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# LARGE LINEAR EQUATION WITH LEFT AND RIGHT FRACTIONAL DERIVATIVES IN A FINITE INTERVAL

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A b s t r a c t. *A class of linear equations with left and right fractional derivatives and singular perturbations is analyzed by the use of generalized functions and Fredholm's theory.*

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## 1. *Introduction*

Equations with left and right fractional derivatives appear as mathematical models in different branches of physics and mechanics (see [12]). We refer to monographs [10], [11], [13], [16], [18], [19], [24] and references therein for equations with the left fractional derivatives. Equations with the both types of fractional derivatives have appeared recently only in a few papers although the interest for models with both types of derivatives increases (cf. [3]– [7], [9], [26]–[28]).

In year 2010 appeared the monograph [14] in which the author solve equations with symetric $D^\alpha sym = \frac{1}{2}\left(D^\alpha_{0+} + D^\alpha_{b-}\right)$ and $D^\alpha anti = \frac{1}{2}\left(D^\alpha_{0+} - D^\alpha_{b-}\right)$ and with complex derivatives: ${}^cD^\alpha_{b-}\ D^\alpha_{0+},\ D^\alpha_{b-}\ D^\alpha_{0+}$. This operators appear when we apply the minimum action principle in constructing mathematical models in fractional mechanics.

The aim of this paper is to reduce the problem of solving differential equations with fractional derivatives, within $\mathcal{D}'_{L^1}((-\infty, b))-$generalized functions with supports contained in $[0, b)$, denoted by $\mathcal{D}'_{L^1}([0, b))$, to the well-known problem of solving Fredholm's type equations with bounded or weakly bounded kernels (cf. [20], [21]).

We consider equation

$$\sum_{i=1}^{p} A_i(D^{\alpha_i}_{0+}Y)(x) + \sum_{j=1}^{q} B_j(D^{\beta_j}_{b^-}Y)(x) + C(x)Y(x) = D(x) \text{ in } \mathcal{D}'_{L^1}([0, b)), \tag{1.1}$$

where $A_i$ and $B_j$ are constants; $\alpha_i = k_i + \gamma_i,\ k_i \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},\ \gamma_i \in [0, 1),\ , i = 1, ...p,\ \alpha_{i+1} > \alpha_i, i = 1, ...p - 1$ and $\beta_j = n_j + \nu_j,\ n_j \in \mathbf{N}_0,\ \nu_j \in [0, 1),\ \beta_{j+1} > \beta_j,\ j = 1, ..., q - 1,\ \nu_q < 1,\ C(x) \in \mathcal{C}^m([0, b))$ and $D(x) \in \mathcal{D}'^{m+k_p}_{L^\infty}([0, b))$ (see, Section 2.2).

Now we can explain that the main contribution of our paper comes from assumptions $C(x) \in \mathcal{C}^m([0, b)),\ D(x) \in \mathcal{D}'^{m+k_p}_{L^\infty}([0, b))$ as well as from a simple procedure of solving (1.1) which will be realized in Sections 3 and 4.

We refer to [24] for explicit methods of solving (1.1) in various classes of function spaces which depend on coefficients and the order of (1.1). Let us mention some of these results

Let $p = q = 1$. With appropriate assumptions on $D$, the case $\alpha = \beta$ and $C = 0$ can be reduced to the generalized Abel integral equation, which is solvable within the space

$$H^*(0, b) = \{f; f = \frac{f^\lambda(x)}{x^{1-\varepsilon_1}(b - x)^{1-\varepsilon_2}}, f^\lambda \in H^\lambda(0, b), \varepsilon_1, \varepsilon_2 \in (0, 1)\},$$

where $H^\lambda(0, b)$ is the space of the Lipshitz functions of order $\lambda$ in $(0, b)$ (cf. Theorem 30.7 in[24]). Note that this case is solved in [28] within a suitable space of generalized functions.

The case $\alpha = \beta$ and $C \neq 0$ as well as the case $k = n$ and $\gamma > \nu$ (with appropriate assumptions on $C$ and $D$) can be reduced to a Noether integral equation of the first kind

$$\int_0^b T(x,t)f(t)dt = f(x), 0 < x < b.$$

Such an integral equation was solved in [24], Theorem 31.11.

In this paper we suppose that $k_p \geq n_q + 1, A_p \neq 0$. We seek for solutions belonging to the space $\mathcal{D}'_{L^1}([0,b))$. Reducing (1.1) to Fredcholm's integral equation of second kind with bounded or weakly bounded singular kernel, we discuss the existance of solutions to (1.1) and note that the solutions are the classical solutions as well, if appropriate conditions hold for $C$ and $D$. We give in Example 1 a unique solution to an equation of a given form over the interval $[0,b)$ for sufficiently small $b$. As an application of Proposition 3.2 we give a complete solution of the linear differential equation in which right fractional derivative do not exist.

## 2. *Preliminaries*

We use the usual notation of distributions theory (see for example [25], [30]): $\mathcal{D}' = \mathcal{D}'(\mathbf{R})$ and $\mathcal{S}' = \mathcal{S}'(\mathbf{R})$ are Schwartz's spaces of distributions; $\mathcal{S}'_+$ is the commutative and associative convolution algebra of tempered distributions supported by $[0,\infty)$. If $T \in \mathcal{S}'$ is a regular distribution defined by a function $f$ so that $f(x)(1+|x|)^{-k} \in L^1(\mathbf{R})$ for some $k > 0$, then we write $T = f$.

The family of distributions $\{f_\beta; \beta \in \mathbf{R}\}$ :

$$f_\beta(t) = \begin{cases} H(t)t^{\beta-1}/\Gamma(\beta), \ \beta > 0, \\ f_{\beta+m}^{(m)}(t), \ \beta \leq 0, \ \beta + m > 0, m \in \mathbf{N}, \end{cases}$$

where $(\cdot)^{(m)}$ is the distributional derivative and $H$ is Heviside's function, is an Abelian group in $\mathcal{S}'_+$ under convolution: $f_{\beta_1} * f_{\beta_2} = f_{\beta_1+\beta_2}, \ f_0 = \delta$ and $f_{-\beta} = \delta^{(\beta)}, \ \beta_1, \beta_2, \beta \in \mathbf{N}_0$. If $f \in \mathcal{S}'_+$, and $\beta < 0$, then $f_\beta * f$ is $-\beta$ fractional derivative and if $\beta \geq 0$, then $f_\beta * f$ is $\beta$ fractional integral of $f$.

### 2.1. *Spaces* $\mathcal{D}'^m_{L^1}([0,b))$ *and* $\mathcal{D}'_{L^1}([0,b))$

Let $b > 0$. We denote by $L^1_0((-\infty,b))$, resp., $L^\infty_0((-\infty,b))$ the space of integrable functions, resp., of bounded functions in $(-\infty,b)$ vanishing in

$(-\infty, 0)$. Let $m \in \mathbf{N}_0$,

$$\mathcal{D}'^m_{L^1}([0,b)) = \{f^{(m)} = \delta^{(m)} * f,\ f \in L^1_0((-\infty, b))\},\ m \in \mathbf{N_0}$$

and

$$\mathcal{D}'^m_{L^\infty}([0,b)) = \{f^{(m)},\ f \in L^\infty_0((-\infty, b))\},\ m \in \mathbf{N_0}.$$

Clearly, $\mathcal{D}'^m_{L^\infty}([0,b)) \subset \mathcal{D}'^m_{L^1}([0,b)$. If $m \le m_1$, then $\mathcal{D}'^m_{L^1}([0,b)) \subset \mathcal{D}'^{m_1}_{L^1}([0,b))$ and the inclusion mapping is continuous. Then, define

$$\mathcal{D}'_{L^1}([0,b)) = \bigcup_{m=0}^{\infty} \mathcal{D}'^m_{L^1}([0,b)).$$

It is a closed subset of $\mathcal{S}'((-\infty, b))$ (where the former space is the strong dual of the test space with the sequence of seminorms defining the structure of $\mathcal{S}$). Since

$$\mathcal{D}'^m_{L^1}([0,b)) \ni v(\cdot) = f^{(m)}(\cdot) = (H(\cdot)f(\cdot)H(b-\cdot))^{(m)},\ f \in L^1_0((-\infty, b)),$$

we will also use the representation

$$v(\cdot) = (H(\cdot)f(\cdot)H(b-\cdot))^{(m)}.$$

If $v \in \mathcal{D}'^m_{L^1}([0,b))$ and $a \in \mathcal{C}^m([0,b))$, then we define $av$ in $\mathcal{D}'^m_{L^1}([a,b))$ by $av = af^{(m)}$.

(We have to use the Leibnitz formula $af^{(m)} = \sum_{j \le m} (-1)^j \binom{m}{j} (a^{(j)} f)^{(m-j)}$). In the same way we define the product $av$ if $a \in \mathcal{C}^\infty([0,b))$ and $v \in \mathcal{D}'_{L^1}([a,b))$.

Let $v_i \in \mathcal{D}'^{m_i}_{L^1}([0,b)), i = 1,2$. Then the convolution $v_1 * v_2$ belongs to $\mathcal{D}'^{m_1+m_2}_{L^1}([0,b))$ and it is defined by $v_1 * v_2 = (f_1 * f_2)^{(m_1+m_2)},\ f_i \in L^1_0((-\infty, b)), i = 1,2$.

2.2. *Left and right fractional derivatives in* $\mathcal{D}'_{L^1}([a,b))$

We introduce a mapping $Q$ as follows. Let $f \in L^1_0((-\infty, b))$. Then $Qf$ is defined in $\mathbf{R}$ by

$$(Qf)(x) = f(b-x),\ 0 \le x < b, (Qf)(x) = 0, x < 0$$

and

$$\text{if } v = f^{(m)} \in \mathcal{D}'^m_{L^1}([0,b)),\ \text{then } Qv = (-1)^m (Qf)^{(m)}.$$

It follows that $Qv \in \mathcal{D}'^{m}_{L^1}([0,b))$ and that $Q$ maps $\mathcal{D}'_{L^1}([0,b))$ onto $\mathcal{D}'_{L^1}([0,b))$. Moreover, $QQ = I$.

Let $v_1$ and $v_2$ be in $\mathcal{D}'_{L^1}([0,b))$, then $Q(Av_1+Bv_2) = AQv_1+BQv_2,\ A,B \in \mathbf{R}$. Let $a \in \mathcal{C}^\infty([0,b)), v \in \mathcal{D}'_{L^1}([0,b))$. Then $Q(av) = Q(a)Q(v)$ and $Q(v^{(m)}) = (-1)^m(Qv)^{(m)},\ m \in \mathbf{N}$.

We recall (cf. [24] and [11]) the definitions of the left and right Riemann-Liouville fractional integrals $I^\alpha_{0+}, I^\alpha_{b-}$ and fractional derivatives $D^\alpha_{0+}, D^\alpha_{b-}$ for $\alpha = k+\gamma,\ \gamma \in (0,1),\ k \in \mathbf{N}_0$, of a function $f$, for $x \in [0,b)$,

$$(I^\alpha_{0+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,\ \ (I^\alpha_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}dt; \tag{2.1}$$

$$(D^\alpha_{0+}f)(x) = \frac{1}{\Gamma(1-\gamma)}\Big(\frac{d}{dx}\Big)^{k+1}\int_0^x \frac{f(t)}{(x-t)^{\gamma}}dt, \tag{2.2}$$

$$(D^\alpha_{b-}f)(x) = \frac{(-1)^{k+1}}{\Gamma(1-\gamma)}\Big(\frac{d}{dx}\Big)^{k+1}\int_x^b \frac{f(t)}{(t-x)^{\gamma}}dt. \tag{2.3}$$

In order to make legitimate definitions (2.1)-(2.3) we assume that $f$ belongs to $AC^{k+1}([0,T]), k \in \mathbf{N_0}$, for every $T \in [0,b)$, which means that the derivatives of $f$ up to order $k$, are continuous and $(k+1)-$th derivative is integrable in $[0,T]$, for every $T \in [0,b)$.

Using the left fractional integral, (2.2) can be written as

$$(D^\alpha_{0+}f) = \Big(\frac{d}{dx}\Big)^{k+1}\Big(I^{1-\gamma}_{0+}f\Big) = f_{-k-1} * f_{1-\gamma} * f. \tag{2.7}$$

Thus, for $v \in \mathcal{D}'^{m}_{L^1}([0,b))$ and $\alpha = k+\gamma,\ k \in \mathbf{N}_0,\ 0 \le \gamma < 1$ we have

$$D^\alpha_{0+}v = \delta^{(k+1+m)} * f_{1-\gamma} * f,$$

Let us remark that $D^k_{0+}v = \delta^{(k+m)} * f = D^{k+m}f$.

$$D^\alpha_{b-}v = QD^\alpha_{0+}Qv.$$

The next lemma gives some properties of operators $D^\alpha_{0+}$ and $D^\alpha_{b-}$, which we need in the sequel. Its proof is simple and thus, omitted.

**Lemma 2.1.**

*1)* $D^{\alpha}_{0+}$ *and* $D^{\alpha}_{b-}$ *map* $\mathcal{D}'^{m}_{L^1}([0,b))$ *into* $\mathcal{D}'^{m+k+1}_{L^1}([0,b))$*, where* $\alpha = k+\gamma,\ k\in \mathbf{N}_0,\ \gamma\in[0,1)$*. In particular, these operators map* $\mathcal{D}'_{L^1}([0,b))$ *into* $\mathcal{D}'_{L^1}([0,b))$.

*2)* $D^{\alpha}_{b-}v = (-1)^{k+1}\delta^{(k+m+1)} * (I^{1-\gamma}_{b-}f)$.

3. *Solutions to linear equation with left and right fractional derivatives*

We consider equation (1.1) with prescribed properties of coefficients. Note that assumption on $D$ implies that $D * f_{\alpha_p+m}$ is a continuous function in $[0,b]$. As regards the supposition $k_p \geq n_q+1$, let us remark that:

if $n_q \geq k_p+1$, then we can transform equation (1.1) to the previous case applying operator $Q$ and obtain

$$\sum_{i=0}^{p} A_i D^{\alpha_i}_{b-} QY + \sum_{j=0}^{q} B_j D^{\beta_j}_{0+} QY + (QC)(x)(QY) = QD. \tag{3.1}$$

Equation (3.1) is also of the form (1.1) with the opposite role of $\beta_q$ and $\alpha_p$.

3.1. *Case* $p=1,\ q=1$

Equation (1.1) in this case becomes

$$(D^{\alpha}_{0+}y)(x) + B(D^{\beta}_{b-}y)(x) + C(x)y(x) = D(x), \quad \text{in } \mathcal{D}'_{L^1}([0,b))\,, \tag{3.2}$$

where $\alpha = k+\gamma,\ \beta = n+\nu;\ n\leq k-1,\ k,n\in\mathbf{N}_0,\ \gamma\in[0,1), \nu\in[0,1), B\in \mathbf{R}, C(x)\in C^m([0,b))$ and $D(x)\in \mathcal{D}'^{m+k}_{L^\infty}([0,b))$.

Assuming that $y$ is of the form $y = \delta^{(m)} * \eta$, with

$$\eta(\cdot) = H(\cdot)H(b-\cdot)\tilde{\eta}(x), \tilde{\eta}\in L^1_0((-\infty,b)),$$

we have

$$D^{\alpha}_{0+}y = \delta^{(m+k+1)} * I^{1-\gamma}_{0+}\eta,$$

for $\gamma\in(0,1)$ and $D^{\alpha}_{0+}y = \delta^{(k+1)} * \eta = \eta^{(n+k)}$ for $\gamma = 0$;

$$D^{\beta}_{b-}y = (-1)^{n+1}\delta^{(m+n+1)} * I^{1-\nu}_{b-}\eta.$$

We rewrite (3.2) (for $\nu \in (0,1)$) as

$$\begin{aligned} &\delta^{(m+k+1)} * I_{0^+}^{1-\gamma}\eta + B(-1)^{n+1}\delta^{(m+n+1)} * I_{b^-}^{1-\nu}\eta \\ &+ \sum_{r=0}^{m} a_r f_{r-m} * (C^{(r)}\eta) = D \end{aligned} \tag{3.3}$$

($a_r = (-1)^r \binom{m}{r}$). Applying ($f_{\alpha+m}*$) to both sides of (3.3) we obtain

$$\eta + (-1)^{n+1} B f_\mu * I_{b^-}^{1-\nu}\eta + \sum_{r=0}^{m} a_r f_{\alpha+r} * (C^{(r)}\eta) = f_{\alpha+m} * D, \tag{3.4}$$

where $\mu = k - n - 1 + \gamma \geq \gamma$ ($k \geq n+1$) and for $x \in [0,b]$,

$$f_\mu * I_{b^-}^{1-\nu}\eta(x) =$$

$$\frac{1}{\Gamma(\mu)\Gamma(1-\nu)} \int_0^b H(x-t)(x-t)^{\mu-1}dt \int_0^b \frac{\eta(\tau)H(\tau-t)}{(\tau-t)^\nu}d\tau, \mu \neq 0. \tag{3.5}$$

and

$$f_0 * I_{b^-}^{1-\nu}\eta(x) = I_{b^-}^{1-\nu}\eta(x)$$

$$f_{\alpha+r} * (C^{(r)}\eta)(x) = \frac{1}{\Gamma(\alpha+r)} \int_0^x C^{(r)}(\tau)\eta(\tau)(x-\tau)^{\alpha+r-1}dt \tag{3.6}$$

($\alpha + r - 1 \geq 0$).
Here and below we consider $L^1-$functions, so a function can take value $\infty$ or $-\infty$ at some points of $[0,b]$.

Since $\mu - 1 \geq \gamma - 1$, (for $\nu \in (0,1)$), it follows that for every $x \in [0,b]$

$$(t,\tau) \mapsto \frac{\eta(\tau)H(x-t)H(\tau-t)}{(x-t)^{1-\mu}(\tau-t)^\nu}, \;\; t \in [0,b], \; \tau \in [0,b],$$

is an integrable function. Thus, we can change the order of integration in (3.5) and with (3.6), equation (3.4) becomes

$$\begin{aligned}\eta(x) &= \frac{(-1)^n B}{\Gamma(\mu)\Gamma(1-\nu)} \int_0^b H(x-t)(x-t)^{\mu-1} \Big( \int_0^b \frac{\eta(\tau)H(\tau-t)}{(\tau-t)^\nu} d\tau \Big) dt \\ &- \sum_{r=0}^m \frac{a_r}{\Gamma(\alpha+r)} \int_0^b H(x-\tau)C^{(r)}(\tau)\eta(\tau)(x-\tau)^{\alpha+r-1} d\tau \\ &+ f_{\alpha+m} * D(x) \\ &= \int_0^b d\tau \Big( \frac{(-1)^n B}{\Gamma(\mu)\Gamma(1-\nu)} \int_0^x \frac{(x-t)^{\mu-1}H(\tau-t)}{(\tau-t)^\nu} dt \\ &- \sum_{r=0}^m \frac{a_r}{\Gamma(\alpha+r)} H(x-\tau)C^{(r)}(\tau)(x-\tau)^{\alpha+r-1} \Big) \eta(\tau) \\ &+ f_{\alpha+m} * D(x),\ x \in [0,b],\ \mu \neq 0;\end{aligned}$$

$$\begin{aligned}\eta(x) &= \int_0^b \Big( \frac{(-1)^n B}{\Gamma(1-\nu)} H(\tau-x) \frac{1}{(\tau-x)^\nu} \\ &- \sum_{r=0}^m \frac{a_r}{\Gamma(\alpha+r)} H(x-\tau)C^{(r)}(\tau)(x+\tau)^{\alpha+r-1} \Big),\ \eta(\tau)d\tau, \mu = 0.\end{aligned}$$

We consider

$$\eta(x) + \int_0^b K(x,\tau)\eta(\tau)d\tau = M(x),\ 0 \le x \le b, \tag{3.7}$$

where $M(x) = f_{\alpha+m} * D(x),\ x \in [0,b]$

$$\begin{aligned}&K(x,\tau) \\ &= \frac{(-1)^{n+1} B}{\Gamma(\mu)\Gamma(1-\nu)} \int_0^x \frac{H(\tau-t)dt}{(t-\tau)^\nu (x-t)^{1-\mu}} \\ &+ \sum_{r=0}^m \frac{a_r}{\Gamma(\alpha+r)} H(x-\tau)C^{(r)}(\tau)(x-\tau)^{\alpha+r-1},\ (x,\tau) \in [0,b]\times[0,b], \mu \neq 0\end{aligned} \tag{3.8}$$

and

$$K(x,\tau) = \frac{(-1)^n B}{\Gamma(1-\nu)} H(\tau - x)\frac{1}{(\tau-x)^{\nu}} + \tag{3.9}$$
$$+ \sum_{r=0}^{m} \frac{a_r}{\Gamma(\alpha+r)} H(x-\tau) C^{(r)}(\tau)(x-\tau)^{\alpha+r-1}, \ \mu = 0.$$

Note that $M$ is a continuous function in $[0,b]$.

Let us analyze the kernel $K$. Suppose that $\mu \neq 0$. The first addend of $K$ contains the integral

$$J(x,\tau) = \int_0^x \frac{H(\tau-t)}{(\tau-t)^{\nu}(x-t)^{1-\mu}} dt, \ (x,\tau) \in [0,b] \times [0,b], \ \mu \neq 0$$

which determines the structure of $K, \ \mu \neq 0$, because the second addend in $K$ is a bounded function on $[0,b] \times [0,b]$.

We will consider separately cases

I: $\mu < \nu$, II: $\nu < \mu$ and III: $\nu = \mu$. (Recall, $\mu = k - n - 1 + \gamma$).

Case I.

Let $(x,\tau) \in [0,b] \times [0,b], x < \tau$. Then, with the change of variable $t = x - (\tau - x)p$, we have (with suitable $C$)

$$J(x,\tau) = \int_0^x \frac{dt}{(\tau-t)^{\nu}(x-t)^{1-\mu}} = (\tau-x)^{\mu-\nu} \int_0^{x/(\tau-x)} \frac{dp}{(1+p)^{\nu} p^{1-\mu}}$$
$$\leq (\tau-x)^{\mu-\nu}\Big(\int_0^1 \frac{dp}{p^{1-\mu}} + \int_1^{\infty} \frac{dp}{p^{1+\nu-\mu}}\Big) \leq C(\tau-x)^{\mu-\nu}. \tag{3.10}$$

Let $(x,\tau) \in [0,b] \times [0,b], x > \tau$. Then, with the change of variable $t = \tau - (x-\tau)p$, we have

$$J(x,\tau) \leq C(x-\tau)^{\mu-\nu}.$$

Case II.

Let $(x,\tau) \in [0,b] \times [0,b], \tau > x$. Then

$$J(x,\tau) \leq \int_0^x \frac{dt}{(\tau-t)^{\nu}(x-t)^{1-\mu}} = (\tau-x)^{\mu-\nu}\Big(\int_0^1 + \int_1^{x/(\tau-x)} \frac{dp}{(1+p)^{\nu}p^{1-\mu}}\Big)$$

$$\leq (\tau - x)^{\mu-\nu}(\int_0^1 \frac{dp}{p^{1-\mu}} + \int_1^{x/(\tau-x)} \frac{dp}{p^{1-\mu+\nu}}).$$

This implies

$$|J(x,\tau)| \leq C,\ (x,\tau) \in [0,b] \times [0,b], \tau > x.$$

Similarly, we have

$$|J(x,\tau)| \leq C,\ (x,\tau) \in [0,b] \times [0,b], x > \tau.$$

Case III.

Let $(x,\tau) \in [0,b] \times [0,b], \tau > x$. We have

$$J(x,\tau) = \int_0^1 \frac{dp}{p^{1-\mu}} + \int_1^{x/(\tau-x)} \frac{dp}{p} \leq C\ ln|\tau - x|.$$

If $(x,\tau) \in [0,b] \times [0,b], \tau < x$, then the same inequality holds, as well.

Since $\nu \in (0,1), \mu > 0$, it follows that $J$ is an integrable function.

Now one has to use the well-known Fredholm's theory of integral equations of second type (see [20], Ch.II and Ch. III, and the Handbook of integral equations [21], Chapter II, especially Section 11) in order to solve equation (3.7). Here we will only present a result which is related to the unique solvability in the case when $\lambda = -1$, respectively, $\lambda_p = (-1)^p$ is not an eigenvalue of the kernel $K$, respectively, iterated kernel $K_p$. Actually, the kernel considered in this work does not have any of properties which can imply a simple analysis of eigenvalues (i.e of zeros of $D(\lambda)$, where $D(\lambda)$ is a power series in $\lambda$ with coefficients constructed by $K$, see II (42) in [20] and [21]). So some of approximation procedures of numerical analysis can serve as a method for explicite approximate solving of the equation. In the end of the paper we will discuss a class of integral equations which can be solved by simpler methods. Very special interesting cases of integral equations with log-type kernel can be foud in [8].

So, we have the following procedure for solving (3.2) given in the form of a theorem:

**Theorem 3.1.**

*a) If $k > n+1$ or $k = n+1$ and $\gamma \geq \nu$, the integral equation (3.7) is of Fredholm's type. Moreover, suppose that $(-1)$ is not an eigenvalue of the kernel $K$. Then the unique solution $\eta$ to (3.7) defines distribution $y = D^m(H\eta) \in \mathcal{D}'^m_{L^1}([0,b))$, the unique solution to equation (3.2) in $\mathcal{D}'^m_{L^1}([0,b))$.*

*b) If $k = n+1$ and $\nu > \gamma > 0$, then (3.7) is a weakly singular Fredholm equation with the kernel given by (3.8). But if $\gamma = 0$, the kernel $K(x,\tau)$ is given by (3.9) and is also weakly singular. Let $K_p$ be $p-$times iterated kernel of the singular kernel $K$, such that $K_p(x,\tau)$ is bounded on $[a,b]\times[a,b]$,*

$$K_p(x,\tau) = \int_0^b K_{p-1}(x,s)K(s,\tau)ds, \quad K_1(x,\tau) = K(x,\tau),\ (x,\tau) \in [0,b]\times[0,b].$$

*If $(-1)^p$ is not an eigenvalue of $K_p$, then we have $\eta$ to be the solution to the $(p-1)$-fold iterated equation (3.7) and $y = D^m(H\eta) \in \mathcal{D}'^m_{L^1}([0,b))$ to be the unique solution to equation (3.2) in $\mathcal{D}'^m_{L^1}([0,b))$.*

P r o o f. *If* $k > n+1$, then $k \geq n+2$ and $\mu \geq 1+\gamma \neq 0$. The kernel $K(x,\tau)$ is of the form (3.8). Since $\mu > \nu$, the function $\mathcal{J}(x,\tau)$ is bounded and with this, $K(x,\tau)$ is bounded, as well. Fredholm's theory can be applied.

*If* $k = n+1$, then $\mu = \gamma$; for $\gamma \neq 0$ and $\gamma > \nu$ $K(x,\tau)$ is also given by (3.8) and is bounded. But if $\gamma \neq 0$ and $\gamma \leq \nu$, the kernel $K(x,\tau)$ is given by (3.8) and is weakly singular. In case $\gamma = 0$, we have $\mu = 0$ and $K(x,\tau)$ is given by (3.9). This kernel is also weakly singular. The theory of Fredholm's equation with weakly singular kernel can be applied (cf. [20], Part III).

In this case there exists $p_0 \in \mathbf{N}$, which depends on $\gamma$ and $\nu$ such that for $p \geq p_0$ the iterated kernels $K_p$ are bounded. Now, if $(-1)^p$ is not an eigenvalue of $K_p$, then the $(p-1)$-fold iterated equation to (3.7) is

$$\varphi(x) = M_p(x) + (-1)^p \int_0^b K_p(x,\tau)\varphi(t)dt,\ 0 \leq x \leq b,$$

where, with $M_1 = M$,

$$M_p(x) = M(x) + \sum_{j=1}^{p-1}(-1)^j \int_0^b K_\nu(x,t)M(t)dt, 0 \leq x \leq b,$$

has a unique solution $\eta$, which is integrable function in $[0, b]$.

(As we mentioned, the previous conclusions are consequences of results exposed in [18], Chapters II, III. Se also [9] and [8].

Now it is clear that $\eta$ is a solution to equation (3.7) and that $D^m(H\eta)$ a unique solution to (3.2).

**Remark 3.1** *1) It is self-understandable that if we have a solution $\eta(x)$ to (3.7) with $C \in \mathcal{C}[(0,b)]$ and $D \in L^\infty([0,b))$ such that $D^\alpha_{0+}\eta$ and $D^\beta_{b-}\eta$ belong to $L^1([0,b))$, then $\eta(x)$ is a classical solution to (3.2) in $L^1([0,b))$.*

*2) To solve integral equation (3.7) one can use the following result (cf. [23], Chapter IV,§1):*

*If $\alpha, \beta$ and $b$ are such that the Kernel $K(x,\tau)$ satisfies one of the conditions:*

*a)* $\int\limits_0^b \int\limits_0^b |K(x,\tau)|^2 dx d\tau < 1, \ M \in L^2([0,b))$;

*b)* $\max |K(x,\tau)| < \frac{1}{b}, \ (x,\tau) \in [0,b]^2, \ M \in \mathcal{C}([0,b])$,

*then the solution to integral equation (3.7) can be expressed by Neumann's series*

$$\eta = M(x) + \sum_{n=1}^{\infty} \int\limits_0^b K_n(x,\tau) M(\tau) d\tau,$$

*where $K_n(x,\tau)$ is the iterated kernel. In case a) the solution belongs to $L^2([0,b))$ and in case b) the solution belongs to $\mathcal{C}([0,b])$.*

*3) The case when $-1$ is an eigenvalue has to be treated by the third Fredholm theorem (Section II in [17]). In the case of a weak singular kernel and $(-1)^p$ being an eigenvalue, one has to use results of Chapter III of [17].*

3.2. *The general case of equation (3.1)*

Let in (1.1), $Y = \delta^{(m)} * \eta \in \mathcal{D}'^{m}_{L^1}([0,b))$. Then we have

$$\sum_{i=1}^{p} A_i f_{-m-\alpha_i} * \eta + \sum_{j=1}^{q} B_j (-1)^{n_j+1} f_{-n_j-m-1} * I_{b^-}^{1-\nu_j}\eta$$

$$+ \sum_{r=0}^{m} a_r f_{r-m} * (C^{(r)}\eta) = D.$$

We apply to this equation $(f_{\alpha_p+m}*)$ and obtain

$$\eta \quad + \sum_{i=1}^{p-1} A_i f_{\alpha_p-\alpha_i} * \eta + \sum_{j=1}^{q} (-1)^{n_j+1} B_j (f_{\mu_{p,j}} * I_{b^-}^{1-\nu_j}\eta) +$$

$$+ \sum_{r=0}^{m} a_r f_{\alpha_p+r} * C^{(r)}\eta = f_{\alpha_p+m} * D,$$

where $\mu_{p,j} = k_p - n_j - 1 + \gamma_p > 0$, because we suppose that $k_p \geq n_q + 1$.

We consider a singular integral equation

$$\eta(x) + \int_0^b K(x,\tau)\eta(\tau)d\tau = M(x), \;\; 0 \leq x \leq b \tag{3.11}$$

where

$$\begin{aligned} K(x,\tau) &= H(x-\tau)\sum_{i=1}^{p-1} \frac{A_i}{\Gamma(\alpha_p-\alpha_i)} \frac{1}{(x-\tau)^{1-(\alpha_p-\alpha_i)}} \\ &+ \sum_{j=1}^{q} \frac{(-1)^{n_j} B_j}{\Gamma(\mu_{p,j})\Gamma(1-\nu_j)} \int_0^x \frac{H(t-\tau)d\tau}{|t-\tau|^{\nu_j}(x-t)^{1-\mu_{p,j}}} \\ &+ \sum_{r=0}^{m} \frac{a_r}{\Gamma(\alpha_p+r)} H(x-\tau)C^{(r)}(\tau)(x-\tau)^{\alpha_p+r-1}, x,\tau \in [0,b], \mu \neq 0; \end{aligned} \tag{3.12}$$

$$K(x,\tau) = H(x-\tau)\sum_{i=1}^{p-1} \frac{A_i}{\Gamma(\alpha_p-\alpha_i)} \frac{1}{(x-\tau)^{1-(\alpha_p-\alpha_i)}}$$

$$+ \sum_{j=1}^{q} \frac{(-1)^n B_j}{\Gamma(1-\nu_j)} H(\tau - x) \frac{1}{(\tau - x)^{\nu_j}} \tag{3.13}$$
$$+ \sum_{r=0}^{m} \frac{a_r}{\Gamma(\alpha + r)} H(x-\tau) C^{(r)}(\tau)(x-\tau)^{\alpha+r-1}, \ \mu = 0.$$

and

$$M(x) = (f_{\alpha_p + m} * D)(x), \ \ 0 \le x \le b$$

is continuous.

Now, with the arguments of previous section, we have the following theorem related to equation (3.1). Again by the use of results from [18], we have the following theorem in which we assume that $\nu_j \in [0,1), j = 1, ..., q$.

**Theorem 3.2.**

*a) Let* $\alpha_p - \alpha_{p-1} \ge 1$ *and: 1)* $k_p - n_q > 1$ *or 2)* $k_p - n_j = 1, \ j \in \{1, ..., q\}$ *and* $\gamma_p > \nu_j$.

*Then the kernel of equation (3.11) given by (3.12) is a Fredholm kernel. Moreover, assume that* $(-1)$ *is not an eigenvalue of this kernel. Then the unique solution* $\eta$ *to (3.11) defines the distribution* $Y = D^m(H\eta)$ *which is a unique solution to equation (1.1) in* $\mathcal{D}'^{m}_{L^1}([0,b))$.

*b) If* $0 < \alpha_p - \alpha_{i_0} < 1, \ 1 \le i_0 < p$ *or if* $k_p - n_{j_0} = 0$*, for a* $j_0$*,* $1 \le j_0 \le q$*, and* $\nu_{j_0} > \gamma_p$ *such that: 1)* $\gamma_p \ne 0$ *or 2)* $\gamma_p = 0$*, then (3.11) with the kernel* $K$ *given in case 1) by (3.2) and in case 2) by (3.13) is a weakly singular Fredholm equation. Let* $K_p, \ p \ge p_0$ *be a bounded iterated kernel of* $K$*. Then* $\eta$ *is the solution to the* $(p-1)$ *fold iterated equation (3.11). Moreover,* $Y = D^m(H\eta)$ *is a unique solution to (1.1) in* $\mathcal{D}'^{m}_{L^1}([0,,b])$.

The procedure of the proof is the same as for the Theorem 3.1. □

**Remark 3.2** *Consider equation (1.1) as the classical one in* $[0,b]$

$$\sum_{i=1}^{p} A_i (D_{0^+}^{\alpha_i}) Y(x) + \sum_{j=1}^{q} B_j (D_{b^-}^{\beta_j} Y)(x) + C(x) Y(x) = D(x),$$

*where* $A_i$ *and* $B_j$ *are constants;* $\alpha_i = k_i + \gamma_i, \ k_i \in \mathbf{N}_0, \ \gamma_i \in [0,1), i = 1, ..., p, \ \alpha_{i+1} > \alpha_i, i = 1, ..., p-1,$ *and* $\beta_j = n_j + \nu_j, \ n_j \in \mathbf{N}_0, \ \nu_j \in$

*$[0,1)$, $\beta_{j+1} > \beta_j$, $j = 1, ..., q-1$, $C \in \mathcal{C}([0,b))$ and $D \in L^\infty([0,b))$. If the solution to (3.11) (with $m = 0$), $Y = \eta$ has fractional derivatives, appearing in the equation, which belong to the space of integrable functions in $[0,b]$, then $Y$ is a classical solution to (1.1) $\in [0,b]$.*

**Example 1.** Let us consider equation

$$D^2 y\,(t) + B[(D^\alpha_{0+}y)\,(t) - (D^\alpha_{b-}y)\,(t)] + \omega^2 y\,(t) = f(t),$$

where $0 < \alpha < 1$. This equation is of the form (1.1) with: $\alpha_2 = 2, A_2 = 1; \alpha_1 = \alpha, A_1 = B; \beta_1 = \alpha, B_1 = -B; c = \omega^2$ and $D\,(t) = f\,(t)$.

We suppose that $f \in \mathcal{D}'_{L^1}([0,b))$ such that $M = f_2 * f$ is continuous. Now the kernel $K$, given by (3.12), reads:

$$K\,(x,\tau) = \frac{bH\,(x-\tau)}{\Gamma\,(2-\alpha)}\,(x-t)^{1-\alpha} + \frac{b}{\Gamma\,(1-\alpha)} \int_0^x \frac{H\,(t-\tau)\,d\tau}{|t-\tau|^\alpha} + H\,(x-\tau)\,\omega^2 (x-\tau),$$

$$(x,\tau) \in [0,b] \times [0,b].$$

We shall find the solution in $[0,b)$ for sufficiently small $b$.Here we have considered the cas when $f$ is a distribution. Since it is known the existence and the unity of the solution of this equation on any interval where the Lipschitz condition holds for $f$, we obtain that the solution obtained in this example can be continued on any finite interval $[0,T], T > 0$, if $f$ is locally Lipschitz in $[0,\infty)$.

We estimate $K$ in $[0,b] \times [0,b]$,

$$|K\,(x,\tau)| \le \frac{b^{2-\alpha}}{\Gamma\,(2-\alpha)} + \frac{b^{2-\alpha}}{\Gamma\,(1-\alpha)\,(1-\alpha)} + \omega^2 b = 2\frac{b^{2-\alpha}}{\Gamma\,(2-\alpha)} + \omega^2 b \equiv N.$$

Let $K_1\,(x,t) = K\,(x,t)$ and

$$K_n\,(x,t) = \int_0^b K_{n-1}\,(x,\tau)\,K\,(\tau,t)\,d\tau, \quad (x,t) \in [0,b] \times [0,b],\ n \ge 2.$$

By the above estimate, we have

$$|K_n\,(x,t)| \le N^n b^{n-1},\ (x,t) \in [0,b] \times [0,b].$$

Let $\omega, b$ and $\alpha$ be such that $Nb < 1$, then corresponding integral equation has a unique solution

$$\eta\,(x) = M\,(x) + \sum_{n=1}^{\infty} (-1)^n \int_0^b K_n\,(x,t)\,M(t)dt,\ x \in [0,b].$$

3.3. *Equation (1.1) in which right fractional derivatives do not exist*

Equation of the form (1.1) in which right fractional derivatives do not exist ($B_j = 0,\ j = 1, ..., q$) have been analysed in many papers and books. Also many methods for explicitly solving such equations have been elaborated. In [13] one can find collected such results and references on them. Also in [21], p. 141-142 one can find explicitly solved generalized Abel integral equation of the second kind.

As a consequence of Theorem 3.2 we have

**Proposition 3.1** *Integral equation (3.11) with the Kernel*

$$\begin{aligned} K(x,\tau) &= H(x-\tau)\sum_{i=1}^{p-1}\frac{A_i}{\Gamma(\alpha_p-\alpha_i)}\frac{1}{(x-\tau)^{1-(\alpha_p-\alpha_i)}} \\ &+ \sum_{r=0}^{m}\frac{a_r}{\Gamma(\alpha_p+r)}H(x-\tau)C^{(r)}(\tau)(x-\tau)^{\alpha_p+r-1}, (x,\tau)\in[0,b]^2 \end{aligned} \tag{3.14}$$

*is: 1) if $\alpha_p - \alpha_i \geq 1,\ i = 1, ..., p-1$, a Voltera integral equation; 2) if $\alpha_p - \alpha_{i_0} < 1,\ 1 \leq i_0 \leq \alpha p - 1$ or $\alpha_p < 1$ is a weakly singular Voltera equation, integrable on $0 \leq x \leq b$, $0 \leq \tau \leq x$.*

*Equation*

$$\sum_{j=0}^{p} A_i(D_{0^+}^{\alpha_i}Y)(x) + C(x)Y(x) = D(x), \tag{3.15}$$

*where $D \in \mathcal{D}\prime_{L^\infty}^{m+k}$, has one and only one solution in $\mathcal{D}\prime_{L^1}^{m}([0,b))$ of the form $Y = D^m(H\eta)$, where*

$$\eta(x) = M(x) + (-1)\int_0^x N(x,\tau,-1)M(\tau)d\tau,\ 0 \leq x \leq b, \tag{3.16}$$

*where*

$$\begin{aligned} N(x,\tau,-1) &= (-1)K(x,\tau) + \sum_{n=1}^{\infty}(-1)^n K_n(x,\tau), \\ K_n(x,\tau) &= \int_\tau^x K(x,t)K_{n-1}(t,\tau)d\tau,\ K_0 = K. \end{aligned}$$

P r o o f. Equation (3.15) is a special case of equation (1.1) with $B_j = 0,\ j = 1, ..., q$. The kernel $K(x,\tau)$ given by (3.14) is the kernel given by (3.12) with $B_j = 0,\ j = 1, ..., q$.

To prove this proposition we have only to apply the theorem for Voltera weakly singular integral equations with the kernels integrable on $0 \le x \le b,\ 0 \le \tau \le x$ and $M(x) \in L^1([0,b))$ (cf. [20], p.13).

*Appendix*

*Theorem A* Voltera equation of the second kind

$$\varphi(x) = f(x) + \lambda \int_0^x N(x,y)\varphi(y)dy$$

has one and only one bounded solution, given by the formula

$$\varphi(x) = f(x) + \lambda \int_0^x \mathcal{N}(x,y,\lambda) f(y)dy,$$

where the resolvent kernel $\mathcal{N}$ is

$$\mathcal{N}(x,y,\lambda) = N(x,y) + \sum_{n=1}^{\infty} \lambda^n N_n(x,y)$$

convergent for all values of $\lambda$. It is assumed that the function $f(x)$ is integrable in the interval $[0,b]$ and the function $N(x,y)$ is integrable in the triangle $0 \le x \le b,\ 0 \le y < x$. ($N_n(x,y) = \int_y^x N(x,s)N_{n-1}(s,y)ds$, integral is a Rieman integral).

*Generalized Abel equation of the second kind* (cf [21], p.141-142),

$$y(x) - \lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} = f(x),\ \alpha = 1 - \frac{m}{n},\ m \in \mathbb{N},\ n \in \mathbb{N}+1,\ m > n$$

has the solution

$$y(x) = f(x) + \int_0^x R(x-t)f(t)dt,$$

where

$$R(x) = \sum_{n=1}^{\infty} \frac{\left(\lambda\Gamma(1-\alpha)x^{1-\alpha}\right)^n}{x\Gamma\left((1-\alpha)\right)}.$$

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