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INCREASING SOLUTIONS OF THOMAS-FERMI TYPE DIFFERENTIAL EQUATIONS – THE SUBLINEAR CASE

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A b s t r a c t. The existence and the asymptotic of solutions of the equations of Thomas-Fermi type is studied in the framework of regular variation (in the sense of Karamata).

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1. *Introduction*

The present paper is devoted to the existence and the asymptotic analysis in the frame of regular variation of increasing positive solutions of nonlinear ordinary differential equations of *Thomas-Fermi type*

$$x'' = \alpha q(t)\phi(x), \tag{A}$$

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where $\alpha = 1$. If $\alpha = -1$ (A) is said to be of *Emden-Fowler type.* The following assumptions are always required for (A):

(i) $q : [a, \infty) \to (0, \infty)$, $a > 0$, is a continuous function which is regularly varying of index σ;

(ii) $\phi : (a, \infty) \to (0, \infty)$, $a > 0$, is a continuous function which is regularly varying of index $\gamma \in (0, 1)$.

Equation (A) is called *superlinear* or *sublinear* according as $\gamma > 1$ or $0 < \gamma < 1$.

We recall that the set of regularly varying functions of index ρ is introduced by J. Karamata in 1930 by the following

Definition 1.1. *A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be regularly varying of index $\rho \in \mathbf{R}$ if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \textit{for all } \lambda > 0. \tag{1.1}$$

If in particular, $\rho = 0$, the function f is called *slowly varying.* With SV and $RV(\rho)$ we denote, respectively, the set of slowly varying and regularly varying functions of index ρ. Thus, the assumptions on q and ϕ and the Definition 1.1 imply

$$q(t) = t^\sigma l(t),\ l(t) \in SV, \quad \phi(x) = x^\gamma L(x),\ L(x) \in SV. \tag{1.2}$$

Notice that the simple oscillating function $f(t) = 2+\sin t$ is not regularly varying. But, by relaxing the limiting requirement (1.1) for the function $f(t)$ by two-sided boundedness condition one obtains the class of regularly bounded functions, denoted by RO.

Definition 1.2. *A measurable function $f : [0, \infty) \to (0, \infty)$ is said to be regularly bounded if*

$$m \le \frac{f(\lambda t)}{f(t)} \le M, \quad 1 \le \lambda \le L < \infty, \quad 0 < m < 1, \quad M > 1. \tag{1.3}$$

Clearly $RV \subset RO$. More generally, all positive measurable functions which are bounded away both from zero and infinity are such as well, so that various simple oscillating functions, as $f(t) = 2 + \sin t$, are regularly bounded.

Comprehensive treatises on regular variation are given in N.H. Bingham et al. [2] and by E. Seneta [15].

To help the reader we present here a fundamental result which will be used throughout the paper.

Proposition 1.1. *(Karamata integration theorem) Let $L(t) \in SV$. Then,*
(i) if $\alpha > -1$,

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) if $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) if $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in SV \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

or

$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in SV \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

The study of nonlinear differential equations in the framework of regular variation was initiated by Avakumović [1] and followed by Marić and Tomić [12, 13, 14]. See also Marić [11, Chapter 3]. These papers and some closely related [16, 17] are concerned exclusively with *decreasing* positive solutions of Thomas-Fermi type equations. No analysis from the viewpoint of regular variation seems to have been made of *increasing* positive solutions of such equations. The reason is that the original Thomas-Fermi singular boundary problem reads

$$x''(t) = t^{-1/2} x(t)^{3/2}, \quad x(0) = 1, \; x(\infty) = 0,$$

showing that the decreasing solutions are of primary interest in physics.

For the recent development of asymptotic analysis of second order differential equations by means of regular variation the reader is referred to [4], [5], [7], [8] and [9].

Our purpose here is to show that effective use of theory of regular variation makes it possible to provide information about the existence and asymptotics of increasing solutions of equation (A). Our results are presented in

Sections 2 and 3. The existence of increasing solutions of (A) which are regularly bounded is established in Section 2, while especially the construction of regularly varying solutions of (A) of the regularity index $\rho = 1$ with the precise asymptotic behavior is carried out in Section 3. The main tool employed in both sections, besides regular variation, is the Schauder-Tychonoff fixed point theorem in locally convex spaces.

In 2007 V.M. Evtukhov and V.M. Kharkov in a remarkable paper [3] studied simultaneously both Thomas-Fermi and Emden-Fowler type of equation (A) and gave sharp conditions for the existence of solutions (which may decrease and increase) belonging to a certain class and possessing certain asymptotic behavior. The condition imposed in [3] on function $q(t)$ means, due to Karamata theorem [2, Theorem 1.6.1], that it is of regular variation. The condition imposed in [3] on function $\phi(x)$ means, due to Lemma 3.2 and 3.3 in [11], that it is either regularly or rapidly varying. These facts are neither used (nor mentioned) by Evtukhov and Kharkov which makes their method of proof different from ours and the statements on solutions somewhat weaker than ours (of course, for the Thomas-Fermi case which we consider here).

For simplicity of notation we introduce the symbol $\sim$ to denote the asymptotic equivalence of two positive functions $f(t)$, $g(t)$ defined in a neighborhood of infinity:

$$f(t) \sim g(t), \quad t \to \infty \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1,$$

and the symbol $g(t) \asymp G(t)$ to denote that there exists two positive constants $m, M, m < M$ such that

$$mG(t) \leq g(t) \leq MG(t) \quad \text{for all sufficiently large } t.$$

The expression "for sufficiently large t" throughout the text will be denoted by $t \in [t_0, \infty)$, where $t_0 \geq a$ need not to be the same at each appearance.

It is known (see e.g. I. T. Kiguradze and T. A. Chanturiya [6]) that if $0 < \gamma < 1$, (i.e. equation (A) is sublinear), all positive solutions of (A) can be extended to $t = \infty$. It is clear that for any positive increasing solution $x(t)$ of (A) existing on $[t_0, \infty)$, $x'(t)$ is positive and increasing, so it tends either to ∞ or to some positive constant as $t \to \infty$. In both cases, $x'(t) \geq k$ for some positive constant k and for $t \geq t_1 \geq t_0$. Accordingly, by an integration we get $x(t) \geq x(t_1) + k(t - t_1)$ which implies that $x(t) \to \infty$ as $t \to \infty$.

Thus, all possible positive increasing solutions of (A) fall into the following two types:

$$\lim_{t\to\infty} x(t) = \infty, \ \lim_{t\to\infty} \frac{x(t)}{t} = \infty, \tag{1.4}$$

$$\lim_{t\to\infty} x(t) = \infty, \ \lim_{t\to\infty} \frac{x(t)}{t} = const > 0. \tag{1.5}$$

A solution $x(t) \in RV(\rho)$ is called a *trivial regularly varying solution of index* ρ, denoted by $x(t) \in tr - RV(\rho)$, if it is expressed in the form $x(t) = t^\rho \xi(t)$ with $\xi(t) \in SV$ satisfying $\lim_{t\to\infty} \xi(t) = const > 0$. Otherwise $x(t)$ is called a *nontrivial* $RV(\rho)$*−solution*, denoted by $x(t) \in ntr - RV(\rho)$.

Thus, increasing solutions of the type (1.5) are trivial RV-solutions of index 1, whereas solution $x(t)$ of the type (1.4) may belong to one of the two essentially different sets:

⋄ the one in which $x(t)/t = \xi(t)$, $\xi(t) \in SV$ and $\xi(t) \to \infty$, $t \to \infty$, and

⋄ the one in which $\xi(t) \to \infty$, $t \to \infty$ but $\xi(t) \notin SV$.

For the elements of the former set then holds $x(t) \in ntr - RV(1)$. However for the elements of the later one we prove their regular boundedness i.e. $x(t) \in RO$.

2. *Existence and asymptotic estimate of regularly bounded solutions of (A)*

To establish the existence of the increasing regularly bounded solutions for the sublinear equation (A), first observe that by [2, Theorem 1.5.12]) for function $x/\phi(x) \in RV(1-\gamma)$, with $1-\gamma > 0$, there exists $\psi \in RV\left(\frac{1}{1-\gamma}\right)$ with

$$\psi\left(\frac{x}{\phi(x)}\right) \sim x, \qquad x \to \infty\,.$$

Here ψ is an *"asymptotic inverse"* of $x/\phi(x)$ and is determined uniquely to within asymptotic equivalence.

Thus we define

$$\rho = \frac{\sigma+2}{1-\gamma} \tag{2.1}$$

and the function $X_0(t)$ on $[a,\infty)$ by

$$\frac{X_0(t)}{\phi(X_0(t))} = \frac{t^2 q(t)}{\rho(\rho-1)} \qquad \text{or} \qquad X_0(t) \sim \psi\left(\frac{t^2 q(t)}{\rho(\rho-1)}\right). \tag{2.2}$$

From the second formula in (2.2) due to [11, Proposition 7] there follows $X_0(t) \in RV(\rho)$.

We begin with the following

Lemma 2.1. *Let*

$$J(t,a) = \int_a^t \int_a^s q(r)\phi(X_0(r))drds\,. \tag{2.3}$$

Then,

$$J(t,a) \sim X_0(t), \qquad t \to \infty.$$

P r o o f. Put, $X_0(t) = t^\rho \eta(t)$, $\eta(t) \in SV$. Hence, by writing $J(t,a)$ in the form

$$J(t,a) = \int_a^t \int_a^s q(r)\frac{\phi(X_0(r))}{X_0(r)} X_0(r)drds = \rho(\rho-1)\int_a^t \int_a^s r^{\rho-2}\eta(r)drds,$$

and applying Karamata theorem twice, one obtains the result. □

Now we can prove the main result of this section

Theorem 2.1. *Suppose that* $\sigma > -\gamma - 1$ *and define* $X_0(t)$ *by* (2.2) *and* ρ *by* (2.1)*. Then, equation* (A) *possesses a positive increasing solution* $x(t)$ *satisfying*

$$x(t) \asymp X_0(t). \tag{2.4}$$

Also, $x(t)$ *is regularly bounded at infinity and satisfies* (1.4)*.*

P r o o f. Note that any solution $x(t)$ of the integral equation

$$x(t) = c + \int_{t_0}^t \int_{t_0}^s q(r)\phi(x(r))drds, \quad c > 0 \tag{2.5}$$

(if it exists) satisfies (A) and is obviously positive and increasing. We shall prove that it indeed exists and possesses properties stated in the Theorem.

Since $\phi \in RV(\gamma)$ with $\gamma > 0$ and $X_0 \in RV(\rho)$ with $\rho > 1$ by [11, Proposition 8] these functions are almost increasing, that is there exist constants $A > 1$ and $B > 1$ such that

$$\phi(x) \le A\,\phi(y) \quad \text{and} \quad X_0(x) \le BX_0(y) \qquad \text{for each} \quad y \ge x > 0\,. \tag{2.6}$$

Due to Lemma 2.1 there exists $t_0 > a$ so that

$$J(t,t_0) \leq 2X_0(t), \qquad t \geq t_0. \tag{2.7}$$

In addition, since (1.1) holds uniformly on each compact λ-set on $(0,\infty)$ ([2, Theorem 1.5.2]) there exists a $t_0 > a$ such that

$$\frac{\lambda^\gamma}{2}\phi(X_0(t)) \leq \phi(\lambda X_0(t)) \leq 2\lambda^\gamma \phi(X_0(t)) \quad \text{for} \;\; t \geq t_0. \tag{2.8}$$

Further since $J(t,t_0) \sim X_0(t)$, $t \to \infty$, there exists $t_1 > t_0$ such that

$$J(t,t_0) \geq \frac{X_0(t)}{2}, \qquad t \geq t_1. \tag{2.9}$$

Let $0 < k < 1$ and $K > 1$ be such that

$$k^{1-\gamma} \leq \frac{1}{4A} \quad \text{and} \quad K \geq \max\left\{(8A)^{\frac{1}{1-\gamma}}, 2kB\frac{X_0(t_1)}{X_0(t_0)}\right\}, \tag{2.10}$$

which is possible due to $0 < \gamma < 1$.

Now we choose t_0 such that (2.7) and (2.8) both hold and define the set $\mathcal{X}$ to be the set of continuous functions $x(t)$ on $[t_0,\infty)$ satisfying

$$\begin{cases} kX_0(t_1) \leq x(t) \leq KX_0(t), & \text{for} \;\; t_0 \leq t \leq t_1\,, \\ kX_0(t) \leq x(t) \leq KX_0(t), & \text{for} \;\; t \geq t_1\,. \end{cases} \tag{2.11}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[t_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. We shall show that the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}x(t) = kX_0(t_1) + \int_{t_0}^{t}\int_{t_0}^{s} q(r)\phi(x(r))drds, \quad t \geq t_0,$$

is a continuous self-map on $\mathcal{X}$ and that $\mathcal{F}(\mathcal{X})$ is relatively compact subset of $C[t_0,\infty)$ and then apply the Schauder-Tychonoff fixed point theorem.

Let $x(t) \in \mathcal{X}$. By using successively (2.6), (2.8) with $\lambda = K$, (2.10) and (2.7) one obtains

$$\mathcal{F}x(t) \;\; \leq \;\; kX_0(t_1) + A\int_{t_0}^{t}\int_{t_0}^{s} q(r)\phi(KX_0(r))drds$$

$$
\begin{aligned}
&\le \frac{K}{2B} X_0(t_0) + 2A\,K^\gamma \int_{t_0}^{t}\int_{t_0}^{s} q(r)\phi(X_0(r))drds \\
&\le \frac{K}{2} X_0(t) + 4A\,K^\gamma X_0(t) \le \frac{K}{2} X_0(t) + \frac{K}{2} X_0(t) = K\,X_0(t), \quad t \ge t_0\,.
\end{aligned}
$$

On the other hand, we have

$$
\mathcal{F}x(t) \ge kX_0(t_1) \quad \text{for} \quad t_0 \le t \le t_1\,,
$$

and using (2.6), (2.8) with $\lambda = k$, (2.10) and (2.9) we get

$$
\begin{aligned}
\mathcal{F}x(t) &\ge \frac{1}{A}\int_{t_0}^{t}\int_{t_0}^{s} q(r)\phi(kX_0(r))drds \ge \frac{k^\gamma}{2A}\int_{t_0}^{t}\int_{t_0}^{s} q(r)\phi(X_0(r))drds \\
&\ge \frac{k^\gamma}{4A} X_0(t) \ge kX_0(t), \qquad t \ge t_1.
\end{aligned}
$$

Therefore, $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore it can be verified that $\mathcal{F}$ is a continuous map and that $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[t_0, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a fixed point $x(t)$ of $\mathcal{F}$ which satisfies the integral equation (2.5) and so equation (A). Thus, $x(t)$ is positive and increasing and the fact that $x(t)$ satisfies (2.11) ensures that $x(t)$ satisfies (1.4). That $x(t)$ is regularly bounded follows directly from the definition of regular boundedness, by using inequalities (2.11) and bearing in mind that $X_0(t)$ is regularly varying. This completes the proof of Theorem 2.1. □

Example 2.1 Consider the differential equation (A) with

$$
\phi(x) = x^\gamma \log(x+1) \quad \text{and} \quad q(t) = \frac{\rho(\rho-1)\log t + 2\rho - 1}{t^{\rho(\gamma-1)+2}(\log t)^\gamma \log(t^\rho \log t + 1)},
$$

where $\rho > 1$ and $0 < \gamma < 1$.

The function $q(t)$ is a regularly varying function of index $\sigma = -\rho(\gamma - 1) - 2$, which satisfies $\sigma > -\gamma - 1$. It is easy to check that

$$
\frac{\rho(\rho-1)}{t^2 q(t)} = \frac{t^{\rho(\gamma-1)}(\log t)^\gamma \log(t^\rho \log t + 1)}{\log t + (2\rho-1)/\rho(\rho-1)} \sim (t^\rho \log t)^{\gamma-1}\log(t^\rho \log t{+}1), \quad t \to \infty.
$$

Therefore, from Theorem 2.1 it follows that equation possesses increasing solutions $x(t)$ satisfying

$$x(t)^{\gamma-1}\log(x(t)+1) \asymp (t^{\rho}\log t)^{\gamma-1}\log(t^{\rho}\log t+1),$$

which implies that

$$x(t) \asymp t^{\rho}\log t.$$

One easily check that $x(t) = t^{\rho}\log t$ is an exact solution.

3. *Existence and asymptotic behavior of regularly varying solutions of index* 1

First observe that the existence and the asymptotic behavior of trivial $RV(1)-$solutions of equation (A), i.e. of the type (1.5), is completely resolved by the following

Theorem 3.2. *Equation* (A) *possesses a trivial* $RV(1)-$*solution if and only if*

$$\sigma+\gamma<-1 \tag{3.1}$$

or

$$\sigma+\gamma=-1 \quad \textit{and} \quad \int_a^{\infty} q(t)\phi(t)dt<\infty\,. \tag{3.2}$$

P r o o f. THE "ONLY IF" PART: Suppose that (A) has a $tr-RV(1)-$solution $x(t)$ on $[t_0,\infty)$ satisfying $x(t)\sim ct$, $t\to\infty$, $c>0$. By integrating twice over $[t_0,t]$ on both sides of equation (A), one obtains

$$x'(t)=x'(t_0)+\int_{t_0}^{t} q(s)\phi(x(s))ds,$$

and

$$x(t)=x(t_0)+x'(t_0)(t-t_0)+\int_{t_0}^{t}\int_{t_0}^{s} q(r)\phi(x(r))drds,$$

which because of (1.2) and Definition 1.1 gives for $t\to\infty$

$$x(t)\sim x'(t_0)t+c^{\gamma}\int_{t_0}^{t}\int_{t_0}^{s} r^{\sigma+\gamma}l(r)L(r)drds, \tag{3.3}$$

If one had $\sigma+\gamma>-1$, a repeated application of Proposition 1.1 would lead to

$$x(t) \sim x'(t_0)t + c^\gamma \frac{t^{\sigma+\gamma+2} l(t)L(t)}{(\sigma+\gamma+1)(\sigma+\gamma+2)}, \qquad t \to \infty\,,$$

with $\sigma+\gamma+2>1$, contradicting the hypothesis. Hence, we must have $\sigma+\gamma\leq -1$. In the case of the equality, due to Proposition 1.1 (iii)

$$L^*(t) = \int_{t_0}^{t} s^{-1} l(s)L(s)ds \in SV$$

In view of Proposition 1.1, formula (3.3) gives $x(t) \sim x'(t_0)t + c^\gamma L^*(t)t$, $t\to\infty$, which noting that $x(t)/t \to c$, $t\to\infty$, proves that $L^*(t)$ has to tend to a constant. Thus either (3.1) or (3.2) holds.

The "if" part": Suppose that (3.1) or (3.2) holds, implying the convergence of the integral

$$J := \int_{t_0}^{\infty} q(s)\phi(s)ds.$$

In addition, since $\phi(x)\in RV(\gamma)$ and since $\phi(x)/x \in RV(\gamma-1)$ with $\gamma-1<0$ is almost decreasing function, there exist constant $W>1$ such that

$$\frac{\phi(y)}{y} \leq W\frac{\phi(x)}{x} \qquad \text{for each} \quad y \geq x > 0$$

and $t_0 \geq a$ such that for $\lambda = c/2$ and $t\geq t_0$ one has

$$\phi\left(\frac{c}{2}t\right) \leq 2\left(\frac{c}{2}\right)^{\gamma}\phi(t) \quad \text{and} \quad J \leq \frac{1}{4W}\left(\frac{c}{2}\right)^{1-\gamma}. \tag{3.4}$$

Let us now define the integral operator

$$\mathcal{F}x(t) = ct - \int_{t_0}^{t}\int_{s}^{\infty} q(r)\phi(x(r))drds, \quad t\geq t_0,$$

and the set

$$\mathcal{X} = \left\{x(t)\in C[t_0,\infty) : \frac{1}{2}ct \leq x(t) \leq ct, \quad t\geq t_0\right\}.$$

If $x(t) \in \mathcal{X}$, then clearly $\mathcal{F}x(t) \le ct$. Also, due to (3.4), we obtain

$$\begin{aligned}\int_{t_0}^{t}\int_{s}^{\infty} q(r)\phi(x(r))drds &= \int_{t_0}^{t}\int_{s}^{\infty} q(r)\frac{\phi(x(r))}{x(r)}\,x(r)drds \\ &\le 2W\int_{t_0}^{t}\int_{s}^{\infty} q(r)\phi\left(\frac{c}{2}r\right)drds \le 4W\left(\frac{c}{2}\right)^{\gamma} J\,t \le \frac{ct}{2}, \quad t \ge t_0\,,\end{aligned}$$

and so $\mathcal{F}x(t) \ge ct/2$ for $t \ge t_0$. This shows that $Fx(t) \in \mathcal{X}$, and hence $\mathcal{F}$ is a self-map of the closed convex set $\mathcal{X}$. Moreover, we can verify that $\mathcal{F}$ is continuous and $\mathcal{F}(\mathcal{X})$ is relatively compact in the topology of the locally convex space $C[t_0, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem $\mathcal{F}$ has a fixed point $x_0(t) \in \mathcal{X}$, which gives birth to a solution of equation (A) such that $x_0(t) \sim ct$ as $t \to \infty$. □

To obtain analogous results for nontrivial regularly varying solutions of index 1 we have to restrict ourselves to the smaller class of $\phi(x) \in RV(\gamma)$ by imposing the following additional requirement

$$u(t) \in SV \cap C^1 \Rightarrow \phi(tu(t)) \sim \phi(t)u(t)^{\gamma}, \quad t \to \infty, \tag{3.5}$$

which amounts to requiring that the slowly varying part $L(x)$ of $\phi(x)$ satisfies

$$u(t) \in SV \cap C^1 \Rightarrow L(tu(t)) \sim L(t), \quad t \to \infty. \tag{3.6}$$

It is easy to check that (3.6) is satisfied by

$$L(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k}, \quad \alpha_k \in \mathbf{R},$$

but not by

$$L(t) = \exp\left(\prod_{k=1}^{N} (\log_k t)^{\beta_k}\right), \quad \beta_k \in (0,1),$$

where $\log_k t = \log\log_{k-1} t$.

Now to prove the main result of the section we need the following

Lemma 3.2. *Let ϕ satisfy* (3.5). *Suppose that*

$$\sigma = -\gamma - 1 \quad \textit{and} \quad \int_a^{\infty} q(t)\phi(t)dt = \infty \tag{3.7}$$

and define the function $X_1(t)$ by

$$X_1(t) = t\left[(1-\gamma)\int_a^t q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}}. \tag{3.8}$$

Then

$$\int_a^t\int_a^s q(r)\phi(X_1(r))drds \sim X_1(t), \qquad t\to\infty\,.$$

P r o o f. Due to (3.7) the integral defining $X_1(t)$ is an SV function $\xi_1(t)$ say, and so $X_1(t) \in ntr - RV(1)$. Using expressions

$$X_1(t) = t\xi_1(t), \quad q(t) = t^{-\gamma-1}l(t), \quad \xi_1(t), l(t) \in SV,$$

and the condition (3.5) we get

$$q(t)\phi(X_1(t)) = q(t)\phi(t\xi_1(t)) \sim q(t)\phi(t)\xi_1(t)^\gamma = t^{-1}l(t)L(t)\xi_1(t)^\gamma, \quad t\to\infty, \tag{3.9}$$

implying that

$$\int_a^t q(s)\phi(X_1(s))ds \sim \int_a^t s^{-1}l(s)L(s)\xi_1(s)^\gamma ds \in SV, \quad t\to\infty.$$

Since

$$\begin{aligned}\int_a^t s^{-1}l(s)\xi_1(s)^\gamma L(s)ds &= \int_a^t q(s)\phi(s)\left[(1-\gamma)\int_a^s q(r)\phi(r)dr\right]^{\frac{\gamma}{1-\gamma}} ds \\ &= \left[(1-\gamma)\int_a^t q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}} = \xi_1(t),\end{aligned}$$

we have finally

$$\int_a^t q(s)\phi(X_1(s))ds \sim \xi_1(t) \in SV, \quad t\to\infty.$$

Integrating from t_0 to t, using Karamata's integration theorem we obtain

$$\int_a^t\int_a^s q(r)\phi(X_1(r))drds \sim t\xi_1(t) = X_1(t), \quad t\to\infty.$$

We now prove the main result of this section.

Theorem 3.3. *Let* ϕ *satisfy* (3.5). *Equation* (A) *possesses nontrivial regularly varying solutions of index* 1 *satisfying* (1.4) *if and only if condition* (3.7) *holds, in which case any such solution* $x(t)$ *has one and the same asymptotic behavior*

$$x(t) \sim t \left[(1-\gamma) \int_a^t q(s)\phi(s)ds \right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.10}$$

Proof. THE "ONLY IF" PART: Suppose that (A) has a nontrivial $RV(1)$-solution $x(t)$ on $[t_0, \infty)$ satisfying (1.4). Let $x(t) = t\xi(t), \xi(t) \in SV$. Then, since

$$x(t)/t \sim \int_{t_0}^t q(s)\phi(x(s))ds = \int_{t_0}^t s^{\sigma+\gamma} l(s)\xi(s)^\gamma L(s\xi(s))ds, \quad t \to \infty, \tag{3.11}$$

and $x(t)/t \to \infty$ as $t \to \infty$, σ must satisfy $\sigma + \gamma \geq -1$. It is impossible, however, that $\sigma + \gamma > -1$. In fact, if this would be the case, then integrating (3.11) from t_0 to t and applying Karamata's integration theorem, we would obtain in view of (3.5).

$$x(t) \sim \frac{t^{\sigma+\gamma+2} l(t)\xi(t)^\gamma L(t\xi(t))}{(\sigma+\gamma+1)(\sigma+\gamma+2)} \in RV(\sigma+\gamma+2), \quad t \to \infty,$$

which is impossible because $\sigma + \gamma + 2 > 1$. Thus, one has $\sigma = -\gamma - 1$ and so because of condition (3.5),

$$\int_{t_0}^t q(s)\phi(x(s))ds = \int_{t_0}^t q(s)\phi(s\xi(s))ds \sim \int_{t_0}^t q(s)\phi(s)\xi(s)^\gamma ds \in SV, \quad t \to \infty.$$

Integrating the above from t_0 to t leads to

$$x(t) \sim t \int_{t_0}^t q(s)\phi(s)\xi(s)^\gamma ds, \quad t \to \infty,$$

or

$$\xi(t) \sim \int_{t_0}^t q(s)\phi(s)\xi(s)^\gamma ds, \quad t \to \infty. \tag{3.12}$$

Let the integral in (3.12) be denoted by $Y(t)$. Then, $Y(t)$ satisfies

$$Y(t)^{-\gamma}Y'(t) \sim q(t)\phi(t), \quad t \to \infty, \tag{3.13}$$

and $Y(t) \to \infty$, $t \to \infty$. Integration of (3.13) over $[t_0, t]$, using $Y(t)^{1-\gamma} \to \infty$, $t \to \infty$, yields the second condition in (3.7) and

$$Y(t) \sim \left[(1-\gamma)\int_{t_0}^{t} q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

from which (3.10) follows immediately.

The "if" part: Suppose that (3.8) holds. Then, by replacing in the proof of Theorem 2.1 the function $X_0(t)$ by $X_1(t)$ defined with (3.8) in Lemma 3.1, application of the Schauder-Tychonoff fixed point theorem provides the existence of an increasing solution $x(t)$ of equation (A) satisfying

$$x(t) \asymp X_1(t). \tag{3.14}$$

We show that the obtained solution $x(t)$ of (A) is regularly varying and satisfies (3.10). Using (2.11) and (3.14), from equation (A) we get

$$x''(t) \asymp q(t)\phi(X_1(t)),$$

or using expressions (3.9)

$$x''(t) \asymp t^{-1}l(t)L(t)\xi_1(t)^{\gamma},$$

and integrating over $[t_0, t]$

$$x'(t) \asymp \int_{t_0}^{t} s^{-1}l(s)L(s)\xi_1(s)^{\gamma}ds.$$

Then, by taking $y(t) = x'(t)$ we obtain

$$t\frac{y'(t)}{y(t)} \asymp l(t)L(t)\xi_1(t)^{\gamma}\left[\int_{t_0}^{t} s^{-1}l(s)L(s)\xi_1(s)^{\gamma}ds\right]^{-1}.$$

Application of Karamata's integration theorem gives

$$\lim_{t\to\infty} l(t)L(t)\xi_1(t)^{\gamma}\left[\int_{t_0}^{t} s^{-1}l(s)L(s)\xi_1(s)^{\gamma}ds\right]^{-1} = 0,$$

which yields $ty'(t)/y(t) \to 0$ as $t \to \infty$, so that $y(t) = x'(t) \in SV$ (see [11, Proposition 10]). Then Karamata's integration theorem gives

$$x(t) \sim \int_{t_0}^{t} x'(s)ds = \int_{t_0}^{t} y(s)ds \sim ty(t), \quad t \to \infty. \tag{3.15}$$

The right-hand side of (3.15) is regularly varying of index 1, so that the one on the left-hand side is also regularly varying of the same index (see [11, Proposition 7]). Therefore, $x(t) \in ntr - RV(1)$ and so has the desired asymptotic behavior. This completes the proof of Theorem 3.2. □

Example 3.1 Consider equation (A) with

$$\phi(x) = x^\gamma \log(x+1) \quad \text{and} \quad q(t) = (t^{\gamma+1}(\log t)^\gamma \log(t \log t + 1))^{-1},$$

where $\gamma \in (0,1)$. Note that $\phi(x)$ fulfills the condition (3.5). Clearly, $q(t)$ is a regularly varying function of index $\sigma = -\gamma - 1$ and satisfies

$$q(t)\phi(t) \sim t(\log t)^{-\gamma}, \quad t \to \infty.$$

This gives

$$\int_{e}^{t} q(s)\phi(s)ds \sim \frac{(\log t)^{1-\gamma}}{1-\gamma}, \quad t \to \infty$$

so that from Theorem 3.2 we conclude that the considered equation possesses nontrivial regularly varying solutions $x(t)$ of index 1, all of which have one and the same asymptotic behavior $x(t) \sim t \log t$, $t \to \infty$. In fact an exact solution is $x(t) = t \log t$.

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