]:

(I) Every [measurable] set is nearly a finite union of intervals.

(II) Every [measurable] function is nearly continuous.

[1] see [10;p.26] and [14;p.71].

(III) Every convergent sequence of [measurable] functions is nearly uniformly convergent.

When [measurable] is interpreted as [Lebesgue measurable] the principles II and III are widely known as theorems of Luzin and Egoroff, respectively (see e.g.,[11],[12],[14]). In Luzin's theorem 'nearly' is interpreted as saying that the Lebesgue measure of the difference of the starting set E and its subset F, where a given function is continuous, can be made arbitrarily small. Similarly, in Egoroff's theorem 'nearly' is interpreted as saying that the Lebesgue measure between the starting set E (of finite measure) and its subset F, on which a given sequence of measurable functions is uniformly continuous, can be made arbitrarily small.

In this note we analyze these principles from a quite different angle motivated by other regularity properties one sometimes needs to put on the sets E and F. For example, one sometimes work with real numbers as strings of digits of some number-system and needs a continuity of a given function on a set F which is regular from the point of view of this number system. A most natural requirement is to ask the set F to include all strings that use certain digits. (Recall, for example, the definition of the Cantor ternary set.) It is not surprising that in order to obtain any positive result in this direction one has to has a bit more extensive notion of a 'number system'. Rather than having a fixed integer $n > 1$ and considering real numbers (from the unit interval) as strings of (finite or infinite) digits taken from the set $\{0, 1, 2, \cdots, n-1\}$,[2] we need to fix a sequence $\vec{n} = \{n_i\}_{i=0}^{\infty}$ of integers > 1 and consider real numbers as infinite strings $\vec{x} = \{x_i\}_{i=0}^{\infty}$ such that $x_i < n_i$ for all i. We call this way of representing real numbers the *$\vec{n}$-ary number system.* (Thus, the case $n_i = 2$ for all i corresponds to the binary system and the case $n_i = 10$ for all i corresponds to the decimal system.) A specific version of the problem which we choose to address here is the following

Problem 1: Describe the smallest $\vec{n}-$ary number system which has the property that every Borel function is continuous on a set of the form $\{\vec{x} : x_i \in J_i \text{ for all } i\}$ where $J_i \subseteq n_i$ has at least two different digits for all i.

There is no real reason behind our choice of the class of Borel functions in this Problem except that it seems like a good test-class. The general version of the Problem 1 would ask for the description of a sufficiently rich algebra of functions for which this version of Littlewood's principle II is true. There is also no real reason behind the requirement that $|J_i| \geq 2$ for all i except

[2] the set which, from now on, will simply be denoted as n.

that it appears to be a good test case. The general form of Problem 1 would also ask for the description of all sequences $\{|J_i|\}_{i=0}^{\infty}$ of cardinalities of sets of the allowable digits that ensure the continuity of a given function. Finally, a general form of Problem 1 would also ask for the description of all (if any) $\vec{n}-$ary number system for which this version of Littlewood's principle $\mathbb{I}$ is true.

In proposition 2.2 we show that a positive answer to Problem 1 even in the class of semi continuous function requires a number system $\vec{n}$ which is double-exponential as a rate of growth. Curiously enough this example points out that there might be a close connection between (say) *Baire hierarchy* of real function and a hierarchy of rapidly growing number-theoretic functions [3]. One such hierarchy is the *Ackermann hierarchy* (sometimes also called the *Grzegorczyk hierarchy*; see [1],[3],[6],[13]) defined by the following double recursive formula

$$A_{n+1}(x+1) = A_n(A_{n+1}(x))$$

starting from the initial function $A_0(x) = x+2$ $(x > 1)$; $A_0(0) = 1, A_0(1) = 2$. Thus, $A_1(x) = 2x$ $(x \neq 0)$, $A_2(x) = 2^x$, and

$$A_3(x) = 2^{2^{\cdot^{\cdot^{2}}}} \}\ {}^{x\ times}, \text{ etc...}$$

Thus, the Proposition 2.2 shows that the A_2-ary system is not sufficiently fast to give us the positive answer to problem 1 and seems unlikely (though this remains yet to be proven) that any of the bounded iterates to A_2 will be sufficiently fast either. It turns out that the next in order number system, the number system of the *tower function* A_3, is sufficiently fast to give us even the positive answer to the most general form of Problem 1. So close connection between A_3 and a mathematical problem is rather rare though the number of places where this function shows up is increasing (see [4] and [8] for a recent and not so recent example, respectively). In a subsequent and more complete paper we shall give a similar analysis of other principles from the real variable theory such as for example the two selection theorems of E. Helly (see [11],[12]). This will require hierarchies of number-theoretic functions that seem unrelated of any of the large spectrum of known hierarchies.

[3]'Order's of infinity' in the sense of [7]

2. *Some Examples*

Our results can best be stated if one (via continued fractions) identifies the reals from the unit interval with infinite strings of integers. This of course ignores the rational numbers which, much like in the theories of Lebesgue measure and Baire category, can indeed be ignored. Some caution is in order however, especially when we make a claim, like for example the one in Proposition 2.2 below, that some function is continuous. What we really mean is the continuity on the irrationals. This does not affect the intended application of Proposition 2.2 which claims the existence of an essentially double-exponential number system for which the Problem 1 has a negative solution in the class of semi-continuous functions.

In this section we give two examples. The first example shows that the notion of [measurability] that would positively resolve the general form of Problem 1 must necessarily be quite different from any of the two classical notions due to H. Lebesgue and R. Baire.

Proposition 1.1. *There is a function f, measurable in the sense on Lebesgue and Baire, which is not continuous on any set of the form $\prod_{i=0}^{\infty} J_i$ such that $|J_i| \geq 2$ for all but finitely many i's.*

P r o o f. For each positive integer i define an equivalence relation $\backsim_i$ between infinite strings of integers by letting $\vec{x} \backsim_i \vec{y}$ if $x_{2^i(2j+1)} = y_{2^i(2j+1)}$ for all but finitely many j's. For each i and each equivalence class e of $\backsim_i$ pick a representative $\vec{z}^{\,i}_e \in e$. Let $d_i(\vec{x})$ be the distance between $\vec{x}$ and the representative $\vec{z} = \vec{z}^{\,i}_e$ of its $\backsim_i$ $-$equivalence class $e = [\vec{x}]$ i.e., the cardinality of the set

$$\{j : x_{2^i(2j+1)} \neq z_{2^i(2j+1)}\}.$$

The function $f(\vec{x}) = \vec{y}$ is defined according to the following two cases:
Case 1: $x_i = x_{i+1}$ for infinitely many i. In this case put $y_i = 0$ for all i.
Case 2: $x_i \neq x_{i+1}$ for all but finitely many i. In this case put

$$y_i = \max\{n : 2^n | d_i(\vec{x})\}.$$

Note that the set of all $\vec{x}$ which fall into the Case 1 is at the same time of full Lebesgue measure and of full Baire category. It follows that f is Lebesgue as well as Baire-measurable.

To show that f satisfies the conclusion of Proposition 1 it suffices to show that f takes a dense-set of values on any product $\prod_{i=0}^{\infty} J_i$ such that $|J_i| \geq 2$ for all but finitely many i. Choose $\vec{w} \in \prod_{i=0}^{\infty} J_i$ such that $w_i \neq w_{i+1}$

for all but finitely many i's. Let e_i be the $\backsim_i$ $-$equivalence class of $\vec{w}$. Then for every finite sequence $n_0, \cdots, n_k$ of integers we can find $\vec{x} \in \prod_{i=0}^{\infty} J_i$ such that for all $i \le k$:
(1) $\vec{x} \in e_i$,
(2) $n_i = \max\{n : \ 2^n | d_i(\vec{x})\}$.

It follows that $f(\vec{x})$ extends the given finite sequence $n_0, \cdots, n_k$ of integers. This shows that the $f-$image of $\prod_{i=0}^{\infty} J_i$ is a dense set of reals and finishes the proof. □

Proposition 2.2. *Suppose that* $\{n_i\}_{i=0}^{\infty}$ *is an infinite sequence of positive integers which is eventually dominated by any of the linear shifts* $\{2^{2^{i-l}}\}_{i=0}^{\infty}$ $(l = 1, 2, \cdots)$ *of the double-exponential sequence. Then there is continuous function* $f : \prod_{i=0}^{\infty} n_i \to [0,1]$ *which takes all the values from the unit interval on any subproduct* $\prod_{i=0}^{\infty} J_i$ *such that* $|J_i| = 2$ *for all* i.

P r o o f. Define a strictly increasing sequence $\{l_p\}$ of nonnegative integers and maps $c_p : \prod_{i=l_p}^{l_{p+1}-1} n_i \to \{0,1\}$ as follows. Let $l_0 = 0$ and suppose $l_0, \cdots l_p$ have been determined. Fix $l > l_p$ such that $n_i \le 2^{2^{i-l_p-1}}$ for all $i \ge l$. For $k > l$, the probability that a given partition $c : \prod_{i=l_p}^{k-1} n_i \to \{0,1\}$ is constant on a given subproduct $\prod_{i=l_p}^{k-1} J_i$ such that $|J_i| = 2$ for all $i \in [l_0, k)$ is equal to $2/2^{2^{k-l_p}}$. Hence the probability that there is such a subproduct on which c is constant is equal to

$$\frac{2}{2^{2^{k-l_p}}} \prod_{i=l_p}^{k-1} \binom{n_i}{2}.$$

This quantity is dominated by

$$M \frac{2}{2^{2^{k-l_p}}} \prod_{i=l_p}^{k-1} \binom{2^{2^{i-l_p-1}}}{2} \le M \frac{2}{2^{2^{k-l_p}}} \frac{2^{\sum_{i=l_p}^{k-1} 2^{i-l_p}}}{2^{k-l_p}} = \frac{M}{2^{k-l_p}},$$

where $M = \prod_{i=l_p}^{l+1} \binom{n_i}{2}$. Let l_{p+1} be the minimal integer $k > l$ such that $M/2^{k-l_p} < 1$ and let $c_p : \prod_{i=l_p}^{l_{p+1}-1} n_i \to \{0,1\}$ be any of the mappings that is not constant on any subproduct of the form $\prod_{i=l_p}^{l_{p+1}-1} J_i$ such that $|J_i| = 2$ for all $i \in [l_p, l_{p+1})$.

The map that would satisfy the conclusion of Proposition 2.2 is viewed as a function $f : \prod_{i=0}^{\infty} n_i \to \{0,1\}^{\mathbb{N}}$ where, for a given string $\vec{x} \in \prod_{i=0}^{\infty} n_i$ the binary string $\vec{y} = f(\vec{x})$ is determined by the rule

$$y_p = c_p(x_{l_p}, \cdots, x_{l_{p+1}-1}).$$

Clearly, f is a continuous map. Given a subproduct $\prod_{i=0}^{\infty} J_i$ such that $|J_i| = 2$ for all i, on any of the partial subproducts $\prod_{i=l_p}^{l_{p+1}-1} J_i$ the map c_p takes both values 0 and 1. It follows that for every infinite binary sequence $\vec{y}$ there is $x \in \prod_{i=0}^{\infty} J_i$ such that $f(\vec{x}) = \vec{y}$. This finishes the proof. $\square$

Remark 2.3. Note that composing the function f of Proposition 2.2 with a characteristic function of a nonempty compact set of irrationals one obtains a semi-continuous map $g : \prod_{i=0}^{\infty} n_i \to [0,1]$ which is not continuous on any subproduct $\prod_{i=0}^{\infty} J_i$ such that $|J_i| = 2$ for all i. Since clearly we can have a similar example where the constant sequence $\{2\}_{i=0}^{\infty}$ is replaced with an arbitrary infinite sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers, this gives us some hints towards the notion of nearness needed for a positive resolution of the general form of Problem 1. It points to us that there must be an important difference in the rate of growth of the $\vec{n}-$ary number system and the output sequence $\{|J_i|\}_{i=0}^{\infty}$. For example, the positive answer to Problem 1 for the algebra of semi-continuous finite step-functions, requires a double-exponential difference in the rate of growths, an estimate which is quite close to the optimal one. We shall give an upper estimate of the difference in rates of growths sufficient for making Baire class-1 functions continuous which is considerably faster than the double-exponential but which will work for the class of all Borel functions and beyond.

3. *An algebra of functions*

We start by describing a strategy S which to every infinite sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers associates in a Lipschitz way an infinite sequence $\{S(m_0, \cdots, m_i)\}_{i=0}^{\infty}$ of positive integers.

Definition 2.1. —it Define $S : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ as follows

$$S(m_0) = 2m_0 - 1$$

$$S(m_0, \cdots, m_{i+1}) = 2(m_{i+1} - 1) \left[\prod_{k=0}^{i} \binom{S(m_0, \cdots, m_k)}{m_k} \right] + 1.$$

Note the following property of S from which one can easily deduce the positive solution to Problem 1 for the class of semi-continuous two-step-functions, and a property which also explains our choice for its definition.

Lemma 2.2. *For every positive integer k, every mapping from the product $\prod_{i=0}^{k} S(m_0, \cdots, m_i)$ into $\{0,1\}$ is constant on a subproduct $\prod_{i=0}^{k} J_i$ such that $|J_i| = m_i$ for all $i \leq k$.* □

In order to treat semi-continuous step-functions with a larger number of steps it seems natural to try iterating the strategy S as follows.

Definition 2.2. *The pth iterate $S^{(p)} : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ of S is defined recursively as follows:*

$$S^{(0)}(m_0, \cdots, m_i) = S(m_0, \cdots, m_i),$$

$$S^{(p+1)}(m_0, \cdots, m_i) = S(S^{(p)}(m_0), S^{(p)}(m_0, m_1), \cdots, S^{(p)}(m_0, \cdots, m_p)).$$

Definition 2.3. *For a set $M \subseteq \mathbb{N}$ define S_M : $BbbN^{<\mathbb{N}} \to \mathbb{N}$ as follows:*

$$S_M(m_0) = S(m_0),$$

$$S_M(m_0, \cdots, m_i) = S^{(p)}(m_0, \cdots, m_i),$$

where

$p = |\bigcup_{k \in M \cap i} \ \prod_{l=0}^{k} S_M(m_0, \cdots, m_l)|.$

Let $S_1 = S_M$ for $M = \mathbb{N}$. We shall say that a product $\prod_{i=0}^{\infty} J_i$ is of *type* $\vec{m} = \{m_i\}_{i=0}^{\infty}$ if $|J_i| = m_i$ for all i. If R is any of the functions S, $S^{(p)}$ or S_M defined above then by $R(\vec{m})$ we denote the infinite sequence $\{R(m_0, \cdots, m_i)\}_{i=0}^{\infty}$. One can view S_M's as some sort of diagonalizations of the sequence of finite iterates $S^{(p)}$. Since $S^{(p)}$ is meant to treat the case of semi-continuous p-step-functions, it is not surprising to have the following result.

Lemma 2.4. *Suppose that f is a given semi-continuous function and that $\{m_i\}_{i=0}^{\infty}$ is a given sequence of positive integers. Then for every infinite $M \subseteq \mathbb{N}$ there is infinite $N \subseteq M$ such that every product of type $S_M(\vec{m})$ contains a subproduct of type $S_N(\vec{m})$ on which f is continuous.* □

Remark 2.5. As indicated above, the proof of Lemma 2.4 involves a natural diagonalization argument which uses Lemma 2.2. The diagonalization argument gives us a considerable amount of flexibility. For example, if we start with a set $M \subseteq \mathbb{N}$ of asymptotic density 1 [4] then we can produce an output set $N \subseteq M$ of asymptotic density 1 as well. More importantly, the diagonalization argument can be repeated indefinitely so that an induction on the Baire rank will lead us to the following more general fact with the same freedom of choosing N inside a given M.

Lemma 2.6. *Suppose that f is a given Borel function and that $\{m_i\}_{i=0}^{\infty}$ is a given sequence of positive integers. Then for every infinite $M \subseteq \mathbb{N}$ there is infinite $N \subseteq M$ such that every product of type $S_M(\vec{m})$ contains a subproduct of type $S_N(\vec{m})$ on which f is continuous.* □

We are now ready to give the definition of the algebra of functions which gives the positive solution to Problem 1 for the principles II and III.

Definition 2.7 *Let $\mathcal{M}(S)$ be the algebra of all real functions f with the property that for every sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers, every infinite $M \subseteq \mathbb{N}$ and every positive integer n there is infinite $N \subseteq M$ such that $N \cap n = M \cap n$ and such that every $S_M(\vec{m})-$product contains a $S_N(\vec{m})-$subproduct on which f is continuous.*

The following is essentially a reformulation of Lemma 2.6.

Theorem 2.8. *Every Borel function belongs to $\mathcal{M}(S)$.* □

Definition 2.9. *Let $\mathcal{M}_{01}(S)$ be the field of sets of reals A whose characteristic functions belong to$\mathcal{M}(S)$.*

It follows that $\mathcal{M}(S)$ is the collection of all functions f with the property that $f^{-1}(-\infty, q) \in \mathcal{M}_{01}(S)$ for all $q \in \mathbb{Q}$.

Theorem 2.10. *$\mathcal{M}_{01}(S)$ is a σ-field of sets which contains all open sets and which is closed under Souslin operation.* □

Theorem 2.11. *Under a suitable axiom of infinity, [5] the field $\mathcal{M}_{01}(S)$ contains every set of reals that can be defined using reals and ordinals as parameters.* □

Remark 2.12. Recall that the classical fields of Lebesgue-measurable and Baire- measurable sets of reals do have the properties listed in Theorems 2.8,2.10 and 2.11. According to the Proposition 1.1, the field $\mathcal{M}_{01}(S)$ does

[4] I.e., $|M \cap \{0, \cdots, n-i\}|/n \to 1$

[5] see [9]

not coincide with neither of these two classical σ-fields.

3. *Rates of growths*

Recall the Ackermann hierarchy of fast-growing number-theoretic functions defined as follows (see e.g.,[3],[13]):

$$A_0(0) = 1, \quad A_0(1) = 2, \quad A_0(x) = 2 + x \quad (x > 1),$$
$$A_{n+1}(x) = A_n^{(x)}(1),$$

where $A_n^{(x)}$ denotes the x^{th} iterate of A_n. (The 0^{th} iterate of any function is the identity function.) Thus,

$$\begin{aligned} &A_1(x) = 2x \quad (x \neq 0),\\ &A_2(x) = 2^x,\\ &A_3(x) = 2^{2^{\cdot^{\cdot^{2}}}} \} \; x \; times. \end{aligned}$$

A useful formula that relates A_n and A_{n+1} is the following:

$$A_{n+1}(x+1) = A_n(A_{n+1}(x)).$$

Many of the standard functions are eventually bounded by the tower function A_3, the first ‘non elementary’ function in the sequence $\{A_n\}$. The purpose of this section is to compare the functions S, $S^{(p)}$ and S_1 of the previous chapter when evaluated at a particular sequence $\{(m_0, \cdots, m_i)\}_{i=0}^{\infty}$. It is natural to choose the starting sequence $\{m_i\}_{i=0}^{\infty}$ to be elementary (i.e., below a bounded iterate of A_2). As it turns out, there is no much difference which elementary sequence $\{m_i\}_{i=0}^{\infty}$ we choose, so we take the constant sequence $m_i = 2$ as our starting sequence. It will be convenient to use the notation 2_n for a sequence of 2's of length $n+1$. Thus we are interested in the rates of growths of the number-theoretic functions $S(2_n)$, $S^{(p)}(2_n)$ and $S_1(2_n)$.

Lemma 3.1. $S(2_n) \leq A_2^{(2)}(2n+1)$.

P r o o f. Note that the first two values $S(2_0) = 3$, $S(2_1) = 7$ satisfy the inequality. For $n \geq 2$ we have

$$S(2_n) = 2(2-1)\prod_{i=0}^{n-1}\binom{S(2_i)}{2} + 1 \;\leq$$

$$\leq 2^2 \frac{1}{2^n} 2^{\Sigma_{i<n} 2^{2i+2}} \leq 2^{2^{2n+1}}.$$

Lemma 3.2. $S^{(p)}(2_n) \leq A_2^{(p+2)}(2n+1)$.

P r o o f. By Lemma 3.1 we have this in case $p = 0$. To see the inductive step, note that for $p > 0$,

$$S^{(p)}(2_n) = 2(S^{(p-1)}(2_n) - 1) \prod_{i=0}^{n-1} \binom{S^{(p)}(2_i)}{S^{(p-1)}(2_i)} + 1$$

$$\leq 2A_2^{(p+1)}(2n+1) \prod_{i=0}^{n-1} \frac{3^{S^{p-1}(2_i)}}{\left[S^{(p-1)}(2_i)\right]^{S^{(p-1)}(2_i)}} \prod_{i=0}^{n-1} [S^{(p)}(2_i)]^{S^{(p-1)}(2_i)}$$

$$\leq 2A_2^{(p+1)}(2n+1) \left[S^{(p)}(2_{n-1})\right]^{\Sigma_{i=0}^{n-1} S^{(p-1)}(2_i)}$$

$$\leq 2A_2^{(p+1)}(2n+1) \left[A_2^{(p+2)}(2n-1)\right]^{A_2^{(p+1)}(2_n)} \leq A_2^{(p+2)}(2n+1).$$

Lemma 3.3. $(n+1)\,[S_1(2_n)]^{(n+1)} + 2n + 6 \leq A_4(n+3)$.

P r o o f. This is easily seen to be true in case $n = 0$, so we concentrate on the inductive step at some $n > 0$:

$$\begin{aligned}(n+1)\,[S_1(2_n)]^{n+1} + 2n + 6 &\leq (n+1)\left[A_2^{(n[S_1(2_{n-1})]^n+2)}(2n+1)\right] + 2n + 6\\ &\leq A_2^{(n[S_1(2_{n-1})]^n+3)}(2n+1)\\ &\leq A_3(n[S_1(2_{n-1})]^n + 2n + 4)\\ &\leq A_3(A_4(n+2)) = A_4(n+3).\end{aligned}$$

□

Remark 3.4. It is easily checked that essentially same bounds are true for the sequence $\{m_i = i\}_{i=0}^{\infty}$. In fact, the function A_4 is so robust that the bound of Lemma 3.3 will essentially[6] be true for every sequence $\{m_i\}_{i=0}^{\infty}$ (in place of $\{m_i = 2\}_{i=0}^{\infty}$) that is *elementary* in A_3, i.e., eventually bounded by some iterate of A_3. Thus we have shown the following interesting connection

[6] More precisely, when $A_4(n+3)$ is replaced by some other shift $A_4(an+b)$.

between Ackermann hierarchy of fast growing number-theoretic functions and Littlewood's principle II from the real variable theory.

Theorem 3.5. *Let $n \geq 3$ be a given integer and let f be a given function in $\mathcal{M}(S)$. Then every A_n-product can be refined to a subproduct where f is continuous and whose type can be taken to be any sequence bounded by an iterate of A_{n-1}.* □

The cases $n \geq 4$ of this result follow easily from what has been said above, so only the case $n = 3$ requires some explanation. This case of Theorem 3.5 follows from the definition of the algebra $\mathcal{M}(S)$ and the following fact which is of independent interest as it gives us what seems to be the minimal number-system for which Problem 1 has a positive solution.

Lemma 3.6. *There is infinite $M \subseteq \mathbb{N}$ with the property that the sequence $S_M(\vec{2})$ is eventually dominated by A_3.*

P r o o f. Recall that $S_M(\vec{2})$ is a step function composed of steps of the form

$$S^{(p)}(\vec{2}) \restriction (\bar{k}, \bar{l}],$$

where $\bar{k} < \bar{l}$ are two consecutive members of M and where

$$p = |\bigcup_{k \in M \cap (\bar{k}+1)} \prod_{l=0}^{k} S_M(2_l)|.$$

By Lemma 3.2 we know that each of the steps $S^{(p)}(\vec{2})$ is bounded by $A_2^{(p+2)}(2n+1)$ so we choose M so thin in order to have the inequality $A_2^{(p+2)}(2n+1) \leq A_2^{(n)}(1)$ for every step $S^{(p)}(\vec{2})$ and every n in the interval it acts. □

4. *Concluding Remarks*

In [4] we have considered a Ramsey theoretic problem which corresponds to the case of two-step-functions in our present discussion. Note that Theorem 3.5 leads to a positive answer to Question 1 of [4; §5].

Note that Lemma 3.6 hints towards a finer hierarchy from that of Ackermann, a hierarchy of fast-growing sequences of integers that will better correspond to Littlewood's principle II. This remains yet to be investigated.

Finally we note that all what has been said above about Litllewood's principle IItransfers readily to his principle III as well. This translation is done by following the standard proof of Egoroff's Theorem (see [11; p.112]) and making all the sets involved clopen when restricted on a suitably chosen subproduct.

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