Divulgaciones Matemáticas Vol. 13 No. 1(2005), pp. 45–53

Alexandroff Topologies Viewed as Closed Sets in the Cantor Cube

Topologías de Alexandroff Vistas como Subconjuntos del Cubo de Cantor

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Abstract

We present some results about the class of Alexandroff topologies (i.e. topologies where the intersection of arbitrary many open sets is open) from the perspective obtained when they are viewed as closed subsets of the Cantor cuber 2^X (the power set of X with the product topology).
Key words and phrases: Alexandroff topologies, lattice of topologies.

Resumen

Presentamos algunos resultados sobre la clase de topologías Alexandroff (es decir, aquellas donde la intersección arbitraria de abiertos es abierto) desde la perspectiva que se obtiene al verlas como conjuntos cerrados del cubo de Cantor 2^X (el conjunto potencia de X con la topología producto).
Palabras y frases clave: topologías de Alexandroff, retículo de topologías.

1 Introduction

Given a topology τ on an infinite set X, by identifying a set with its characteristic function, we can view τ as a subset of the cantor cube 2^X (i.e.

Received 2004/05/07. Revised 2005/07/04. Accepted 2005/07/06.
MSC (2000): Primary 54A10; Secondary 54A05, 54F05.

$\{0,1\}^X$ with the product topology). Under this identification, we can talk about open, closed, clopen, meager, etc. topologies. For instance, a topology over $\mathbb{N}$ (or any countable set X) is said to be analytic, when it is an analytic set as a subset of the cantor set $2^{\mathbb{N}}$ (i.e. a continuous image of the irrationals [4]). The systematic study of analytic topologies initiated in [9, 10] shows that by restricting our attention to analytic topologies some pathologies are avoided and we get, for instance, a smoother theory of countable sequential spaces. In this paper we will analyze, from this perspective, a particular class of topologies, namely, the so called Alexandroff topologies.

A topology τ over X is said to be an Alexandroff topology if it is closed under arbitrary intersection. Watson [11] attributed this notion to both Alexandroff and Tucker and thus called them AT topologies; we will use his notation denotating by $AT(X)$ the collection of all AT topologies on X. It is easy to verify that a topology is AT if for every $x \in X$ the set $N_x^\tau = \bigcap\{V : x \in V \text{ and } V \in \tau\}$ is an open set. N_x^τ is called the irreducible (or minimal) neighborhood of x (when there is no danger of confusion, we will just write N_x instead of N_x^τ). AT topologies are specially relevant for the study of non T_1 topologies (notice that the only T_1 AT topology is the discrete topology). They play an important role in the study of the lattice of all topologies over a set [8, 11, 12] and recently have received more attention due to its connection with digital topologies [5, 6].

The starting point for this work is the fact that AT topologies correspond exactly to those which are closed as subsets of 2^X. The collection of clopen topologies is particularly simple, since they essentially correspond to topologies over finite sets (both results were known for the case of a countable set X [9]). To state one of the contributions of this paper we need to recall some notions. To each topology τ it is associated the following binary relation:

$$x \leq_\tau y \text{ if } x \in cl_\tau(\{y\}) \tag{1}$$

where cl_τ denotes the closure operator of τ. It is easy to verify that $\leq_\tau$ is transitive and reflexive, so it is a pre-order on X which is called the pre-order induced by τ (it is also called the specialization pre-order of τ). An AT topology τ is uniquely determined by its associated $\leq_\tau$ (see theorem 3.1). Moreover, it is known that the map $\rho \mapsto \leq_\rho$ is a complete lattice isomorphism between the lattice of AT topologies and the lattice of pre-orders over X [8]. More information about the relation $\leq_\tau$ can be found in [3, II 1.8]. The pre-order $\leq_\tau$ allows to introduce an equivalence relation $\approx$ over the lattice $TOP(X)$ of all topologies over X as follows

$$\tau \approx \rho \text{ if } \leq_\tau = \leq_\rho \tag{2}$$

For instance, the collection of all T_1 topologies forms an equivalence class. We got interested in this equivalence relation after reading S. Watson's work [12] and we are thankful to him for allowing us to have a copy of it.

One of the results of this paper says that $\tau \approx \rho$ iff τ and ρ have equal closure in 2^X. From this fact, we easily get that τ is T_1 iff τ is dense in 2^X. Another curious result is that an AT topology is compact iff X is an isolated point of τ in 2^X.

Since $AT(X)$ can be viewed as a subset of the hyperspace $K(2^X)$ of all compact subsets of 2^X with the Vietoris topology, it is natural to study the topological properties of $AT(X)$ as a subspace of $K(2^X)$. We will show in the last section that $AT(X)$ is homeomorphic to the collection of pre-orders over X (with the topology inherited from the cantor cube $2^{X\times X}$).

Our notation and terminology is standard [2, 7]. For our purposes it is convenient to present the product topology in the following way. For every $K \subseteq F \subseteq X$, we define the interval

$$[K, F] = \{A \in 2^X : K \subseteq A \subseteq F\} \tag{3}$$

The collection of all intervals $[K, F]$ with K finite and F cofinite is a basis for the product topology on 2^X. Notice that these intervals are in fact clopen in 2^X.

2 Closed and Open Topologies

We start with a characterization of closed topologies.

Theorem 2.1. *Let τ be a topology over X. The following are equivalent*
(i) τ is AT.
(ii) $\tau \subseteq 2^X$ is closed.

Proof. $(i) \Rightarrow (ii)$: Let A_α be a net of open sets in τ, and suppose that A_α converges (in 2^X) to a set A. We will show that $A \in \tau$. Let $x \in A$ and N_x be the irreducible neighborhood of x (since τ is AT). It suffices to show that $N_x \subseteq A$. Since A_α converges to A and $x \in A$, then there is β such $x \in A_\alpha$ for all $\alpha > \beta$. Since each A_α is open, then $N_x \subseteq A_\alpha$ for all $\alpha > \beta$. Therefore $N_x \subseteq A$.

$(ii) \Rightarrow (i)$: Let $x \in X$ and A_α be the net of all open neighborhoods of x ordered by reversed inclusion. Then A_α converges (in 2^X) to $\bigcap A_\alpha$. Since τ is closed, then $\bigcap A_\alpha \in \tau$ and therefore τ is AT. $\square$

The following proposition is probably well known and its proof is straightforward.

Proposition 2.2. *Let $f, g : 2^X \times 2^X \to 2^X$ $h : 2^X \to 2^X$ be the functions defined by $f(A, B) = A \cap B$, $g(A, B) = A \cup B$ and $h(A) = X - A$. Then f, g and h are continuous and open. Moreover, h is a homeomorphism.*

Proposition 2.3. *(i) Let $\mathcal{F} \subseteq 2^X$ be a closed subset closed under finite intersections. Then $\mathcal{F}$ is closed under arbitrary intersections.*
(ii) Let $\mathcal{F} \subseteq 2^X$ be a closed subset closed under finite unions. Then $\mathcal{F}$ is closed under arbitrary unions.

Proof. (i) Let A_α with $\alpha \in J$ be any collection of sets in $\mathcal{F}$, and let K be the collection of finite subsets of J ordered by inclusion. Let $B_S = \bigcap_{\alpha \in S} A_\alpha$, then $(B_S : S \in K)$ is a net in $\mathcal{F}$. Clearly, $\lim_S B_s = \bigcap_\alpha A_\alpha$. The proof of (ii) is similar. □

Theorem 2.4. *Let τ be a topology over X. The closure $\overline{\tau}$ of τ in 2^X is a topology and therefore is the smallest AT topology containing τ.*

Proof. By proposition 2.3, it is enough to show that $\overline{\tau}$ is closed under finite unions and intersections. Let $A, B \in \overline{\tau}$ and let $\{A_\alpha\}$, $\{B_\beta\}$ be two nets in τ converging to A and B respectively. By proposition 2.2, the intersection and union functions are continuous, thus $A_\alpha \cap B_\beta$ converges to $A \cap B$ and $A_\alpha \cup B_\beta$ converges to $A \cup B$. Finally, it follows from theorem 2.1 that $\overline{\tau}$ is AT □

Now we will show that open topologies correspond to topologies over finite sets.

Theorem 2.5. *Let τ be a topology over X. The following are equivalent.*

(i) τ is open in 2^X.

(ii) τ is clopen in 2^X.

(iii) $\emptyset, X$ are interior points of τ in 2^X.

(iv) There is a finite τ-clopen set F whose complement is τ-discrete.

Proof. Clearly (i) implies (iii). Suppose now that (iii) holds, then there are finite sets K and L such $\{A \in 2^X : K \subseteq A\} \subseteq \tau$ and $\{A \in 2^X : A \cap L = \emptyset\} \subseteq \tau$. Let $F = K \cup L$. Thus F is τ-clopen and $X - F$ is discrete, hence (iv) holds. Finally we will show that (iv) implies (ii). For every $K \subseteq F$ with $K \in \tau$, let V_K be the basic clopen set $\{A \subseteq X : K \subseteq A \ \& \ A \cap (F - K) = \emptyset\}$. Then by (iv) $V_K \subseteq \tau$. On the other hand, given $A \in \tau$ put $K = A \cap F$, then $A \in V_K$. Therefore $\tau = \bigcup\{V_K : \ K \subseteq F, \ K \in \tau\}$. Since F is finite, then τ is clopen in 2^X. □

3 The induced pre-order

The following result is well known [8] and shows that AT topologies are completely determined by its induced pre-order $\leq_\tau$ given by (1).

Theorem 3.1. *A topology τ over X is AT iff there is a unique binary relation $\leq$ over X which is transitive, reflexive and such that for all $A \subseteq X$ the following holds:*

$$A \in \tau \;\textit{ iff }\; \{y \in X : x \leq y\} \subseteq A \;\textit{ for every } x \in A.$$

In this case, the irreducible neighborhood of x is

$$N_x^\tau = \{y \in X : x \leq y\}.$$

and moreover

$$cl_\tau(A) = \bigcup_{x \in A} cl_\tau(\{x\}) = \bigcup_{x \in A} \{y \in X : y \leq x\}.$$

Thus $\leq$ is precisely $\leq_\tau$. Furthermore, τ is T_0 iff $\leq$ is antisymmetric. □

For a given pre-order $\leq$ over X the AT topology given by theorem 3.1 will be called the *associated AT topology* of $\leq$ and will be denoted $\tau(\leq)$. Thus the previous theorem implies that $\tau(\leq_\rho) = \rho$ for any AT topology ρ. The following is a useful fact.

Proposition 3.2. *If $\tau \subseteq \rho$, then $\leq_\rho \subseteq \leq_\tau$. Moreover, for AT topologies the converse also holds.*

Proof. The first claim is obvious. For the second claim, let τ and ρ be AT topologies over X such that $\leq_\rho \subseteq \leq_\tau$. Notice that $N_x^\rho \subseteq N_x^\tau$ for all $x \in X$. It suffices to show that N_x^τ is ρ-open. Let $z \in N_x^\tau$, then $N_z^\rho \subseteq N_z^\tau \subseteq N_x^\tau$. □

It is clear from theorem 3.1 that in the equivalence class (with respect to relation $\approx$ defined in (2)) of a topology ρ there is a unique AT member, namely $\tau(\leq_\rho)$. It is known that such AT topology is the largest element of the equivalence class [3, p. 45], that is to say, $\rho \subseteq \tau(\leq_\rho)$ for every topology ρ. This will be a consequence of the following

Theorem 3.3. *Let τ and ρ be topologies over X. Then*

$$\tau \approx \rho \;\textit{ iff }\; \overline{\tau} = \overline{\rho}$$

where $\overline{\tau}$ is the closure of τ in 2^X. In particular, $\overline{\tau}$ is the largest topology within the equivalence class of τ.

Proof. Let τ be a topology over X. By theorem 2.1, closed topologies are AT and, by theorem 3.1, they are uniquely determined by their induced pre-order. Thus it is clearly sufficient to show that $\leq_\tau = \leq_{\overline{\tau}}$. By proposition 3.2, it suffices to show that $\leq_\tau \subseteq \leq_{\overline{\tau}}$. Suppose $x \not\leq_{\overline{\tau}} y$ and let $V \in \overline{\tau}$ such that $x \in V$ and $y \notin V$. By the definition of the product topology, there is $W \in \tau$ such $x \in W$ and $y \notin W$, thus $x \not\leq_\tau y$. □

Since τ and $\overline{\tau}$ induce the same pre-order, then we immediately get the following corollaries

Corollary 3.4. *Let τ be a topology over X. Then*
(i) τ is T_0 iff $\overline{\tau}$ is T_0.
(ii) τ is T_1 iff τ is dense in 2^X.

For a given pre-order $\leq$ over X, we will denote by $Min(\leq)$ the collection of minimal elements of X, i.e. those $x \in X$ such that there is no $y < x$. Analogously we denote by $Max(\leq)$ the collection of maximal elements of X. Notice that $Min(\leq)$ is closed and $Max(\leq)$ is open in the AT topology $\tau(\leq)$. Notice also that $Max(\leq)$ is contained in every $\tau(\leq)$-dense set. It is clear that for an *AT* topology τ, the collection of τ-closed sets also form a topology, called the cotopology of τ, which we will denote by $co\tau$. Notice that $x \leq_{co\tau} y$ iff $y \leq_\tau x$. We will use this observation in the proof of the next result.

Proposition 3.5. *Let τ be a T_0 AT topology over X.*

(i) $\emptyset$ is isolated in τ iff $Max(\leq_\tau)$ is a finite τ-dense set iff there is a finite τ-dense set.

(ii) X is isolated in τ iff $Min(\leq_\tau)$ is finite and $X = \bigcup_{x \in Min} N_x$ iff there is a finite set L such that $X = \bigcup_{x \in L} N_x$.

Proof. To show (i) notice that if $F \subseteq X$ is a cofinite set, then $[\emptyset, F] \cap \tau = \{\emptyset\}$ iff $X \setminus F$ is τ-dense. On the other hand, suppose L is a finite τ-dense set. Since every point in $Max(\leq_\tau)$ is τ-open, then $Max(\leq_\tau) = Max(L, \leq_\tau)$. To see (ii), just applied (i) to the cotopology $co\tau$ and recall that the map $A \mapsto X \setminus A$ is a homeomorphism of 2^X, so X is isolated in τ iff $\emptyset$ is isolated in $co\tau$. □

Theorem 3.6. *Let τ be a T_0 AT topology. The following are equivalent*
(i) (X, τ) is compact.
(ii) X is an isolated element of τ in 2^X.

Proof. Suppose (X, τ) is compact. Consider the open covering of X given by the irreducible neighborhoods N_x with $x \in X$. Then by compactness there is a finite set K such that the N_x's with $x \in K$ is a finite covering of X. Then by proposition 3.5(ii) X is isolated in τ. Conversely, suppose X is isolated in τ and let $\mathcal{U}$ be an open covering of X. Then by proposition 3.5(ii) any finite subset of $\mathcal{U}$ containing $Min(\leq_\tau)$ is a covering of X. $\square$

4 Convergency of AT topologies

Let $PO(X)$ denote the collection of all pre-orders over X with the topology its inherited from $2^{X\times X}$. Here we view a pre-order as a binary relation, thus as a subset of $X \times X$. It is easily shown that $PO(X)$ is a compact subset of $2^{X\times X}$.

We have already mentioned that the map from $PO(X)$ onto $AT(X)$ given by $\leq\mapsto \tau(\leq)$ is a lattice isomorphism. We will show below that this map is moreover a homeomorphism when $AT(X)$ given the topology inherited from $K(2^X)$ (the hyperspace of compact subsets of 2^X with the Vietoris topology). For the particular case of a countable set X, a different proof of this result was given in [1].

First we fix the natural basis for the Vietories topology on $K(2^X)$. Since 2^X is compact, the following sets form a basis for the Vietoris topology on $K(2^X)$

$$\{\mathcal{C} \in K(2^X) :\ \mathcal{C} \cap [K, F] \neq \emptyset\} \ \ , \ \ \{\mathcal{C} \in K(2^X) :\ \mathcal{C} \subseteq \bigcup_{i=1}^{n} [K_i, F_i]\}$$

where $K, K_i \subseteq X$ are finite and $F, F_i \subseteq X$ are cofinite and the interval $[K, F]$ is defined by equation (3).

Theorem 4.1. *The map from $PO(X)$ to $K(2^X)$ that sends $\leq$ to $\tau(\leq)$ is continuous and injective. Thus it is a homeomorphism of $PO(X)$ onto $AT(X)$. In particular, $AT(X)$ is a compact set with the Vietoris topology.*

Proof. From theorem 3.1 it follows that T is injective and onto $AT(X)$. So it remains to show that it is continuous. Let $\{\leq_d\}_{d\in D}$ be a net in $PO(X)$ converging to a preorder $\leq$, where $(D, \preceq)$ is a directed set. This means that for every finite $S \subseteq X$ there is $d_0 \in D$ such that

$$\{(x, y) \in S^2 :\ x \leq_d y\} = \{(x, y) \in S^2 :\ x \leq y\} \ \text{ for all } d \succeq d_0 \tag{4}$$

(viewing a pre-order as binary relations, i.e. as a subset of X^2). Let τ_d and τ denote the AT topologies associated to $\leq_d$ and $\leq$, respectively. There are two cases to consider:

Suppose first that $\tau \subseteq \mathcal{U} := \bigcup_{i=1}^{n}[K_i, F_i]$, where $K_i \subseteq X$ is finite and $F_i \subseteq$ is cofinite for $i \leq n$. Let $S = \bigcup_i (K_i \cup (X \setminus F_i))$. Since S is finite, there is d_0 such that (4) holds. We will show that $\tau_d \subseteq \mathcal{U}$ for all $d \succeq d_0$. Fix $d \succeq d_0$, $V \in \tau_d$ and let $T = V \cap S$. Let N_T be the union of N_x^τ for $x \in T$. Since $N_T \in \tau$, then there is i such that $K_i \subseteq N_T \subseteq F_i$. It suffices to show that $K_i \subseteq V \subseteq F_i$. In fact, let $y \in K_i$, then there is $x \in T$ such that $x \leq y$. Since $(x, y) \in S^2$, then $x \leq_d y$. Since $V \in \tau_d$ and $x \in T \subseteq V$, then $y \in V$. On the other hand, if there is $x \in V \setminus F_i$, then $x \in T$, but this is impossible, as $T \subseteq N_T \subseteq F_i$.

Now suppose $\tau \cap [K, F] \neq \emptyset$ for some finite K and cofinite F. Consider $S = K \cup (X \setminus F)$. As before fix d_0 such that (4) holds. It is routine to show that $V_d = \bigcup_{x \in K} N_x^{\tau_d}$ belongs to $[K, F]$ for all $d \succeq d_0$. $\square$

References

[1] Díaz, J. A., *Complementación en el retículo de topologías*, Tesis de maestría, Universidad de Los Andes, Departamento de Matemáticas, Mérida, Venezuela, 1999.

[2] Engelking, R., *General Topology*, PWN, Warszawa, 1977.

[3] Johnstone, P., *Stone Spaces*, Cambridge studies in advanced mathematics 3, Cambridge University Press, 1986.

[4] Kechris, A. S., *Classical descriptive set theory*, Springer-Verlag, 1994.

[5] T. Y. Kong, R. Kopperman, and P. Meyer. *A topological approach to digital topology*, Amer. Math. Monthly, (December):901–917, 1991.

[6] Kong, T. Y. , Roscoe, A. W. , Rosenfeld, A., *Concepts of digital topology*, Topology and its Applications, 46:219–262, 1992.

[7] Munkres, J. R., *Topology*, Prentice Hall, Englewood Clifss, N. J., 1975.

[8] Steiner, A. K., *The lattice of topologies: structure and complementation*, Trans. Amer. Math. Soc., 122(2):379–398, 1966.

[9] Todorčević, S., Uzcátegui, C. , *Analytic topologies over countable sets*, Top. and its Appl., 111(3):299–326, 2001.

[10] Todorčević, S., Uzcátegui, C. , *Analytic k-spaces*, Top. and its Appl., 146-147:511–526, 2005.

[11] Watson, S., *The number of complements*, Top. and its Appl., (55):101–125, 1994.

[12] Watson, S., *The construction of topological spaces: Order*, `http://math.yorku.ca/watson`.