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REGULARLY VARYING SOLUTIONS OF GENERALIZED THOMAS-FERMI EQUATIONS

T. KUSANO, V. MARIĆ AND T. TANIGAWA

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A b s t r a c t. The existence of slowly and regularly varying solutions of index 1 (in the sense of Karamata) and their asymptotics of generalized Thomas-Fermi equations presented below are studied.

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1. *Introduction*

The objective of this paper is to discuss the existence of regularly varying solutions in the sense of Karamata for generalized Thomas-Fermi equations of the form

$$(|x'|^{\alpha}\text{sgn}\, x')' = q(t)|x|^{\beta}\text{sgn}\, x, \tag{A}$$

where α and β are distinct positive constants, and $q : [a, \infty) \to (0, \infty)$, $a > 0$, is a continuous function. It is often useful to rewrite equation (A) in the form

$$((x')^{\alpha *})' = q(t)x^{\beta *},$$

in terms of the asterisk notation

$$u^{\gamma *} = |u|^{\gamma}\text{sgn}\, u, \quad u \in \mathbf{R}, \quad \gamma > 0. \tag{1.1}$$

In addition, we show how an application of the theory of regular variation gives the possibility of obtaining the precise asymptotic behaviour of solutions in question for some subclasses of equation (A).

For $\alpha = 1$, $\beta > 1$ equation (A) is reduced to the one of Thomas-Fermi type and for $\alpha = \beta$ it is reduced to the half-linear one:

$$((x')^{\alpha *})' = q(t)x^{\alpha *}. \tag{B}$$

The latter fact is suggestive to call (A) super-half-linear if $\alpha < \beta$ and sub-half-linear if $\alpha > \beta$.

Noting that if $x(t)$ satisfies (A), then so does $-x(t)$, we will focus our attention on positive solutions of (A) which can be extended to infinity. Such solutions are called *proper* positive solutions of (A). It should be noticed that not all positive solutions are proper. In fact, it is known ([5]) that the super-half-linear equation (A) always has a solution $x(t)$ which is positive on a finite interval $[t_0, t_1)$ and satisfies

$$\lim_{t \to t_1 - 0} x(t) = \lim_{t \to t_1 - 0} x'(t) = \infty,$$

and that the sub-half-linear equation (A) always has a solution $x(t)$ on an infinite interval $[t_0, \infty)$ such that

$$x(t) > 0 \text{ on some finite interval } [t_0, t_1), \quad \text{and} \quad x(t) = 0 \text{ on } [t_1, \infty).$$

Let $x(t)$ be a positive solution of (A) on $[t_0, \infty)$. Since equation (A) implies that $(x'(t))^{\alpha *}$ is increasing, it follows that either $x'(t) < 0$ on the entire interval $[t_0, \infty)$ or $x'(t) > 0$ on $[t_1, \infty)$ for some $t_1 > t_0$. In the former case $x'(t) \to 0$ as $t \to \infty$, and $x(t)$ *decreases* to a finite nonnegative limit as $t \to \infty$.

In the latter case $x'(t)$ increases to a finite or infinite positive limit $x'(\infty)$ as $t \to \infty$. If $x'(\infty) < \infty$, then $x(t)$ satisfies $\lim\limits_{t \to \infty} x(t)/t = x'(\infty)$, that is, $x(t)$ *increases* and is asymptotic to a constant multiple of t as $t \to \infty$. If

$x'(\infty) = \infty$, then $\lim_{t\to\infty} x(t)/t = \infty$, which means that $x(t)$ increases faster than any constant multiple of t as $t \to \infty$.

The problem of existence of proper solutions for equation (A) have been studied by Mizukami, Naito and Usami [5]. Their main results are as follows.

Proposition 1.1. *Equation (A), either super-half-linear or sub-half-linear, has a proper solution $x(t)$ such that*

$$\lim_{t\to\infty} x(t) = \text{ const} > 0 \tag{1.2}$$

if and only if

$$\int_a^\infty \left(\int_t^\infty q(s)ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{1.3}$$

Proposition 1.2. *Equation (A), either super-half-linear or sub-half-linear, has a proper solution $x(t)$ such that*

$$\lim_{t\to\infty} \frac{x(t)}{t} = \text{ const } > 0 \tag{1.4}$$

if and only if

$$\int_a^\infty t^\beta q(t)dt < \infty. \tag{1.5}$$

Proposition 1.3. *Super-half-linear equation (A) has a proper solution $x(t)$ such that*

$$\lim_{t\to\infty} x(t) = 0 \tag{1.6}$$

if and only if

$$\int_a^\infty \left(\int_t^\infty q(s)ds \right)^{\frac{1}{\alpha}} dt = \infty. \tag{1.7}$$

Proposition 1.4. *Sub-half-linear equation (A) has a proper solution $x(t)$ such that*

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty \tag{1.8}$$

if and only if

$$\int_a^\infty t^\beta q(t)dt = \infty. \tag{1.9}$$

Our analysis is based on the use of the theory of regular variation (in the sense of Karamata).

The most complete presentation of theory of regular variation and its applications can be found in the book [1] of Bingham, Goldie and Teugels. For a comprehensive survey of results on the asymptotic analysis of ordinary differential equations in the framework of regular variation up to 2000 the reader is referred to the monograph [4] of Marić.

For the reader's benefit we state the definition and some basic properties of regularly varying functions.

Definition 1.1. *A measurable function* $f : [0,\infty) \to (0,\infty)$ *is said to be* regularly varying of index $\rho \in \mathbf{R}$ *if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \textit{for any} \quad \lambda > 0. \tag{1.10}$$

One of the most important properties of regularly varying solutions is the following representation theorem.

Proposition 1.5. $f(t) \in RV(\rho)$ *if and only if it is expressed in the form*

$$f(t) = c(t) \exp\left\{\int_{t_0}^t \frac{\delta(s)}{s} ds\right\}, \quad t \geq t_0, \tag{1.11}$$

for some $t_0 > 0$ *and some measurable functions* $c(t)$ *and* $\delta(t)$ *such that*

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty), \quad \lim_{t\to\infty} \delta(t) = \rho. \tag{1.12}$$

The totality of regularly varying solutions of index ρ is denoted by $RV(\rho)$. In particular SV stands for $RV(0)$, and members of $SV = RV(0)$ are called *slowly varying* functions. If $c(t) \equiv c_0$ in (1.11), $f(t)$ is referred to as a *normalized* regularly varying function of index ρ. By definition any $f(t) \in RV(\rho)$ is written as $f(t) = t^\rho L(t)$ with $L(t) \in SV$, and so the class SV of slowly varying functions is of fundamental importance in the study of regularly varying functions. Typical examples of slowly varying functions are: all functions tending to some positive constants,

$$\prod_{n=1}^{N}(\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbf{R}, \quad \text{and} \quad \exp\left\{\prod_{n=1}^{N}(\log_n t)^{\beta_n}\right\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm.

The following result concerns operations which preserve slow variation.

Proposition 1.6. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^\alpha$ for any $\alpha \in \mathbf{R}$, $L_1(t)+L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$ as $t \to \infty$) are slowly varying.*

A slowly varying function $L(t)$ may grow to infinity or decay to zero as $t \to \infty$. However, the order of growth or decay of $L(t)$ at infinity is severely limited as the following proposition shows.

Proposition 1.7. *If $L(t)$ is slowly varying, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} L(t) = \infty, \quad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

The following result which encompasses the integration theorem due to Karamata is useful in handling slowly (and regularly) varying functions analytically.

Proposition 1.8. *Let $L(t)$ be a slowly varying function. Then. we have as $t \to \infty$*

$$\int_{t_0}^{t} s^{\gamma} L(s)ds \sim \frac{t^{\gamma+1}}{\gamma+1} L(t) \quad \text{if} \quad \gamma > -1; \tag{i}$$

$$\int_{t}^{\infty} s^{\gamma} L(s)ds \sim -\frac{t^{\gamma+1}}{\gamma+1} L(t) \quad \text{if} \quad \gamma < -1; \tag{ii}$$

If $\gamma = -1$ the occurring integrals are new slowly varying functions (iii).

Here and throughout the paper the symbol $\sim$ denotes the asymptotic equivalence:

$$f(t) \sim g(t) \quad \text{as} \quad t \to \infty \Leftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1. \tag{1.13}$$

We note that a function $f(t)$ satisfying

$$\lim_{t\to\infty} \frac{f(t)}{t^\rho} = \text{ const } > 0 \tag{1.14}$$

is a simple example of regularly varying functions of index ρ. Such a function is called a *trivial* regularly varying function of index ρ. If in particular $\rho = 0$, $f(t)$ is called a *trivial* slowly varying function. A function $f(t) \in RV(\rho)$ not satisfying (1.14) is said to be a *nontrivial* regularly varying function of index ρ. Proposition 1.1 (resp. Proposition 1.2) shows that one can completely characterize the existence of a trivial slowly varying solution (resp. of a trivial regularly varying solution of index 1) for equation (A), either super-half-linear or sub-half-linear.

Here we formulate some conditions under which equation (A) possesses a nontrivial slowly varying solution and a nontrivial regularly varying solution of index 1, for both super-half-linear and sub-half-linear cases of (A) by making extensive use of the existence results explained in Section 2 for half-linear equations (B). Our results are presented in Section 3 and 4 devoted, respectively, to super-half-linear and sub-half-linear equations of the form (A). It is hoped that this paper could provide a clue to the construction of regularly varying solutions of general index $\rho \neq 0, 1$ for generalized Thomas-Fermi type equations.

Since the inequalities occurring in the paper hold for $t \geq T$, we shall omit the adjective occasionally.

2. *Half-linear equation*

2.1. Slowly varying solutions. In this preparatory section we present basic existence theorems of slowly varying solutions (SV-solutions for short) and of regularly varying solutions of index 1 (RV(1)-solutions for short) for the half-linear equation (B), upon which the proofs of our main results for (A) essentially depend. We note that the existence of such solutions for (B) was established for the first time by Jaroš, Kusano and Tanigawa [4]. Their results are framed here in two propositions which follow.

Proposition 2.1. *Equation (B) possesses a slowly varying solution $x(t)$ which decreases and for $t \geq T$ has the form*

$$x(t) = \exp\left\{-\int_T^t \frac{(Q(s) - v(s))^{1/\alpha}}{s} ds\right\}, \tag{2.1}$$

where $v(t)$ is some solution tending to zero of the integral equation

$$v(t) = \alpha t^\alpha \int_t^\infty \frac{|v(s) - Q(s)|^{1+1/\alpha}}{s^{\alpha+1}} ds, \tag{2.2}$$

if and only if

$$Q(t) := t^\alpha \int_t^\infty q(s) ds \to 0, \quad as \quad t \to \infty. \tag{2.3}$$

P r o o f. For the proof of the "only if" part of the proposition see [3. Th. 2.2']. We prove the "if" part in a more direct way than in [3].

Suppose that (2.3) holds. Let l be a positive constant such that

$$\lambda := \left(1 + \frac{1}{\alpha}\right) l^{\frac{1}{\alpha}} < 1, \tag{2.4}$$

and choose $T > a$ large enough so that for $t \geq T$,

$$Q(t) \leq l. \tag{2.5}$$

Let V be the set

$$V = \{v(t) \in C_0[T, \infty) : 0 \leq v(t) \leq l, t \geq T\}, \tag{2.6}$$

where $C_0[T, \infty)$ is the Banach space of all continuous functions on $[T, \infty)$ that tend to 0 as $t \to \infty$ with the norm $\|v\|_0 = \sup_{t \geq T} |v(t)|$, and define the integral operator $\mathcal{F}$ by

$$\mathcal{F}v(t) = \alpha t^\alpha \int_t^\infty \frac{|v(s) - Q(s)|^{1+\frac{1}{\alpha}}}{s^{\alpha+1}} ds. \tag{2.7}$$

Then it can be shown that $\mathcal{F}$ is a self-map on V and satisfies

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_0 \leq \lambda \|v_1 - v_2\|_0.$$

In view of (2.4) this means that $\mathcal{F}$ is a contraction on V, and there exists a fixed point $v(t) \in V$ of $\mathcal{F}$, which is a solution of the integral equation (2.2). Then in virtue of Elbert lemma, [2], $x(t)$ is a solution of equation (B) because the function $u(t) = (v(t) - Q(t))/t^\alpha$ satisfies the generalized Riccati equation associated with (B):

$$u' + \alpha |u|^{1+\frac{1}{\alpha}} = q(t). \tag{2.8}$$

Since $v(t) - Q(t) \to 0$ as $t \to \infty$, $x(t)$ is a slowly varying function due to the representation (1.11) with $\rho = 0$. This establishes the existence of an SV-solution for (B).

The following result is useful.

Corollary 2.1. *Let $\phi(t)$ be a positive continuous function which decreases to 0 as $t \to \infty$ and satisfies for $t \geq T$*

$$t^{\alpha} \int_t^{\infty} q(s)ds \leq \phi(t). \tag{2.9}$$

Then, equation (B) possesses a slowly varying solution $x(t)$ expressed in the form (2.1) for some $T > a$, where $v(t)$ is a solution of (2.2) and satisfies

$$0 \leq v(t) \leq \phi(t)^{1+\frac{1}{\alpha}}. \tag{2.10}$$

For the proof first notice that, because of (2.3), such a function $\phi(t)$ always exists. Then define the set V_{ϕ} by

$$V_{\phi} = \{v(t) \in C_0[T, \infty) : 0 \leq v(t) \leq \phi(t), t \geq T\}$$

and follow the argument used in the proof of Proposition 2.1.

2.2. Regularly varying solution of index 1. Our next task is to discuss the existence of an $RV(1)$-solution of equation (B).

Proposition 2.2. *Equation (B) possesses a regularly varying solution of index 1 which increases and for $t \geq T$ has the form*

$$X(t) = \exp\left\{ \int_T^t \left(\frac{1 - Q(s) + w(s)}{s^{\alpha}} \right)^{\frac{1}{\alpha}} ds \right\}, \tag{2.11}$$

where $w(t)$ is some solution for $t \geq T$ tending to zero as $t \to \infty$ of the integral equation

$$w(t) = \frac{\alpha}{t} \int_T^t F(s, w(s))ds \tag{2.12}$$

with

$$F(t, w) = 1 + \left(1 + \frac{1}{\alpha}\right) w - (1 - Q(t) + w)^{1+\frac{1}{\alpha}}, \tag{2.13}$$

if and only if (2.3) holds.

P r o o f. Only the proof of the "if" part will be given. (For the "only if" part see [3. Th. 3.1 with $c = 0$]).

Suppose that (2.3) holds. Substituting $u(t) = (1 - Q(t) + w(t))/t^{\alpha}$ in the Riccati equation (2.8), we obtain the following differential equation for $w(t)$:

$$(tw)' = \alpha\left(1 + \left(1 + \frac{1}{\alpha}\right)w - |1 - Q(t) + w|^{1+\frac{1}{\alpha}}\right).$$

Whence, in virtue of (1.11) and Elbert lemma, $X(t)$ is an $RV(1)$ solution of the equation (B) provided that we prove the existence for $t \geq T$ of a solution $w(t)$ of (2.12) which tend to zero as $t \to \infty$.

To that end, choose $0 < m < \frac{1}{4}$ such that for $t \geq T$

$$|w(t)| \leq m, \quad Q(t) \leq m^2 \quad \text{and} \quad \mu := \frac{(\alpha+1)(\alpha+4)}{\alpha} m < 1 \tag{2.14}$$

and define the set W by

$$W = \{w(t) \in C_0[T, \infty) : |w(t)| \leq m, t \geq T\}. \tag{2.15}$$

Notice that on this set the preceding differential equation is reduced to $(tw)' = \alpha F(t, w)$, whose integrated version being (2.12) and $F(t, w)$ is defined by (2.13).

Further, define the integral operator $\mathcal{G}$

$$\mathcal{G}w(t) = \frac{\alpha}{t}\int_T^t F(s, w(s))ds, \quad t \geq T. \tag{2.16}$$

Since $Q(t) - w(t) \to 0$, as $t \to \infty$ we can write (2.13) as

$$F(t, w) = \left(1 + \frac{1}{\alpha}\right)\left(Q(t) - \frac{1}{2\alpha}(Q(t) - w)^2 R(t, w))\right), \tag{2.17}$$

where

$$R(t, w) = \sum_{n=2}^{\infty}(-1)^n \frac{2\alpha^2}{1+\alpha}\binom{1 + \frac{1}{\alpha}}{n}(Q(t) - w)^{n-2}, \tag{2.18}$$

and

$$\frac{\partial F}{\partial w}(t, w) = \frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right)(Q(t) - w)S(t, w), \tag{2.19}$$

where

$$S(t,w)=\sum_{n=1}^{\infty}(-1)^{n-1}\alpha\binom{\frac{1}{\alpha}}{n}(Q(t)-w)^{n-1}. \tag{2.20}$$

Since $R(t,w)\to 1$ and $S(t,w)\to 1$ as $t\to\infty$, we have for $t\ge T_o$, and some T_o

$$0\le R(t,w)\le 2 \quad\text{and}\quad 0\le S(t,w)\le 2. \tag{2.21}$$

Then from (2.15), (2.17) and (2.21) one has for $t\ge T$

$$|F(t,w)|\le\frac{m}{\alpha}\quad\text{and}\quad|\mathcal{G}w(t)|\le m.$$

Hence $\mathcal{G}$ is a self-map.

Likewise, from (2.15), (2.19) and (2.20) one obtains for $t\ge T$

$$\left|\frac{\partial F}{\partial w}\right|\le\frac{\mu}{\alpha}.$$

Thus

$$|\mathcal{G}w_1(t)-\mathcal{G}w_2(t)|\le\frac{\alpha}{t}\int_T^t\left|\frac{\partial F}{\partial w}\right||w_1-w_2|ds\le\mu\|w_1-w_2\|_0.$$

This shows that $\mathcal{G}$ is a contraction on W, so that there exists a fixed point $w(t)$ of $\mathcal{G}$ in W, which is a solution of the integral equation (2.12) on $[T,\infty)$, which tends to 0 as $t\to\infty$ (QED).

Often, a more precise estimate for $w(t)$ is needed. The following result may help in some situations.

Corollary 2.2. *Suppose that there exists a (continuous) slowly varying function $\psi(t)$ on $[a,\infty)$ which tends to 0 as $t\to\infty$ such that*

$$Q(t):=t^{\alpha}\int_t^{\infty}q(s)ds\le\psi(t)\quad\textit{for all large}\, t.$$

Then, equation (B) possesses a regularly varying solution of index 1 expressed in the form (2.11) for some $T>a$, where $w(t)$ is a solution of the integral equation (2.12) such that

$$w(t)=O(\psi(t))\quad\textit{as}\quad t\to\infty.$$

The proof mimics the one of Corollary 2.1 with the integral operator $\mathcal{G}$ given by (2.16) on the set

$$W_\psi = \{w(t) \in C_0[T,\infty) : |w(t)| \leq \sqrt{\psi(t)}, \quad t \geq T\}.$$

The condition on $\psi(t)$ makes possible the use of Karamata's integration theorem ((i) of Proposition 1.8) to the effect that there exists a constant $\gamma \geq 1$ such that

$$\frac{1}{t}\int_a^t \psi(s)ds \leq \gamma\psi(t), \quad t \geq a$$

which is needed to show that $\mathcal{G}$ is a self-map.

3. *Super-half-linear equation*

3.1. Slowly varying solutions. This section is devoted to the super-half-linear equation (A), i.e. when $\alpha < \beta$, where $q : [a,\infty) \to (0,\infty)$ is a continuous function. We begin with the existence of slowly varying solutions.

Theorem 3.1. *Equation (A) possesses a decreasing slowly varying solution if*

$$Q(t) := t^\alpha \int_t^\infty q(s)ds \to 0 \quad as \quad t \to \infty. \tag{3.1}$$

This solution is a trivial or a nontrivial one according as the integral

$$\int_a^\infty \frac{Q(t)^{1/\alpha}}{t}dt \tag{3.2}$$

converges or diverges.

P r o o f. Let l be a positive constant such that

$$\lambda := \left(1 + \frac{1}{\alpha}\right) l^{\frac{1}{\alpha}} < 1. \tag{3.3}$$

and choose $T > a$ so that

$$Q(t) \leq l, \quad t \geq T. \tag{3.4}$$

Let Ξ denote the set of continuous functions $\xi(t)$ on $[T,\infty)$ which are non-increasing and satisfy

$$1 \geq \xi(t) \geq \exp\left\{-\int_T^t \frac{Q(s)^{\frac{1}{\alpha}}}{s} ds\right\}, \quad t \geq T. \tag{3.5}$$

It is clear that Ξ is a closed convex subset of the locally convex space $C[T,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T,\infty)$. For any $\xi(t) \in \Xi$ define

$$q_\xi(t) = q(t)\xi(t)^{\beta-\alpha}, \quad Q_\xi(t) = t^\alpha \int_t^\infty q_\xi(s)ds, \tag{3.6}$$

and consider the family of half-linear differential equations

$$(x'(t)^{\alpha*})' = q_\xi(t)x(t)^{\alpha*}, \quad \xi(t) \in \Xi. \tag{3.7}$$

Since $Q_\xi(t) \leq Q(t)$ for any $\xi(t) \in \Xi$, we have

$$Q_\xi(t) \leq l, \quad t \geq T, \quad \text{and} \quad \lim_{t\to\infty} Q_\xi(t) = 0.$$

It follows from Proposition 2.1 that each member of (3.7) possesses a slowly varying solution $x_\xi(t)$ for $t \geq T$ having the representation

$$x_\xi(t) = \exp\left\{-\int_T^t \left(\frac{Q_\xi(s) - v_\xi(s)}{s^\alpha}\right)^{\frac{1}{\alpha}} ds\right\}, \tag{3.8}$$

where $v_\xi(t)$ is positive and satisfies

$$v_\xi(t) = \alpha t^\alpha \int_t^\infty \frac{|v_\xi(s) - Q_\xi(s)|^{1+\frac{1}{\alpha}}}{s^{\alpha+1}} ds. \tag{3.9}$$

Each solution $x_\xi(t)$ is decreasing and satisfies

$$1 \geq x_\xi(t) \geq \exp\left\{-\int_T^t \frac{Q_\xi(s)^{\frac{1}{\alpha}}}{s} ds\right\} \geq \exp\left\{-\int_T^t \frac{Q(s)^{\frac{1}{\alpha}}}{s} ds\right\}. \tag{3.10}$$

Our basic idea which pervades throughout most of the future considerations in this paper, is to show that there is at least one function $\xi(t) \in \Xi$

such that $\xi(t) = x_\xi(t)$ where $x_\xi(t)$ is defined by (3.8) and satisfies one member of the family (3.7), which is at the same time a solution of equation (A). In that the Schauder-Tychonoff fixed point theorem is our main tool. However, it is the choice of the set Ξ which makes the procedure feasible.

Let us define the mapping $\Phi : \Xi \to C[T, \infty)$ by

$$\Phi\xi(t) = x_\xi(t), \quad t \geq T. \tag{3.11}$$

It can be verified that Φ is a self-map on Ξ and sends Ξ continuously into a relatively compact subset of Ξ.

(i) Φ maps Ξ into itself. This is an immediate consequence of (3.10).

(ii) $\Phi(\Xi)$ is relatively compact in $C[T, \infty)$. The inclusion $\Phi(\Xi) \subset \Xi$ implies that $\Phi(\Xi)$ is locally uniformly bounded on $[T, \infty)$. The inequality

$$0 \geq (\Phi\xi)'(t) = -x_\xi(t)\frac{(v_\xi(t) - Q_\xi(t))^{\frac{1}{\alpha}}}{t} \geq -\frac{Q_\xi(t)^{\frac{1}{\alpha}}}{t} \geq -\frac{Q(t)^{\frac{1}{\alpha}}}{t}, \quad t \geq T,$$

holding for all $\xi(t) \in \Xi$, shows that $\Phi(\Xi)$ is locally equicontinuous on $[T, \infty)$. The relative compactness of $\Phi(\Xi)$ then follows from the Arzela-Ascoli lemma.

(iii) Φ is a continuous mapping. Let $\{\xi_n(t)\}$ be a sequence in Ξ converging to $\xi(t) \in \Xi$ as $n \to \infty$ uniformly on compact subintervals of $[T, \infty)$. We have to prove that $\Phi\xi_n(t)$ converges to $\Phi\xi(t)$ on any compact subinterval of $[T, \infty)$. We first note that

$$|\Phi\xi_n(t) - \Phi\xi(t)| =$$

$$\left|\exp\left\{-\int_T^t \left(\frac{Q_{\xi_n}(s) - v_{\xi_n}(s)}{s^\alpha}\right)^{\frac{1}{\alpha}} ds\right\} - \exp\left\{-\int_T^t \left(\frac{Q_\xi(s) - v_\xi(s)}{s^\alpha}\right)^{\frac{1}{\alpha}} ds\right\}\right|$$

$$\leq \int_T^t \left|\left(\frac{Q_{\xi_n}(s) - v_{\xi_n}(s)}{s^\alpha}\right)^{\frac{1}{\alpha}} - \left(\frac{Q_\xi(s) - v_\xi(s)}{s^\alpha}\right)^{\frac{1}{\alpha}}\right| ds. \tag{3.12}$$

We also remark that if $\alpha \geq 1$, then

$$\left|\left(\frac{Q_{\xi_n}(s) - v_{\xi_n}(s)}{s^\alpha}\right)^{\frac{1}{\alpha}} - \left(\frac{Q_\xi(s) - v_\xi(s)}{s^\alpha}\right)^{\frac{1}{\alpha}}\right|$$
$$\leq \left(\left|\frac{v_{\xi_n}(s) - v_\xi(s)}{s^\alpha}\right| + \left|\frac{Q_{\xi_n}(s) - Q_\xi(s)}{s^\alpha}\right|\right)^{\frac{1}{\alpha}},$$

and if $\alpha < 1$, then

$$\left|\left(\frac{Q_{\xi_n}(s) - v_{\xi_n}(s)}{s^\alpha}\right)^{\frac{1}{\alpha}} - \left(\frac{Q_\xi(s) - v_\xi(s)}{s^\alpha}\right)^{\frac{1}{\alpha}}\right| \leq C_2 \left(\left|\frac{v_{\xi_n}(s) - v_\xi(s)}{s^\alpha}\right| + \left|\frac{Q_{\xi_n}(s) - Q_\xi(s)}{s^\alpha}\right|\right),$$

where $C_2 = l^{\frac{1}{\alpha}-1}/\alpha T^{1-\alpha}$. Combining the above remark with (3.12), we see that the continuity of Φ is assured if it is proved that the two sequences

$$\frac{1}{t^\alpha}|v_{\xi_n}(t) - v_\xi(t)|, \quad \frac{1}{t^\alpha}|Q_{\xi_n}(t) - Q_\xi(t)|, \tag{3.13}$$

converge to 0 as $n \to \infty$ uniformly on any compact subinterval of $[T, \infty)$. The convergence of the second sequence in (3.13) follows from the Lebesgue dominated convergence theorem applied to the right hand integral of the inequality

$$\frac{1}{t^\alpha}|Q_{\xi_n}(t) - Q_\xi(t)| \leq \int_t^\infty q(s)|\xi_n(s)^{\beta-\alpha} - \xi(s)^{\beta-\alpha}|ds.$$

To evaluate the first sequence, using (3.9), we obtain

$$\frac{1}{t^\alpha}|v_{\xi_n}(t) - v_\xi(t)| \leq \alpha \int_t^\infty \left|\frac{|v_{\xi_n}(s) - Q_{\xi_n}(s)|^{1+\frac{1}{\alpha}} - |v_\xi(s) - Q_\xi(s)|^{1+\frac{1}{\alpha}}}{s^{\alpha+1}}\right| ds$$

$$\leq \alpha\lambda \int_t^\infty \frac{|v_{\xi_n}(s) - v_\xi(s)|}{s^{\alpha+1}} ds + \alpha\lambda \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{\alpha+1}} ds, \quad t \geq T. \tag{3.14}$$

The substitution

$$z(t) = \int_t^\infty \frac{|v_{\xi_n}(s) - v_\xi(s)|}{s^{\alpha+1}} ds$$

transforms (3.14) into the differential inequality

$$-tz'(t) \leq \alpha\lambda z(t) + \alpha\lambda \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{\alpha+1}} ds$$

or

$$-(t^{\alpha\lambda} z(t))' \leq \alpha\lambda t^{\alpha\lambda-1} \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{\alpha+1}} ds. \tag{3.15}$$

Integrating (3.15) from t to ∞ and noting that $t^{\alpha\lambda}z(t) \to 0$ as $t \to \infty$, we have

$$z(t) \le \frac{1}{t^{\alpha\lambda}} \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{\alpha(1-\lambda)+1}} ds,$$

which, combined with (3.14), yields

$$\frac{1}{t^\alpha}|v_{\xi_n}(t) - v_\xi(t)| \le \frac{\alpha\lambda}{t^{\alpha\lambda}} \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{\alpha(1-\lambda)+1}} ds + \alpha\lambda \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{\alpha+1}} ds,$$

for $t \ge T$. This clearly ensures that $|v_{\xi_n}(t) - v_\xi(t)|/t^\alpha \to 0$ uniformly on compact subintervals of $[T, \infty)$. Thus, the continuity of Φ has been established.

This enables us to apply the Schauder-Tychonoff fixed point theorem to Φ, which leads us to the conclusion that there exists a $\xi_0(t) \in \Xi$ such that $\xi_0(t) = \Phi\xi_0(t) = x_{\xi_0}(t)$ for $t \ge T$. By the definition (3.11) of Φ, the function $\xi_0(t)$ is slowly varying and satisfies

$$((\xi_0'(t))^{\alpha *})' = q_{\xi_0}(t)\xi_0(t)^\alpha = q(t)\xi_0(t)^{\beta-\alpha}\xi_0(t)^\alpha = q(t)\xi_0(t)^\beta$$

for $t \ge T$, which means that $\xi_0(t)$ is a solution of equation (A) of the form (3.8).

To prove the second statement of the theorem, observe that each solution $x_\xi(t)$ is decreasing and satisfies (3.10) which then holds for $\xi_0(t)$ i.e.

$$1 \ge \xi_0(t) \ge \exp\left\{ -\int_T^t \frac{Q(s)}{s}^{1/\alpha} ds \right\}.$$

Therefore, if the occurring integral (3.2) is convergent, $\xi_0(t)$ tends to a positive constant and so is trivial. It is easy to see that this condition is also necessary: Suppose $\xi_0(t) \to c > 0$, as $t \to \infty$. Then, the convergence of (3.2) follows from Proposition 1.1.

It is clear that if (3.2) diverges then $\xi_0(t)$ cannot be trivial (and tends to zero) for otherwise, due to Proposition 1.1, the integral would converge giving a contradiction. Conversely, if $\xi_0(t)$ is nontrivial, then the definition of Ξ implies the divergence of (3.2). This completes the proof.

If one further restricts the coefficient $q(t)$ of equation (A) one can obtain some additional properties of its (nontrivial) slowly varying solutions. There holds

Theorem 3.2. *Let $q(t) \in RV(-\gamma - 1)$ for some $\gamma > 0$ then equation (A) (with $\alpha < \beta$) may possess a nontrivial slowly varying solution $x(t)$ only* *if* $\gamma = \alpha$ *i.e. when* $q(t) = t^{-\alpha-1}L(t)$*, where* $L(t) \to 0$*, as* $t \to \infty$ *to satisfy condition (3.1). Then it has the exact asymptotic representation for* $t \to \infty$*, of the form*

$$x(t) \sim \left[\frac{\beta-\alpha}{\alpha^{1+\frac{1}{\alpha}}}\int_T^t (sq(s))^{\frac{1}{\alpha}}ds\right]^{\frac{\alpha}{\alpha-\beta}} = \left[\frac{\beta-\alpha}{\alpha^{1+\frac{1}{\alpha}}}\int_T^t \frac{L(s)^{\frac{1}{\alpha}}}{s}ds\right]^{\frac{\alpha}{\alpha-\beta}}. \qquad (3.16)$$

P r o o f. Suppose that (A) has a nontrivial SV-solution $x(t)$. By Proposition 1.3 $q(t)$ must satisfy $\int^{\infty}\left(\int_t^{\infty} q(s)ds\right)^{\frac{1}{\alpha}} dt = \infty$, which preclude the possibility that $\gamma > \alpha$. On the other hand, integrating (A) from t to ∞ we have

$$(-x'(t))^{\alpha} = \int_t^{\infty} q(s)x(s)^{\beta}ds,$$

which, via Karamata's integration theorem, yields

$$(-x'(t))^{\alpha} \sim \frac{1}{\gamma}tq(t)x(t)^{\beta},$$

or

$$-x(t)^{-\frac{\beta}{\alpha}}x'(t) \sim \frac{1}{\gamma^{\frac{1}{\alpha}}}(tq(t))^{\frac{1}{\alpha}} \quad \text{as} \quad t \to \infty. \qquad (3.17)$$

Assume now that $\gamma < \alpha$. Integrating (3.17) from T to t gives

$$x(t) \sim \frac{\beta-\alpha}{\alpha\gamma^{\frac{1}{\alpha}}}\left[\int_T^t (sq(s))^{\frac{1}{\alpha}}ds\right]^{\frac{\alpha}{\alpha-\beta}} \quad \text{as} \quad t \to \infty. \qquad (3.18)$$

Since $q(t) \in RV(-\gamma-1)$, we see from (3.18) that $x(t) \in RV\left(\frac{\alpha-\gamma}{\beta-\alpha}\right)$, that is, $x(t)$ is not slowly varying, a contradiction. Thus the case $\gamma < \alpha$ is also impossible, and hence it must hold that $\gamma = \alpha$, i.e. $q(t) \in RV(-\alpha-1)$ which gives (3.16).

Remark 3.1. Proposition 1.8(iii) shows that $x(t)$ in (3.16) is indeed an SV-function.

Remark 3.2. It is tacitly assumed here that condition (3.1) holds to ensure the existence of an SV-solution. It is, however, not used in the proof. Consequently, Theorem 3.2 is valid regardless what result guarantees the existence of a nontrivial solution. A similar remark hold obviously for Theorems 3.4, 4.2 and 4.4.

Example 3.1. Consider the equation

$$(x'(t)^{\alpha *})' = q(t)x(t)^{\beta *}, \quad q(t) = \frac{\alpha r(t)}{t^{\alpha+1}(\log t)^{\alpha}(\log\log t)^{2\alpha-\beta}}, \tag{3.19}$$

where $r(t)$ is a positive continuous function such that $\lim_{t\to\infty} r(t) = \rho > 0$.

As is easily seen,

$$Q(t) = t^{\alpha}\int_t^{\infty} q(s)ds \sim \frac{\rho}{(\log t)^{\alpha}(\log\log t)^{2\alpha-\beta}} \to 0, \quad \text{as} \quad t\to\infty$$

which means that (3.1) holds. Since

$$\frac{Q(t)^{1/\alpha}}{t} \sim \frac{\rho^{1/\alpha}}{t\log t(\log\log)^{2-\frac{\beta}{\alpha}}} \quad \text{and} \quad \int^{\infty} \frac{\rho^{\frac{1}{\alpha}}ds}{s\log s(\log\log s)^{2-\frac{\beta}{\alpha}}} = \infty,$$

Theorem 3.1 ensures that equation (3.19) has a nontrivial slowly varying $x(t)$, which, by Theorem 3.2, satisfies

$$x(t) \sim \frac{\rho^{\frac{1}{\alpha-\beta}}}{\log\log t}, \quad t\to\infty.$$

If in particular,

$$r(t) = 1 + \frac{1}{\log t} + \frac{2}{\log t\log\log t},$$

then, equation (3.19) possesses an exact SV-solution $x(t) = 1/\log\log t$.

3.2. Regularly varying solutions of index 1. Let

$$L(t) = b\exp\left\{\int_a^t \frac{\delta(s)}{s}ds\right\}, \quad t\geq a, \quad b>0, \tag{3.20}$$

be a (normalized) slowly varying function increasing to ∞ as $t\to\infty$, which implies that $L(t)$ is continuously differentiable and

$$\delta(t) = t\frac{L'(t)}{L(t)} > 0 \quad \text{and} \quad \int_a^{\infty}\frac{\delta(s)}{s}ds = \infty. \tag{3.21}$$

We prove

Theorem 3.3. *Suppose that there exists a constant $K > 0$ such that for all large t*

$$Q(t) := t^{\alpha} \int_t^{\infty} q(s)(sL(s))^{\beta-\alpha} ds \le K\delta(t). \tag{3.22}$$

Then, equation (A) possesses an increasing regularly varying solution $X(t)$ of index 1 such that $X(t) \le tL(t)$ for all large t.

P r o o f. Denote by Ξ the set of positive continuous nondecreasing functions $\xi(t)$ on $[T, \infty)$ satisfying for $t \ge T$

$$\left(\frac{3}{4T^{\alpha}}\right)^{\frac{1}{\alpha}} t \le \xi(t) \le \frac{tL(t)}{TL(T)}. \tag{3.23}$$

Define for any $\xi(t) \in \Xi$

$$q_{\xi}(t) = q(t)\xi(t)^{\beta-\alpha}, \quad Q_{\xi}(t) = t^{\alpha} \int_t^{\infty} q_{\xi}(s) ds \tag{3.24}$$

and consider the family of half-linear equations

$$(x'(t)^{\alpha *})' = q_{\xi}(t) x(t)^{\alpha *}, \quad \xi(t) \in \Xi. \tag{3.25}$$

Using (3.22), (3.23) and (3.24) we see that for $t \ge T$

$$Q_{\xi}(t) \le \frac{K\delta(t)}{(TL(T))^{\beta-\alpha}}. \tag{3.26}$$

Since $\delta(t) \to 0$, an application of Proposition 2.2 shows that every member of the family (3.25) possesses for $t \ge T$ an increasing $RV(1)$ solution $X_{\xi}(t)$ of the form

$$X_{\xi}(t) = \exp\left\{ \int_T^t \left(\frac{1 - Q_{\xi}(s) + w_{\xi}(s)}{s^{\alpha}} \right)^{1/\alpha} ds \right\} \tag{3.27}$$

where $w_{\xi}(t)$ is a solution of the integral equation

$$w_{\xi}(t) = \frac{\alpha}{t} \int_T^t F_{\xi}(s, w_{\xi}(s)) ds. \tag{3.28}$$

Here $F_\xi(t,w)$ stands for the function

$$F_\xi(t,w_\xi) = 1 + \left(1+\frac{1}{\alpha}\right) w_\xi - (1 - Q_\xi(t) + w_\xi)^{1+\frac{1}{\alpha}}. \tag{3.29}$$

Since by (2.13)

$$F_\xi(t,w_\xi) \le \left(1+\frac{1}{\alpha}\right) Q_\xi,$$

there follows from (3.28) and (3.26)

$$w_\xi(t) \le \frac{(\alpha+1)K}{(TL(T))^{\beta-\alpha}} \frac{1}{t} \int_T^t \delta(s)ds. \tag{3.30}$$

Now, we define the mapping $\Psi : \Xi \to C[t,\infty)$ by

$$\Psi\xi(t) = X_\xi(t) \quad \text{for} \quad t \ge T \tag{3.31}$$

and, as before, show that it satisfies the hypotheses of the Schauder-Tychonoff theorem.

(i) $\Psi(\Xi) \subset \Xi$.

By using (3.30) we obtain the following estimate for the solution $X_\xi(t)$ given by (3.27)

$$X_\xi(t) \le \exp\left\{\int_T^t (1+|w_\xi(s)|)^{\frac{1}{\alpha}} \frac{ds}{s}\right\} \le \exp\left\{\int_T^t \left(1 + \frac{AC(\alpha)}{s^2}\int_T^s \delta(z)dz\right)\frac{ds}{s}\right\} \tag{3.32}$$

where $C(\alpha)$ is some positive constant and

$$A = \frac{K(\alpha+1)}{(TL(T))^{\beta-\alpha}}.$$

Further, by integrating partially in the last integral we get

$$\int_T^t \left(\frac{1}{s^2}\int_T^s \delta(z)dz\right) ds \le \int_T^t \frac{\delta(z)}{z} dz$$

and so by (3.32) and (3.20) and choosing T in such a way that $AC(\alpha) \le 1$, there follows

$$X_\xi(t) \le \frac{tL(t)}{TL(T)} \quad \text{for} \quad t \ge T. \tag{3.33}$$

On the other hand, since $X'(t)$ increases, and for e.g. $Q_\xi(t) < 1/4$

$$X'_\xi(t) \geq X'_\xi(T) = \frac{(1 - Q_\xi(T))^{1/\alpha}}{T} \geq \left(\frac{3}{4}\right)^{1/\alpha} \frac{1}{T}, \tag{3.34}$$

for $t \geq T$ and all $\xi(t) \in \Xi$, we easily have the lower bound

$$X_\xi(t) \geq \left(\frac{3}{4T^\alpha}\right)^{\frac{1}{\alpha}} t. \tag{3.35}$$

Whence (i) holds

(ii) $\Psi(\Xi)$ is relatively compact in $C[T, \infty)$.

The local uniform boundedness follows from (i). We get local equicontinuity as follows: From (3.34), (3.33) and since e.g. $|w_\xi(t)| < 1$ for all $\xi(t) \in \Xi$ and $t \geq T$, one gets

$$\left(\frac{3}{4}\right)^{1/\alpha} \frac{1}{T} \leq X'_\xi(t) \leq X_\xi(t) \frac{(1 + |w_\xi(t)|)^{1/\alpha}}{t} \leq \frac{2^{1/\alpha} L(t)}{T L(T)}.$$

Then, an application of the Arzela-Ascoli lemma proves (ii).

(iii) The continuity of Ψ can be proved in the following manner. We let $\{\xi_n(t)\}$ be a sequence in Ξ converging to $\xi(t) \in \Xi$ as $n \to \infty$ uniformly on compact subintervals of $[T, \infty)$, and prove that $\{\Psi\xi_n(t)\}$ converges to $\Psi\xi(t)$ uniformly on any compact subinterval of $[T, \infty)$. Arguing as in the proof of the continuity of Φ in Theorem 3.1, it suffices to verify that the two sequences

$$\frac{1}{t^\alpha} |w_{\xi_n}(t) - w_\xi(t)| \quad \text{and} \quad \frac{1}{t^\alpha} |Q_{\xi_n}(t) - Q_\xi(t)|$$

converge to 0 uniformly on compact subintervals of $[T, \infty)$. The second sequence can be handled easily. To deal with the first sequence, using (3.28) and the inequality

$$|F_{\xi_n}(s, w_{\xi_n}(s)) - F_\xi(s, w_\xi(s))| \leq \left(1 + \frac{1}{\alpha}\right) |w_{\xi_n}(t) - w_\xi(t)|$$

$$+ \left(1 + \frac{1}{\alpha}\right) m_\alpha(|w_{\xi_n}(t) - w_\xi(t)| + |Q_{\xi_n}(t) - Q_\xi(t)|), \quad m_\alpha = \left(\frac{5}{4}\right)^{\frac{1}{\alpha}},$$

we obtain

$$|w_{\xi_n}(t) - w_\xi(t)| \leq \frac{\alpha}{t} \int_T^t |F_{\xi_n}(s, w_{\xi_n}(s)) - F_\xi(s, w_\xi(s))| ds \tag{3.36}$$

$$\leq \frac{M_\alpha}{t}\int_T^t |w_{\xi_n}(s) - w_\xi(s)|ds + \frac{N_\alpha}{t}\int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds,$$

where

$$M_\alpha = (1+\alpha)(1+m_\alpha), \quad N_\alpha = 1+\alpha.$$

The substitution $z(t) = \int_T^t |w_{\xi_n}(s) - w_\xi(s)|ds$ transforms (3.36) into

$$\left(\frac{z(t)}{t^{M_\alpha}}\right)' \leq \frac{N_\alpha}{t^{M_\alpha+1}}\int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds,$$

which, integrated over $[T,t]$, yields

$$z(t) \leq \frac{N_\alpha}{M_\alpha} t^{M_\alpha}\int_T^t \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{M_\alpha}}ds. \tag{3.37}$$

Using (3.37) in (3.36), we have

$$|w_{\xi_n}(t) - w_\xi(t)| \leq N_\alpha t^{M_\alpha - 1}\int_T^t \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{M_\alpha}}ds + \frac{N_\alpha}{t}\int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds,$$

or

$$\frac{1}{t^\alpha}|w_{\xi_n}(t) - w_\xi(t)| \leq N_\alpha t^{M_\alpha - N_\alpha}\int_T^t \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{M_\alpha}}ds + \frac{N_\alpha}{t^{N_\alpha}}\int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds,$$

for $t \geq T$. This shows that $|w_{\xi_n}(t) - w_\xi(t)|/t^\alpha \to 0$ uniformly on any compact subinterval of $[T,\infty)$, and establishes the continuity of Ψ.

The Schauder-Tychonoff theorem then guarantees the existence of a function $\xi_0(t) \in \Xi$ such that $\xi_0(t) = X_{\xi_0}(t)$ for $t \geq T$. This means that $\xi_0(t)$ satisfies

$$(\xi_0'(t)^\alpha)' = q_{\xi_0}(t)\xi_0(t)^\alpha = q(t)\xi_0(t)^{\beta-\alpha}\xi_0(t)^\alpha = q(t)\xi_0(t)^\beta, \quad t \geq T,$$

that is, $\xi_0(t)$ is a solution of equation (A) on $[T,\infty)$. It is clear that $\xi_0(t)$ is a regularly varying function of index 1. Thus (A) possesses an $RV(1)$-solution, with the wanted upper bound due to (3.23) and with an representation of the form (3.27). (QED)

Example 3.2. Let $\alpha < \beta$ and consider the equation

$$(x'(t)^{\alpha*})' = q(t)x(t)^{\beta*}, \quad q(t) = \frac{\alpha}{t^{\beta+1}(\log t)^{\beta-\alpha|1}}\left(1+\frac{1}{\log t}\right)^{\alpha-1}. \quad (3.38)$$

It is clear that (3.38) has a trivial $RV(1)$-solution since $q(t)$ satisfies $\int^{\infty} t^{\beta}q(t)dt < \infty$. (cf. Proposition 1.2). It also satisfies (3.22) with $L(t) = \log t$ where $\delta(t) = 1/\log t$. (In this case condition (3.22) implies the former one i.e. (1.5)). However, equation (3.38) also has a nontrivial $RV(1)$-solution $x(t) = t\log t$.

Like in the case of slowly varying solutions, we shall restrict further the coefficient $q(t)$ to obtain an additional information on nontrivial $RV(1)$-solutions.

We prove

Theorem 3.4. *Suppose that $q(t) \in RV(-\beta-1)$ i.e. of the form $q(t) = t^{-\beta-1}L^*(t)$ and such that $\int^{\infty}\frac{L^*(t)}{t}dt$ converges. Then, for any nontrivial RV(1) solution $X(t)$ of equation (A) $(\beta > \alpha)$ there holds for $t \to \infty$*

$$X(t) \sim t\left[\frac{\beta-\alpha}{\alpha}\int_t^{\infty} s^{\beta}q(s)ds\right]^{\frac{1}{\alpha-\beta}} = t\left[\frac{\beta-\alpha}{\alpha}\int_t^{\infty}\frac{L^*(s)}{s}ds\right]^{\frac{1}{\alpha-\beta}}. \quad (3.39)$$

P r o o f. Suppose equation (A) has a nontrivial $RV(1)$-solution $X(t) = ty(t)$, implying $y(t) \in SV$ and $y(t) = X(t)/t \to \infty$, as $t \to \infty$. Hence, integrating on both sides of (A) over (T, t) one obtains

$$X'(t) \sim \left[\int_T^t \frac{L^*(s)y(s)^{\beta}}{s}ds\right]^{\frac{1}{\alpha}}, \quad \text{as} \quad t \to \infty$$

where, due to Karamata theorem, Proposition 1.8 (iii), the right hand side function is a slowly varying function.

Another integration and the use Proposition 1.8 (i), leads to

$$X(t) \sim t\left[\int_T^t \frac{L^*(s)y(s)^{\beta}}{s}ds\right]^{\frac{1}{\alpha}}, \quad \text{as} \quad t \to \infty,$$

so that

$$y(t) \sim \left[\int_T^t \frac{L^*(s)y(s)^\beta}{s} ds \right]^{\frac{1}{\alpha}}.$$

To determine the behaviour of $y(t)$ put

$$z(t) = \int_T^t \frac{L^*(s)y(s)^\beta}{s} ds$$

so that $y(t) \sim z(t)^{\frac{1}{\alpha}}$ and

$$\frac{z'(t)}{z(t)^{\beta/\alpha}} \sim \frac{L^*(t)}{t}.$$

Noting that $\beta > \alpha$, and $z(t) \to \infty$ as $t \to \infty$, an integration over $[t, \infty)$ leads for $t \to \infty$, to

$$y(t) \sim \left[\frac{\beta - \alpha}{\alpha} \int_t^\infty \frac{L^*(s)}{s} ds \right]^{\frac{1}{\alpha - \beta}}$$

and (3.39) follows.

4. *Sub-half-linear equation*

4.1. Slowly varying solutions. Sub-half-linear equations of the form (A) with $\alpha > \beta$ are under consideration.

Let $M(t)$ be a normalized slowly varying function on $[a, \infty)$ which decreases to 0 as $t \to \infty$

$$M(t) = b \exp \left\{ \int_a^t \frac{\delta(s)}{s} ds \right\}, \quad t \geq a, b > 0. \tag{4.1}$$

This implies

$$\delta(t) = t\frac{M'(t)}{M(t)} < 0, \quad \text{and} \quad \int_a^\infty \frac{\delta(s)}{s} ds = -\infty. \tag{4.2}$$

Theorem 4.1. *Suppose that there exists a constant $K > 0$ such that*

$$Q(t) := t^\alpha \int_t^\infty q(s)(M(s))^{\beta - \alpha} ds \leq K(-\delta(t))^\alpha \quad \textit{for all large} \quad t. \tag{4.3}$$

Then, equation (A) possesses a decreasing slowly varying solution $x(t)$ such that $x(t) \geq M(t)$ for all large t.

P r o o f. Choose $T > a$ so that, in addition to (4.3), the following inequalities hold for $t \geq T$:

$$K(M(T))^{\alpha-\beta} \leq 1, \tag{4.4}$$

$$\left(1 + \frac{1}{\alpha}\right)(-\delta(t))^{\alpha} \leq l, \tag{4.5}$$

where $l \in (0,1)$ is a given constant.

We denote by Ξ the set of continuous positive nonincreasing functions $\xi(t)$ on $[T, \infty)$ satisfying

$$1 \geq \xi(t) \geq \frac{M(t)}{M(T)}, \quad t \geq T. \tag{4.6}$$

For any $\xi(t) \in \Xi$ define

$$q_{\xi}(t) = q(t)\xi(t)^{\beta-\alpha}, \quad Q_{\xi}(t) = t^{\alpha} \int_t^{\infty} q_{\xi}(s)ds, \tag{4.7}$$

and consider the family of half-linear equations

$$((x'(t))^{\alpha *})' = q_{\xi}(t)(x(t))^{\alpha *}, \quad \xi(t) \in \Xi. \tag{4.8}$$

Since $\xi(t)^{\beta-\alpha} \leq (M(t)/M(T))^{\beta-\alpha}$, using (4.7), (4.4) and (4.3), we have

$$Q_{\xi}(t) \leq K(M(T))^{\alpha-\beta}(-\delta(t))^{\alpha} \leq (-\delta(t))^{\alpha}. \tag{4.9}$$

Because of (4.3) Proposition 2.1 is applicable, and each equation of (4.8) has a decreasing SV-solution $x_{\xi}(t)$ of the form

$$x_{\xi}(t) = \exp\left\{-\int_T^t \frac{(Q_{\xi}(s) - v_{\xi}(s))^{1/\alpha}}{s} ds\right\}, \quad t \geq T, \tag{4.10}$$

where $v_{\xi}(t)$ satisfies

$$v_{\xi}(t) = \alpha t^{\alpha} \int_t^{\infty} \frac{|v_{\xi}(s) - Q_{\xi}(s)|^{1+\frac{1}{\alpha}}}{s^{\alpha+1}} ds, \tag{4.11}$$

and $0 \le v_\xi(t) \le (-\delta(t))^{1+\frac{1}{\alpha}}$ for $t \ge T$. We note that $x_\xi(t)$ is decreasing and satisfies

$$1 \ge x_\xi(t) \ge \exp\left\{-\int_T^t \frac{Q_\xi(s)^{\frac{1}{\alpha}}}{s}ds\right\} \ge \exp\left\{\int_T^t \frac{\delta(s)}{s}ds\right\} = \frac{M(t)}{M(T)}. \quad (4.12)$$

We now define the mapping $\Phi : \Xi \to C[T,\infty)$ by

$$\Phi\xi(t) = x_\xi(t), \quad t \ge T, \quad (4.13)$$

and want to show that the Schauder-Tychonoff fixed point theorem is applicable to Φ.

(i) $\Phi(\xi) \subset \Xi$. This is a trivial consequence of (4.12).

(ii) $\Phi(\Xi)$ is relatively compact in $C[T,\infty)$. This follows from the fact that $\Phi(\Xi)$ is locally uniformly bounded and locally equicontinuous on $[T,\infty)$. The uniform boundedness is a direct consequence of the inclusion $\Phi(\xi) \subset \Xi$, while the equicontinuity is assured by the inequality

$$0 \ge (\Phi\xi)'(t) = -x_\xi(t)\left(\frac{Q_\xi(t) - v_\xi(t)}{t^\alpha}\right)^{1/\alpha} \ge -\frac{Q_\xi(t)^{\frac{1}{\alpha}}}{t} \ge \frac{\delta(t)}{t}, \quad t \ge T.$$

(iii) Φ is continuous. We omit the proof since it is essentially the same as that of the continuity of Φ defined by (3.11) (cf. the proof of Theorem 3.1). Consequently, Φ has a fixed point $\xi_0(t) \in \Xi$, which provides a slowly varying solution $x(t) = \xi_0(t)$ of the sub-half-linear equation (A), with the representation of the form (4.10) satisfying $x(t) \ge M(t)$ due to (4.6).

If one again restricts the attention to the case $q(t) \in RV(\gamma)$, it is easily obtained, due to (4.2) and by the use of Proposition 1.8, (i), that condition (4.3) implies (1.5) so that equation (A) has a trivial SV-solution. Nevertheless it might have also a nontrivial one as it is illustrated by

Example 4.1. Let $\alpha > \beta$ and consider the equation

$$(x'(t)^{\alpha *})' = q(t)x(t)^{\beta *}, \quad q(t) = \frac{\alpha}{t^{\alpha+1}(\log t)^{2\alpha-\beta}}\left(1 + \frac{2}{\log t}\right). \quad (4.14)$$

This equation has a trivial SV-solution and a nontrivial one at the same time. In fact, by inspection a nontrivial SV function $1/\log t$ satisfies (4.14), and since $\left(\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}}$ is integrable on $[e,\infty)$, the existence of a trivial SV-solution follows from Proposition 1.1.

By choosing

$$M(t) = \frac{1}{\log t}, \quad \text{for which} \quad \delta(t) = -\frac{1}{\log t},$$

a simple computation shows that condition (4.3) is fulfilled and so Theorem 4.1 ensures the existence of a slowly varying solution for equation (4.14). No criterion is available for deciding whether the obtained solution is a trivial SV-function or a nontrivial one.

The following result provides some further information about the nontrivial SV-solutions:

Theorem 4.2. *Let $q(t) \in RV(-\alpha - 1)$ i.e. $q(t) = t^{-\alpha-1}L(t)$, then, for any nontrivial slowly varying solution $x(t)$ of equation (A) $(\alpha > \beta)$ there holds for $t \to \infty$,*

$$x(t) \sim \left[\frac{\alpha - \beta}{\alpha^{1+\frac{1}{\alpha}}} \int_t^\infty (sq(s))^{1/\alpha} ds\right]^{\frac{\alpha}{\alpha-\beta}} = \left[\frac{\alpha - \beta}{\alpha^{1+\frac{1}{\alpha}}} \int_t^\infty \frac{L(s)^{\frac{1}{\alpha}}}{s} ds\right]^{\frac{\alpha}{\alpha-\beta}}. \quad (4.15)$$

P r o o f. Let $x(t)$ be a nontrivial SV-solution of equation (A). Then $x(t) \to 0$ and $x'(t) \to 0$ as $t \to \infty$. Integrating twice over (t, ∞) on both sides of (A), one obtains (4.15).

4.2. Regularly varying solutions of index 1.

Theorem 4.3. *Let Equation (A) has an increasing regularly varying solution of index* 1 *if*

$$Q(t) := t^\alpha \int_t^\infty s^{\beta-\alpha} q(s) ds \to 0, \quad \textit{as} \quad t \to \infty. \quad (4.16)$$

P r o o f. Let $m \in \left(0, \frac{1}{4}\right)$ be a constant such that

$$\frac{(\alpha+1)(\alpha+4)}{\alpha} m \le 1. \quad (4.17)$$

Put

$$\mu_\alpha = \left(\frac{3}{2}\right)^{\frac{1}{\alpha}}, \quad \nu_\alpha = \left(\frac{3}{4}\right)^{\frac{1}{\alpha}}.$$

Choose $T > a$ so that

$$\nu_\alpha T^{\mu_\alpha - 1} \ge 1, \quad (4.18)$$

and

$$Q(t) \leq m^2 \quad \text{for} \quad t \geq T. \tag{4.19}$$

Let Ξ denote the closed convex subset of $C[T, \infty)$ consisting of continuous nondecreasing function $\xi(t)$ on $[T, \infty)$ such that

$$t \leq \xi(t) \leq t^{\mu_\alpha}, \quad t \geq T. \tag{4.20}$$

Define the functions $q_\xi(t)$ and $Q_\xi(t)$ by

$$q_\xi(t) = q(t)\xi(t)^{\beta-\alpha}, \quad Q_\xi(t) = t^\alpha \int_t^\infty q_\xi(s)ds, \tag{4.21}$$

and consider the family of half-linear equations

$$(x'(t)^{\alpha *})' = q_\xi(t)x(t)^{\alpha *}, \quad \xi(t) \in \Xi. \tag{4.22}$$

Since $\xi(t)^{\beta-\alpha} \leq t^{\beta-\alpha}$, we have for $t \geq T$, and each $\xi(t) \in \Xi$,

$$Q_\xi(t) \leq Q(t) \leq m^2, \tag{4.23}$$

and so by Proposition 2.2 each equation of (4.22) possesses for $t \geq T$ an $RV(1)$-solution $X_\xi(t)$ of the form

$$X_\xi(t) = \exp\left\{ \int_T^t \left(\frac{1 - Q_\xi(s) + w_\xi(s)}{s^\alpha} \right)^{\frac{1}{\alpha}} ds \right\}, \tag{4.24}$$

where $w_\xi(t)$ solves the integral equation

$$w_\xi(t) = \frac{\alpha}{t} \int_T^t F_\xi(s, w_\xi(s))ds,$$

and satisfies $|w_\xi(t)| \leq m$.
(For the definition of $F_\xi(s, w)$ see (3.29).) It is easy to see that $X_\xi(t)$ given by (4.24) satisfies

$$\nu_\alpha \frac{t}{T} \leq X_\xi(t) \leq \left(\frac{t}{T}\right)^{\mu_\alpha},$$

which, in view of (4.18), implies

$$t \leq T^{\mu_\alpha} X_\xi(t) \leq t^{\mu_\alpha}, \quad t \geq T. \tag{4.25}$$

Let us define Ψ to be the mapping which assigns to each $\xi(t) \in \Xi$ the function $\Psi\xi(t)$ given by

$$\Psi\xi(t) = T^{\mu_\alpha} X_\xi(t), \quad t \geq T. \tag{4.26}$$

(i) Ψ maps Ξ into itself. This is a trivial consequence of (4.25).

(ii) $\Psi(\Xi)$ is relatively compact. The inclusion $\Psi(\Xi) \subset \Xi$ implies that $\Psi(\Xi)$ is locally uniformly bounded on $[T, \infty)$. For any $\xi(t) \in \Xi$ we easily have

$$\frac{3}{4} \leq (\Psi\xi)'(t) \leq \frac{5}{4} t^{\mu_\alpha - 1}, \quad t \geq T,$$

which shows that $\Psi(\Xi)$ is locally equicontinuous on $[T, \infty)$. The conclusion then follows from the Arzela-Ascoli lemma.

(iii) Ψ is continuous. Letting $\{\xi_n(t)\}$ be a sequence in Ξ converging to $\xi(t) \in \Xi$ uniformly on compact subintervals of $[T, \infty)$, we have to prove that $\{\Psi\xi_n(t)\}$ converges to $\Psi\xi(t)$ uniformly on any compact subinterval of $[T, \infty)$. Since

$$|\Psi\xi_n(t) - \Psi\xi(t)| = T^{\mu_\alpha} \left| \exp\left\{ \int_T^t \left(\frac{1 - Q_{\xi_n}(s) + w_{\xi_n}(s)}{s^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \right.$$

$$\left. - \exp\left\{ \int_T^t \left(\frac{1 - Q_\xi(s) + w_\xi(s)}{s^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \right|$$

$$\leq t^{\mu_\alpha} \int_T^t \left| \left(\frac{1 - Q_{\xi_n}(s) + w_{\xi_n}(s)}{s^\alpha} \right)^{\frac{1}{\alpha}} - \left(\frac{1 - Q_\xi(s) + w_\xi(s)}{s^\alpha} \right)^{\frac{1}{\alpha}} \right| ds, \quad t \geq T,$$

to show the continuity of Ψ it suffices to verify that the sequences

$$\frac{1}{t^\alpha} |Q_{\xi_n}(t) - Q_\xi(t)| \quad \text{and} \quad \frac{1}{t^\alpha} |w_{\xi_n}(t) - w_\xi(t)| \tag{4.27}$$

converge to 0 uniformly on compact subintervals of $[T, \infty)$. We need only to deal with the second sequence in (4.27). A straightforward computation leads to the inequality

$$|w_{\xi_n}(t) - w_\xi(t)| \leq \frac{2k}{t} \int_T^t |w_{\xi_n}(s) - w_\xi(s)| ds + \frac{k}{t} \int_T^t |Q_{\xi_n}(s) - Q_\xi(s)| ds, \tag{4.28}$$

for $t \geq T$, where $k = 1 + \frac{1}{\alpha}$. Putting $z(t) = \int_T^t |w_{\xi_n}(s) - w_\xi(s)|ds$, (4.28) is transformed into

$$\left(\frac{z(t)}{t^{2k}}\right)' \leq \frac{k}{t^{2k+1}} \int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds,$$

whence it follows that

$$z(t) \leq t^{2k}\int_T^t \frac{k}{s^{2k+1}}\int_T^s |Q_{\xi_n}(r) - Q_\xi(r)|drds \leq t^{2k}\int_T^t \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{2s^{2k}}ds, \ \ t \geq T. \tag{4.29}$$

Using (4.29) in (4.28), we obtain

$$\frac{1}{t^\alpha}|w_{\xi_n}(t)-w_\xi(t)| \leq \frac{kt^{2k}}{t^{\alpha+1}}\int_T^t \frac{|Q_{\xi_n}(s) - Q_\xi(s)|}{s^{2k}}ds+\frac{k}{t^{\alpha+1}}\int_T^t |Q_{\xi_n}(s)-Q_\xi(s)|ds,$$

for $t \geq T$, which ensures that $|w_{\xi_n}(t)|/t^\alpha \to 0$ uniformly on every compact subinterval of $[T, \infty)$. Thus, Ψ is a continuous mapping. Consequently, by the Schauder-Tychonoff fixed point theorem Ψ has a fixed point $\xi_0(t) \in \Xi$, which by definition satisfies

$$\xi_0(t) = T^{\mu_\alpha} X_{\xi_0}(t), \quad t \geq T. \tag{4.30}$$

From (4.30) we conclude that

$$\begin{aligned}(\xi_0'(t)^\alpha)' &= T^{\alpha\mu_\alpha}(X_{\xi_0}'(t)^\alpha)' = T^{\alpha\mu_\alpha} q_{\xi_0}(t)X_{\xi_0}(t)^\alpha \\ &= q_{\xi_0}(t)(T^{\mu_\alpha}X_{\xi_0}(t))^\alpha = q_{\xi_0}(t)\xi_0(t)^\alpha = q(t)\xi_0(t)^\beta\end{aligned}$$

for $t \geq T$, which means that $\xi_0(t)$ is a solution of equation (A) on $[T, \infty)$. It is obvious that $\xi_0(t)$ is a regularly varying function of index 1 with the representation of the form (4.24). (QED)

Example 4.2. Let $\alpha > \beta$ and consider the equation

$$(x'(t)^{\alpha*})' = q(t)x(t)^{\beta*}, \quad q(t) = \frac{L(t)}{t^{\beta+1}}, \tag{4.31}$$

where $L(t)$ is a slowly varying function such that

$$\lim_{t\to\infty} L(t) = 0 \quad \text{and} \quad \int^\infty \frac{L(t)}{t}dt = \infty.$$

Since

$$t^{\alpha}\int_{t}^{\infty} s^{\beta-\alpha}q(s)ds \sim \frac{L(t)}{\alpha} \to 0, \quad t\to\infty,$$

Theorem 4.3 implies that (4.31) has a regularly varying solution of index 1. In view of Proposition 1.4, this solution is automatically a nontrivial one because $\int^{\infty} t^{\beta}q(t)dt = \infty$.

If we choose in particular

$$L(t) = \frac{r(t)(\log\log t)^{\alpha-\beta-1}}{\log t}$$

with $r(t) = \alpha\left(1-\frac{1}{\log t}\right)\left(1+\frac{1}{\log t\log\log t}\right)^{\alpha-1}$, Theorem 4.3 is still applicable, giving the existence of a nontrivial $RV(1)$-solution. Indeed $t\log\log t$ is such a solution.

If one restricts $q(t)$ to the Karamata class as in previous cases, one can obtain further information on solutions:

Take $q(t) = t^{\gamma}L(t)$ where γ is a constant and $L(t)$ is a slowly varying function. Condition (4.16) then reduces to

$$Q(t) \sim \frac{1}{\alpha-(\gamma+\beta+1)}t^{\gamma+\beta+1}L(t) \to 0.$$

This is possible if a) $\gamma+\beta+1<0$ and if b) $\gamma+\beta+1=0$ and $L(t)\to 0$. Then an application of Propositions 1.2 and 1.4 leads to the conclusion that the $RV(1)$-solution whose existence is proved in Theorem 4.3, is a trivial one in case a) and a nontrivial one in case b), provided that $L(t)/t = t^{\beta}q(t)$ is not integrable on $[a,\infty)$.

In the later case we obtain

Theorem 4.4. *Suppose that* $q(t)\in RV(-\beta-1)$ *i.e.* $q(t) = t^{-\beta-1}L(t)$ *and* $L(t)\to 0$ *then, for any nontrivial RV(1)-solution of equation (A)* ($\alpha > \beta$) *there holds for* $t\to\infty$

$$x(t) \sim t\left[\frac{\alpha-\beta}{\alpha}\int_{T}^{t} s^{\beta}q(s)ds\right]^{\frac{1}{\alpha-\beta}} = t\left[\frac{\alpha-\beta}{\alpha}\int_{T}^{t}\frac{L(s)}{s}ds\right]^{\frac{1}{\alpha-\beta}}.$$

For the proof an argument similar to the proof of Theorem 3.3 is used.

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Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka, 841-0180, Japan
kusanot@zj8.so-net.ne.jp

Department of Mathematics
Faculty of Education
Kumamoto University
Kumamoto 860-8555, Japan
tanigawa@educ.kumamoto-u.ac.jp

Serbian Academy of Sciences and Arts
Kneza Mihaila 35
11000 Beograd, Serbia
vojam@uns.ac.rs