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On a Hedlund's theorem and place-dependent cellular automata

Sobre un teorema de Hedlund y autómatas celulares dependientes de la posición

I. Arcaya (arcaya@euler.ciens.ucv.ve)
Universidad Central de Venezuela. Facultad de Ciencias.
Escuela de Matemáticas. Ciudad Universitaria. Caracas, Venezuela.

N. Romero (nromero@uicm.ucla.edu.ve)
Universidad Centroccidental Lisandro Alvarado.
Departamento de Matemática. Decanato de Ciencias y Tecnología.
Apartado Postal 400.
Barquisimeto, Venezuela.

Abstract

The main goal of this paper is to show extensions of the well known cellular automata characterization given by Hedlund in one of his classic articles. We first extend that result to discrete dynamical systems over the space of d-dimesional sequences with values on a finite alphabet and defined by means of a finite number of local rules. In addition, using the barrier concept provided by the Set Theory, we extend the notion of local rule with values in any discrete topological space, then we generalize the extended result in this context.
Key words and phrases: Shift mapping, Local rule, Cellular automata, Barriers.

Resumen

El principal objetivo de este artículo es mostrar extensiones de la conocida caracterización de los autómatas celulares dada por Hedlund en uno de sus clásicos artículos. Primero extendemos este resultado a la clase de sistemas dinámicos discretos sobre el espacio de sucesiones d-dimensionales con valores en un alfabeto finito y definido a partir de un número finito de reglas locales. Seguidamente, usando el concepto

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de barrera proveido por la Teoría de Conjuntos, extendemos la noción de regla local con valores en cualquier espacio topológico discreto, luego generalizamos el resultado extendido en este contexto.
Palabras y frases clave: aplicación shift, regla local, autómata celular, barreras.

1 Introduction

Cellular Automata (CA) are discrete dynamical systems acting on the *configuration space* $\mathcal{A}^{\mathbb{Z}^d}$ of all d-dimensional sequence $x : \mathbb{Z}^d \to \mathcal{A}$, where $\mathcal{A}$ is a finite alphabet, $\mathbb{Z}^d$ is the d-dimensional integer lattice, and the *global transition maps* defining CA are given by the action of a *local rule* which determines the evolution of each *cell* $x(n)$ ($n \in \mathbb{Z}^d$) of the *configuration* $x \in \mathcal{A}^{\mathbb{Z}^d}$ depending on the values of cells on a uniform neighborhood. More explicitly, $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is a *cellular automaton* if and only if there are: a finite and nonempty set $\mathbb{V}$ and a local rule $f : \mathcal{A}^{\mathbb{V}} \to \mathcal{A}$, where $\mathcal{A}^{\mathbb{V}}$ is the set of all functions from $\mathbb{V}$ to $\mathcal{A}$, such that for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $n \in \mathbb{Z}^d$:

$$F(x)(n) = f(x|_{\mathbb{V}+n}), \tag{1}$$

where $x|_{\mathbb{V}+n} : \mathbb{V} \to \mathcal{A}$ is given by $x|_{\mathbb{V}+n}(k) = x(n+k)$ for all $k \in \mathbb{V}$. In other words, the value of cell n in the configuration $F(x)$ depends of the values of the cells $n+k$, $k \in \mathbb{V}$, in the configuration x.

CA have been used to model a huge number of discrete dynamical systems of relevant significance, in fact CA have at present spread to a wide spectrum of disciplines including physics, chemistry, biochemistry, biology, economy and even sociology. The apparent simplicity of CA does not imply trivial asymptotic behavior of its orbits; actually, a global description of the temporal (and spatial) evolution of cells can be extremely difficult.

On the configuration space $\mathcal{A}^{\mathbb{Z}^d}$ we consider the product topology, that is the finest topology so that, for each $n \in \mathbb{Z}^d$, the projection $\pi_n : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}$, $\pi_n(x) = x(n)$, is a continuous map. It is well known, that even if $\mathcal{A}$ is any discrete topological space, the family of cylinders $C(\mathbb{U}, h) = \{x \in \mathcal{A}^{\mathbb{Z}^d} : x|_{\mathbb{U}} = h\}$, where $\mathbb{U}$ is a finite and nonempty subset of $\mathbb{Z}^d$ and h is a function from $\mathbb{U}$ into $\mathcal{A}$, is a clopen (closed and open set) basis for that topology. It is well know that for any topological space $\mathcal{A}$, the function $D : \mathcal{A}^{\mathbb{Z}^d} \times \mathcal{A}^{\mathbb{Z}^d} \to [0, +\infty)$ given by $D(x,y) = 2^{-i}$ where $i = \inf\{\|n\| : n \in \mathbb{Z}^d,\ x(n) \neq y(n)\}$ and

$$\|n\| = \max\{|n_i| : 1 \le i \le d\} \ \text{ if } \ n = (n_1, \cdots, n_d),$$

define a metric compatible with the product topology on $\mathcal{A}^{\mathbb{Z}^d}$. In the particular case when the alphabet $\mathcal{A}$ is finite, this metric is called Cantor metric and $\mathcal{A}^{\mathbb{Z}^d}$ is a Cantor set; that means compact, perfect and totally disconnected.

Recently other topological structures on $\mathcal{A}^{\mathbb{Z}^d}$ have been considered, for example the Besicovitch and Weyl topologies have been used to explain chaotic behavior of CA, see for example [2], [3], [5] and references therein. We do not consider these topological structures in this work.

For instance, consider $\mathcal{A}$ any discrete topological space which we assume finite. It is obvious that any local rule is a continuous function and CA are continuous transformations of the configuration space $\mathcal{A}^{\mathbb{Z}^d}$. Also it is simple to verify that CA commute with each *shift map* $\sigma_j : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ $(j = 1, \cdots, d)$ which is defined, for each $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $n \in \mathbb{Z}^d$, as $\sigma_j(x)(n) = x(n + e_j)$ where e_j is the j-th canonical vector of $\mathbb{Z}^d$. Notice that shift maps are cellular automata. As homeomorphisms on $\mathcal{A}^{\mathbb{Z}^d}$ the shift maps have an important impact in the developing of the dynamical systems theory. In a classical work, Hedlund [6] gave them remarkable treatment and produced very important results. One of them, see Theorem 3.4 in [6], characterized CA in terms of shift maps, in fact: every continuous transformation of $\mathcal{A}^{\mathbb{Z}^d}$ is a cellular automaton if and only if it commutes with every shift map on $\mathcal{A}^{\mathbb{Z}^d}$.

Recently, see [7], Hedlund's theorem (also called Curtis-Hedlund-Lyndon Theorem) have been extended to continuous transformation of $\mathcal{A}^{\mathbb{Z}^d}$ with $\mathcal{A}$ any discrete topological space. This extension maintains the topological structure on $\mathcal{A}^{\mathbb{Z}^d}$ and uses the concept of barriers of the Set Theory to generalize the notion of local rule in the definition of cellular automata.

In this work, instead of considering CA, we deal with discrete dynamical systems on $\mathcal{A}^{\mathbb{Z}^d}$ where the global transition map depends on a finite number of local rules acting on possibly different neighborhoods. This kind of dynamical systems are called *place-dependent cellular automata*; this notion extends the classical definition of CA, see [1]. The main goal here is to show a version of Theorem 3.4 in [6] for place-dependent cellular automata; in addition, we also prove the corresponding extension when the alphabet $\mathcal{A}$ is any discrete topological space.

This paper is organized as follows. In section 2 we explicitly introduce the concept of place-dependent cellular automata and show the version of Hedlund's theorem for this kind of transformation on $\mathcal{A}^{\mathbb{Z}^d}$ with $\mathcal{A}$ finite; the last section is devoted to extend, using the notion of barriers provided by Set Theory, the concept of place-dependent cellular automata when $\mathcal{A}$ is any discrete topological space; next we prove the corresponding version of the extension of Hedlund's theorem in this case.

2 Place-dependent cellular automata and Hedlund's Theorem. Case $\mathcal{A}$ finite.

Consider a finite alphabet $\mathcal{A} = \{0, \cdots, N-1\}$ and a positive integer n. Let $\mathbb{V}_i$ be a finite and nonempty subset of $\mathbb{Z}^d$ and $f_i : \mathcal{A}^{\mathbb{V}_i} \to \mathcal{A}$ the local rule acting on $\mathbb{V}_i$ ($i = 1, \cdots, n$). By means of these local rules we will define global transition maps on $\mathcal{A}^{\mathbb{Z}^d}$; in order to do it we need a new set of d-dimensional indexes. That set is related with the number of parallelepipeds with d-volume equal to n and sides of integer length. Observe that if $d = 2$, that number is $\Phi_2(n)$: the number of divisors of n. When $d = 3$ the number of parallelepipeds of 3-volume equal to n and sides with integer length is given by $\Phi_3(n) = \sum_{\ell|n} \Phi_2(\ell)$, where $\ell|n$ means that the integer ℓ is a divisor of n. In general, and employing a recursive argument, for any $d \geq 3$, the number of d-parallelepipeds with sides of integer length whose product is equal to n is given by:

$$\Phi_d(n) = \sum_{\ell|n} \Phi_{d-1}(\ell). \tag{2}$$

In this way, (2) describes the number of forms in which a d-dimensional index can be arrange according to the local rules $f_i : \mathcal{A}^{\mathbb{V}_i} \to \mathcal{A}$ ($i = 1, \cdots, n$).

Take positive integers $n_1, \cdots, n_d$ such that $n = \prod_{i=1}^{d} n_i$. Observe that this decomposition may not be of prime factors. Now we use the set of d-dimensional indexes

$$I(n_1, \cdots, n_d) = \{(r_1, \cdots, r_d) \in \mathbb{Z}^d : 0 \leq r_j \leq n_j - 1,\ 1 \leq j \leq d\};$$

to arrange (in some way) the local rules; clearly the cardinal of $I(n_1, \cdots, n_d)$ is equal to n.

Definition 2.1. Given local rules $f_r : \mathcal{A}^{\mathbb{V}_r} \to \mathcal{A}$ with $r \in I(n_1, \cdots, n_d)$, the $(n_1, \cdots, n_d)$-*cellular automaton* generated by them, also called $(n_1, \cdots, n_d)$-*place dependent cellular automaton*, is the transformation $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ defined, for each $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$, by

$$F(x)(m) = f_r\left(x\big|_{\mathbb{V}_r + \sum_{j=1}^{d} q_j n_j e_j} \right), \tag{3}$$

whenever $m_j = q_j n_j + r_j$ with $q_j \in \mathbb{Z}$, $j = 1, \cdots, d$, and $r = (r_1, \cdots, r_d)$.

Observe that for any $(n_1, \cdots, n_d)$-cellular automaton the temporal evolution of each cell $x(m)$ of the configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$ depends on a particular local rule which is given by the location of m in $\mathbb{Z}^d$. Clearly every CA on $\mathcal{A}^{\mathbb{Z}^d}$ is a $(1, \cdots, 1)$-cellular automaton.

Example 2.1. Consider the alphabet $\mathcal{A} = \{0, 1\}$ and local rules:

$$f_0 : \mathcal{A}^{\mathbb{V}_0} \to \mathcal{A},\ f_1 : \mathcal{A}^{\mathbb{V}_1} \to \mathcal{A} \text{ with } \mathbb{V}_0 = \{-1, 0, 1\} \text{ and } \mathbb{V}_1 = \{0, 1, 2\},$$

given, for all $h \in \mathcal{A}^{\mathbb{V}_0}$ and $g \in \mathcal{A}^{\mathbb{V}_1}$, by:

$$f_0(h) = h(-1) + h(1) \pmod 2 \text{ and } f_1(g) = g(2).$$

Then the place-dependent cellular automaton defined by these local rules is given, for every $x \in \{0, 1\}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$, by:

$$F(x)(k) = \begin{cases} x(2n-1) + x(2n+1) \pmod 2 & \text{if } k = 2n \\ x(2n+2), \text{ if } k = 2n+1 \end{cases}.$$

Observe that if $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is the shift map, then

$$(\sigma F)(x)(k) = F(x)(k+1) = \begin{cases} x(2n+2), \text{ if } k = 2n \\ x(2n+1) + x(2n+3) \pmod 2, \text{ if } k = 2n+1 \end{cases}$$

and

$$(F\sigma)(x)(k) = \begin{cases} x(2n) + x(2n+2) \pmod 2, \text{ if } k = 2n \\ x(2n+3), \text{ if } k = 2n+1 \end{cases}.$$

Notice that in general $(\sigma F)(x) \neq (F\sigma)(x)$, so from the Hedlund's theorem (cf. Theorem 3.4, [6]) it follows that F is not a cellular automaton.

We show now, see figure below, the temporal evolution of a particular configuration. Let $x = \{x(n)\}_{n\in\mathbb{Z}}$ be the configuration in $\{0, 1\}^{\mathbb{Z}}$ given by: $x(n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$ and $x(0) = 1$. It is easy to see that for each $n \geq 1$ and $k \geq 1$, $F^k(x)(n) = 0$. So the figure below shows part of this evolution for non positive cells of that configuration. In the grill of the figure we arrange the temporal evolution of x and $F^k(x)$ for $k = 0, 1, \cdots, 15$, in the following way: each square of the grill is colored: white or black; white means that the state of this cell is 0, otherwise black; on the k-th row on the grill ($0 \leq k \leq 15$) each square represents the cell of $F^k(x)$. Thus, the state of the cell $F^k(x)(\ell)$ ($-15 \leq \ell \leq 0$) is located in the $|\ell|$-th square (from the left to the right) on the k-th row. Observe that by means of this process the positive orbit between times 0 and 15 describes a geometric shape known as a Sierpinski triangle.

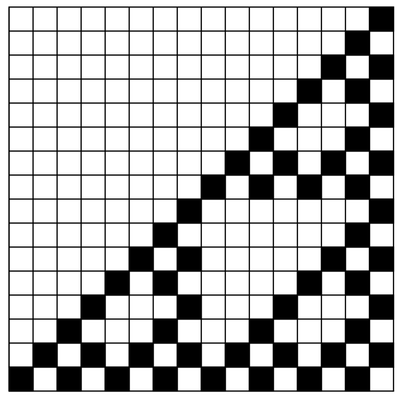

Figure 1: Sierpinski triangle obtained from the partial temporal evolution of x by the place-dependent cellular automaton of example 2.1.

Proposition 2.1. *If $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is the $(n_1, \cdots, n_d)$-cellular automaton generated by the local rules $f_r : \mathcal{A}^{\mathbb{V}_r} \to \mathcal{A}$ with $r \in I(n_1, \cdots, n_d)$, then*

(i) F is continuous; and

(ii) $F\sigma_j^{n_j} = \sigma_j^{n_j} F$ for every $1 \leq j \leq d$.

Proof. Clearly any local rule is a continuous map. Take $\epsilon > 0$ and $\ell \geq 1$ such that $2^{-\ell} < \epsilon$. Let m be the positive integer such that, for every $n \in \mathbb{Z}^d$ with $\|n\| \leq \ell$ and $r \in I(n_1, \cdots, n_d)$, $n + \mathbb{V}_r \subset \{n \in \mathbb{Z}^d : \|n\| \leq m\}$. Thus, for every $x, y \in \mathcal{A}^{\mathbb{Z}^d}$ with $D(x, y) < 2^{-m}$ it follows that $x\big|_{\mathbb{V}_r+n} = y\big|_{\mathbb{V}_r+n}$ for all $\|n\| \leq \ell$. Since $F(x)(n) = F(y)(n)$ for any $\|n\| \leq \ell$, then $D(F(x), F(y)) < 2^{-\ell} < \epsilon$; this proves item (i).

On the other hand, take $x = \{x(m)\}_{m\in\mathbb{Z}^d} \in \mathcal{A}^{\mathbb{Z}^d}$. If $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$ satisfies $m_j = q_j n_j + r_j$ with $q_j \in \mathbb{Z}$ and $0 \leq r_j < n_j$ for all $j = 1, \cdots, d$, then for any $\ell \in \mathbb{V}_r$, with $r = (r_1, \cdots, r_d)$, it holds:

$$
\begin{aligned}
\sigma_j^{n_j}(x)\big|_{\mathbb{V}_r+\sum_{i=1}^d q_i n_i e_i}(\ell) &= \sigma_j^{n_j}(x)(\ell + \sum_{i=1}^d q_i n_i e_i) \\
&= x\left(\ell + \sum_{i\neq j} q_i n_i e_i + (q_j+1) n_j e_j\right) \\
&= x\big|_{\mathbb{V}_r+\sum_{i\neq j} q_i n_i e_i + (q_j+1) n_j e_j}(\ell),
\end{aligned}
$$

for every $j = 1, \cdots, d$. Clearly this implies (ii). □

An important property of any transformation F of $\mathcal{A}^{\mathbb{Z}^d}$ commuting with power of the shift maps is the following. Suppose that positive integers $n_1, \cdots, n_d$ are given and $F\sigma_j^{n_j} = \sigma_j^{n_j} F$ for $1 \le j \le d$. Now consider the partial functions of F; that is, for every $m \in \mathbb{Z}^d$ define $F_m : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}$ by $F_m(x) = F(x)(m)$, for all $x \in \mathcal{A}^{\mathbb{Z}^d}$. Thus one can write $F = \{F_m\}_{m \in \mathbb{Z}^d}$. From the commuting property of F it is easy to verify that for every $m \in \mathbb{Z}^d$ it holds $F_m \sigma_j^{n_j} = F_{m+n_j e_j}$. Using this relationship between the partial functions of F and $\sigma_1^{n_1}, \cdots, \sigma_d^{n_d}$, it follows that

$$F_m = F_r \sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d}, \tag{4}$$

for all $m = (m_1, \cdots, m_d)$ where $m_j = q_j n_j + r_j$ with $q_j \in \mathbb{Z}$, $0 \le r_j < n_j$, $1 \le j \le d$ and $r = (r_1, \cdots, r_d)$. Therefore, we only need the partial functions F_r, with $r \in I(r_1, \cdots, r_d)$, to express any other partial function of F.

Theorem 2.1 (Hedlund's Theorem Extension)**.** *Every continuous transformation $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ commuting with $\sigma_j^{n_j}$ for some integer $n_j \ge 1$ and each $1 \le j \le d$, is a $(n_1, \cdots, n_d)$-cellular automaton.*

Proof. Given $r \in I(n_1, \cdots, n_d)$, let F_r be the r-th partial function of F. Consider $\ell = \max\{n_1, \cdots, n_d\}$. Since F is uniformly continuous ($\mathcal{A}^{\mathbb{Z}^d}$ is compact), there exists $k \ge 1$ such that $D(F(x), F(y)) < 2^{-\ell}$ whenever $D(x, y) < 2^{-k}$. In particular, this implies that for all $r \in I(n_1, \cdots, n_d)$, $F_r(x) = F_r(y)$ for all $x, y \in \mathcal{A}^{\mathbb{Z}^d}$ with $x(n) = y(n)$ for every $\|n\| \le k$. In other words, if $\mathbb{V} = \{n \in \mathbb{Z}^d : \|n\| \le k\}$, then for every $h : \mathbb{V} \to \mathcal{A}$, the partial function F_r is constant on the cylinder $C(\mathbb{V}, h)$ for all $r \in I(n_1, \cdots, n_d)$.

Take $r \in I(n_1, \cdots, n_d)$ and define $f_r : \mathcal{A}^{\mathbb{V}} \to \mathcal{A}$ as $f_r(h) = F_r(x)$ for every $x \in C(\mathbb{V}, h)$. Observe that f_r is well defined. On the other hand, from (4) it follows that for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $m = (q_1 n_1, \cdots, q_d n_d) + r$ with $q_j \in \mathbb{Z}$ ($j = 1, \cdots, d$):

$$F(x)(m) = F_m(x) = F_r \sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d}(x).$$

But the value of $\sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d}(x)$ on $\ell \in \mathbb{V}$ is just $x(\ell + \sum_{j=1}^d q_j n_j e_j)$; this implies that $F(x)(m) = f_r\left(x|_{\mathbb{V}_r + \sum_{j=1}^d q_j n_j e_j}\right)$ and hence F is a $(n_1, \cdots, n_d)$-cellular automaton. □

Remark 2.1. Consider positive integers $n_1, \cdots, n_d$ and local rules $f_r : \mathcal{A}^{\mathbb{V}_r} \to \mathcal{A}$ with $r \in I(n_1, \cdots, n_d)$. Let $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ be the transformation given by:

$$F(x)(m) = f_r\left(x\big|_{\mathbb{V}_r+m}\right), \tag{5}$$

for all $x \in \mathcal{A}^{\mathbb{Z}^d}$, $m = (q_1n_1, \cdots, q_dn_d) + r$ with $q_j \in \mathbb{Z}$ for all $j = 1, \cdots, d$, and $r \in I(n_1, \cdots, n_d)$. It is easy to see that $F\sigma_j^{n_j} = \sigma_j^{n_j}F$ for all $j = 1, \cdots, d$. So, theorem 2.1 it follows that F is a $(n_1, \cdots, n_d)$-cellular automaton; that is, for each $r \in I(n_1, \cdots, n_d)$ there exists a local rule $g_r : \mathcal{A}^{\mathbb{U}_r} \to \mathcal{A}$ such that, for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $m = \sum_{j=1}^d q_jn_je_j + r$, it holds

$$F(x)(m) = g_r\left(x\big|_{\mathbb{U}_r+\sum_{j=1}^d q_jn_je_j}\right). \tag{6}$$

On the other hand, take the $(n_1, \cdots, n_d)$-cellular automaton given by (6). For each $r \in I(n_1, \cdots, n_d)$ define $\mathbb{V}_r = \mathbb{U}_r - r = \{u - r : u \in \mathbb{U}_r\}$ and the local rule $f_r : \mathcal{A}^{\mathbb{V}_r} \to \mathcal{A}$ by $f_r(h) = g_r(\widetilde{h})$, where $\widetilde{h} : \mathbb{U}_r \to \mathcal{A}$ is defined as $\widetilde{h}(u) = h(u - r)$ for all $u \in \mathbb{U}_r$. Since $x\big|_{\mathbb{U}_r+\sum_{j=1}^d q_jn_je_j} = x\big|_{\mathbb{V}_r+m}$ for all $m = \sum_{j=1}^d q_jn_je_j + r$, then F is also expressed as in (5).

3 Place-dependent cellular automata and Hedlund's Theorem. Case $\mathcal{A}$ discrete.

Let $\mathcal{A}$ be any discrete topological space; consider on $\mathcal{A}^{\mathbb{Z}^d}$ the product topology. With the same tools of [7] we will extend the concept of place-dependent cellular automata; that is, we will use the notion of barriers to define: generalized local rules and generalized place-dependent extended cellular automata. The barriers were used in [4] to obtain a canonical form for continuous functions $\phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ commuting with the shift $S(A) = A \setminus \{\min A\}$, where $[\mathbb{N}]^\infty$ denotes the set of all infinite subsets of $\mathbb{N}$.

Now we recall the notion of barrier and extend the concept of local rule. Next we will provide an example of a transformation $F : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ commuting with a nontrivial power of the shift such that F cannot be expressed in terms of a finite number of local rules, as defined in the previous section; see remark above.

Let $\mathcal{F}(\mathbb{Z}^d, \mathcal{A})$ be the family of all functions from $\mathbb{V}$ to $\mathcal{A}$, where $\mathbb{V}$ is any finite and nonempty subset of $\mathbb{Z}^d$. This family is partially ordered by the relation:

$$f \sqsubset g \text{ if and only if } dom(f) \subset dom(g) \text{ and } g\big|_{dom(f)} = f, \tag{7}$$

where $dom(f)$ denotes the domain of f and $g\big|_{dom(f)}$ is the restriction of g to $dom(f)$.

Definition 3.1. An antichain of $\mathcal{F}(\mathbb{Z}^d, \mathcal{A})$ is any collection in $\mathcal{F}(\mathbb{Z}^d, \mathcal{A})$ of non-comparable elements respect to $\sqsubset$. A collection $\mathcal{B} \subset \mathcal{F}(\mathbb{Z}^d, \mathcal{A})$ is a barrier in $\mathcal{A}^{\mathbb{Z}^d}$ if $\mathcal{B}$ is an antichain of $\mathcal{F}(\mathbb{Z}^d, \mathcal{A})$, and for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ there exists a unique $f \in \mathcal{B}$ such that $x\big|_{dom(f)} = f$. Denote this function by f_x.

Since any function $f \in \mathcal{F}(\mathbb{Z}^d, \mathcal{A})$ is identified with a unique cylinder $C(\mathbb{V}, f)$, the concept of barriers can be expressed in terms of cylinders. Observe that if $\mathbb{V}$ is a finite and nonempty subset of $\mathbb{Z}^d$, then $\mathcal{A}^{\mathbb{V}}$ is a barrier. This kind of barriers is called uniform.

Definition 3.2. A *generalized local rule* of $\mathcal{A}^{\mathbb{Z}^d}$ is any function $\phi : \mathcal{B} \to \mathcal{A}$, where $\mathcal{B}$ is a barrier of $\mathcal{F}(\mathbb{Z}^d, \mathcal{A})$.

Obviously any local rule of the previous section is a generalized local rule, its domain is a uniform barrier.

Example 3.1. Let $\mathcal{A} = \mathbb{N}$ endowed with the discrete topology. Consider, for each $a \in \mathbb{N} \setminus \{0\}$ and $j \in \{1, 2, \cdots, a\}$, the set

$$\mathcal{B}_j^a = \left\{ f \in \mathbb{N}^{[-j,j]} : f(0) = j \text{ and } \sum_{|m| \leq j} f(m) = a \right\},$$

where $[-j, j] = \{-j, \cdots, j\}$. Let $\mathcal{B}_0 = \bigcup_{\substack{a \in \mathbb{N} \setminus \{0\} \\ 1 \leq j \leq a}} \mathcal{B}_j^a \cup \{f_0\}$, and $f_0 : \{0\} \to \mathbb{N}$ given by $f_0(0) = 0$. For every pair of different functions $f : [-j, j] \to \mathbb{N}$ and $g : [-\ell, \ell] \to \mathbb{N}$ in $\mathcal{B}_0$, the corresponding cylinders $C([-j, j], f)$ and $C([-\ell, \ell], g)$ are disjoint, this implies that $\mathcal{B}_0$ is an antichain. Given a configuration $x \in \mathbb{N}^{\mathbb{Z}}$ with $x(0) = j$, it follows that $f_x : [-j, j] \to \mathbb{N}$, with $f_x(m) = x(m)$ for every $m \in [-j, j]$, is the only function in $\mathcal{B}_0$ satisfying $x|_{dom(f_x)} = f_x$; therefore $\mathcal{B}_0$ is a barrier of $\mathbb{N}^{\mathbb{Z}}$. In particular, the function $\phi_0 : \mathcal{B}_0 \to \mathbb{N}$ defined, for each $f \in \mathcal{B}_0$, as

$$\phi_0(f) = \sum_{|m| \leq f(0)} f(m)$$

is a generalized local rule. Now consider the barrier $\mathcal{B}_1 = \mathbb{N}^{\mathbb{V}_1}$ with $\mathbb{V}_1 = \{1\}$, and the generalized local rule (actually a local rule) given by $\phi_1(g) = g(1)$ for all $g \in \mathcal{B}_1$.

Let $F = \{F_n\}_{n\in\mathbb{Z}}$ be the transformation of $\mathbb{N}^{\mathbb{Z}}$ defined as follow:

$$F(x)(n) = F_n(x) = \begin{cases} \displaystyle\sum_{|m|\leq x(2q)} x(2q+m), & \text{if } n = 2q \\ x(2q+1), & \text{if } n = 2q+1, \end{cases}.$$

It is easy to verify that F is continuous and $F\sigma^2 = \sigma^2 F$, where σ is the shift on $\mathbb{N}^{\mathbb{Z}}$. In fact, F can be expressed, for all $x \in \mathbb{N}^{\mathbb{Z}}$, by means of

$$F_n(x) = \begin{cases} F_0\sigma^{2q}(x), & \text{if } n = 2q \\ F_1\sigma^{2q}(x), & \text{if } n = 2q+1 \end{cases},$$

where $F_0(x) = \sum_{|j|\leq x(0)} x(j)$ and $F_1(x) = x(1)$. That is, $F(x)(n) = \phi_r(f^r_{x,2q})$, whenever $n = 2q+r$ with $r \in \{0,1\}$ and $f^r_{x,2q}$ is the function in $\mathcal{B}_r$ associated to the configuration $\sigma^{2q}(x)$; that is, $\sigma^{2q}(x)\big|_{dom(f^r_{x,2q})} = f^r_{x,2q}$.

Observe that the temporal evolution of the cells depends on different number of values of the cells in nonuniform neighborhoods; therefore F is not a place-dependent cellular automaton.

Definition 3.3. Given positive integers $n_1, \cdots, n_d$, F from $\mathcal{A}^{\mathbb{Z}^d}$ into itself is a $(n_1,\cdots,n_d)$-*generalized cellular automaton*, if for every $r \in I(n_1,\cdots,n_d)$ there exists a generalized local rules $\phi_r : \mathcal{B}_r \to \mathcal{A}$ such that for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ it holds:

$$F(x)(m) = \phi_r(f^r_{x,q_1,\cdots,q_d}), \tag{8}$$

whenever $m = \sum_{j=1}^d q_j\, n_j + r$ with $r \in I(n_1,\cdots,n_d)$, and

$$(\sigma_1^{q_1 n_1}\cdots\sigma_d^{q_d n_d})(x)\big|_{dom(f^r_{x,q_1,\cdots,q_d})} = f^r_{x,q_1,\cdots,q_d}. \tag{9}$$

Clearly the transformation of the preceding example is a 2-generalized cellular automaton on $\mathbb{N}^{\mathbb{Z}}$.

Theorem 3.1 (Extension of Hedlund's Theorem)**.** *Let $\mathcal{A}$ be a discrete topological space. Then*

(i) Every $(n_1,\cdots,n_d)$-generalized cellular automaton F on $\mathcal{A}^{\mathbb{Z}^d}$ is continuous and $F\sigma_j^{n_j} = \sigma_j^{n_j}F$ for every $j = 1,\cdots,d$.

(ii) Given positive integers $n_1,\cdots,n_d$, every continuous transformation F on $\mathcal{A}^{\mathbb{Z}^d}$ commuting with $\sigma_j^{n_j}$, $j = 1,\cdots,d$, is a $(n_1,\cdots,n_d)$-generalized cellular automaton.

Proof. (i) Let F be the $(n_1, \cdots, n_d)$-generalized cellular automaton on $\mathcal{A}^{\mathbb{Z}^d}$ given by the generalized local rules $\phi_r : \mathcal{B}_r \to \mathcal{A}$, with $r \in I(n_1, \cdots, n_d)$; see (8) and (9). Take x in $\mathcal{A}^{\mathbb{Z}^d}$ and a cylinder $C(\mathbb{U}, h)$ containing $F(x)$. Thus, for every $u \in C(\mathbb{U}, h)$ with $u = \sum_{j=1}^{d} u_j n_j e_j + r$ and $r \in I(n_1, \cdots, n_d)$, it follows that $h(u) = \phi_r(f^r_{x,u_1,\cdots,u_d})$. Let C_x be the set of all $y \in \mathcal{A}^{\mathbb{Z}^d}$ such that for each $u \in \mathbb{U}$ it holds

$$x\big|_{dom(f^r_{x,u_1,\cdots,u_d})+\sum_{j=1}^{d} u_j n_j e_j} = y\big|_{dom(f^r_{x,u_1,\cdots,u_d})+\sum_{j=1}^{d} u_j n_j e_j}.$$

Clearly C_x is a cylinder containing x. It is easy to verify that for every $y \in C_x$, $F(y) \in C(\mathbb{U}, h)$. Thus, F is continuous.

Now take $m = \sum_{i=1}^{d} q_i n_i e_i + r$ with $r \in I(n_1, \cdots, n_d)$, and $j \in \{1, \cdots, d\}$. As

$$m + n_j e_j = \sum_{i \neq j} q_i n_i e_i + (q_j + 1) n_j e_j + r, \text{ and}$$

$$\sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d}(\sigma_j^{n_j}(x)) = \sigma_1^{q_1 n_1} \cdots \sigma_{j-1}^{q_{j-1} n_{j-1}} \sigma_j^{(q_j+1) n_j} \sigma_{j+1}^{q_{j+1} n_{j+1}} \cdots \sigma_d^{q_d n_d}(x),$$

then from (8) and (9) it follows that

$$f^r_{\sigma_j^{n_j}(x), q_1, \cdots, q_d} = f^r_{x, q_1, \cdots, q_{j-1}, q_j+1, q_{j+1}, \cdots, q_d};$$

this clearly implies $F\sigma_j^{n_j} = \sigma_j^{n_j} F$.

(ii) Let $n_1, \cdots, n_d$ be positive integers and $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ a continuous transformation commuting with $\sigma_j^{n_j}$ for each $1 \leq j \leq d$. For every $r \in I(n_1, \cdots, n_d)$ let F_r be the function from $\mathcal{A}^{\mathbb{Z}^d}$ into $\mathcal{A}$ such that for any $m = (q_1 n_1, \cdots, q_d n_d) + r$ in $\mathbb{Z}^d$ it holds $F_m = F_r \sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d}$. Since F is continuous, for each $a \in \mathcal{A}$ and every $r \in I(n_1, \cdots, n_d)$, $F_r^{-1}(\{a\})$ is a disjoint union of cylinders of $\mathcal{A}^{\mathbb{Z}^d}$. Denote by $\mathcal{B}_r^a$ the collection of functions $f : \mathbb{U} \to \mathcal{A}$ ($\mathbb{U}$ a finite and nonempty subset of $\mathbb{Z}^d$) such that $C(\mathbb{U}, f)$ is a cylinder of the disjoint union determined by $F_r^{-1}(\{a\})$. Clearly $F_r(C(\mathbb{U}, f)) = \{a\}$ for all $f \in \mathcal{B}_r^a$; moreover, if $f, g \in \mathcal{B}_r^a$ with $f \neq g$, then f and g are not comparable in the partial orden $\sqsubset$ defined by (7), and for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ with $F_r(x) = a$, there exists a unique function $f_x \in \mathcal{B}_r^a$ such that $x\big|_{dom(f_x)} = f_x$. In this way, for each $r \in I(n_1, \cdots, n_d)$, $\mathcal{B}_r = \bigcup_{a \in \mathcal{A}} \mathcal{B}_r^a$ defines a barrier of $\mathcal{A}^{\mathbb{Z}^d}$. Now, for each $r \in I(n_1, \cdots, n_d)$ consider the function $\phi_r : \mathcal{B}_r \to \mathcal{A}$ defined by $\phi_r(f) = a$ whenever f belongs to $\mathcal{B}_r^a$.

Finally, take $x \in A^{\mathbb{Z}^d}$ and $m = (q_1 n_1, \cdots, q_d n_d) + r$ with $r \in I(n_1, \cdots, n_d)$. Making $a = F(x)(m) = F_m(x) = F_r(\sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d})(x)$, we consider the unique $f^r_{x,q_1,\cdots,q_n} \in \mathcal{B}^a_r$ such that

$$\sigma_1^{q_1 n_1} \cdots \sigma_d^{q_d n_d}(x)\big|_{dom(f^r_{x,q_1,\cdots,q_n})} = f^r_{x,q_1,\cdots,q_n},$$

then it follows that $F(x)(m) = \phi_r(f^r_{x,q_1,\cdots,q_n})$. The proof is completed. □

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