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On extension of scalar valued positive definite functions on ordered groups

Extensión de funciones definidas positivas a valores escalares en grupos ordenados

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Abstract

We prove that every scalar valued positive definite function f, on a generalized interval of an ordered group, has a positive definite extension to the whole group. We also prove that if f is continuous (measurable), then every positive definite extension of f is continuous (measurable). Additionally we obtain a representation result for the measurable case.
Key words and phrases: locally compact abelian group, ordered group, positive definite function, extension.

Resumen

Probamos que toda función f, a valores escalares y definida positiva en un intervalo generalizado de un grupo ordenado, tiene una extensión definida positiva a todo el grupo. También probamos que si f es continua (medible), entonces toda extensión definida positiva de f es continua (medible). Además obtenemos un resultado de representación para el caso medible.
Palabras y frases clave: grupo abeliano localmente compacto, grupo ordenado, función definida positiva, extensión.

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1 Introduction.

Let a be a real number such that $0 < a \leq +\infty$ and let $I = (-a, a)$. A function $f : I \to \mathbb{C}$ is said to be *positive definite* if for any positive integer n and any $x_1, \ldots, x_n$ in $\mathbb{R}$ such that $x_i - x_j \in I$ for $i, j = 1, \ldots, n$ and $c_1, \ldots, c_n$ in $\mathbb{C}$ we have that

$$\sum_{i,j=1}^{n} f(x_i - x_j)\, c_i\, \overline{c_j} \geq 0.$$

M. G. Kreĭn [6] proved that every continuous positive definite function on $I = (-a, a)$ can be extended to a continuous positive definite function on the whole line.

The concept of positive definite function can be extended in a natural way to abelian groups: Let $(\Omega, +)$ be an abelian group, let Δ be a subset of Ω. A function $f : \Delta \to \mathbb{C}$ is said to be *positive definite* if for any positive integer n and any $\omega_1, \ldots, \omega_n$ in Ω such that $\omega_i - \omega_j \in \Delta$ for $i, j = 1, \ldots, n$ and $c_1, \ldots, c_n$ in $\mathbb{C}$ we have that

$$\sum_{i,j=1}^{n} f(\omega_i - \omega_j)\, c_i\, \overline{c_j} \geq 0.$$

Therefore it is natural to ask the following question: Does every continuous positive definite function, on a symmetric neighborhood of the neutral element of a locally compact abelian group, has a positive definite extension to the whole group?

The answer to this question is, in general, negative. A simple counterexample on the unit circle was given by A. Devinatz in [4], in this paper it is shown that, under certain conditions, a continuous positive definite function on a rectangle has a continuous positive definite extension to the whole plane. W. Rudin [9] showed that there exist continuous positive definite functions on a rectangle which do not have positive definite extension to the whole plane.

In his book [10, Exercise 4.2.9], Z. Sasvári shows that a positive definite function on an interval of an ordered group can be extended to a positive definite function on the whole group. In this paper we will extend this result to a more general kind of sets, that we call generalized intervals and no continuity assumption is needed for this result. Moreover we show that if the starting function is continuous (measurable) then every positive definite extension is continuous (measurable). Also an extension of the decomposition result given first by F. Riesz in [7] and refined by M. Crum in [3] is given.

For the basic facts about harmonic analysis on locally compact groups we refer the reader to the book of W. Rudin [8].

2 Preliminaries.

Let $(\Gamma, +)$ be an abelian group with neutral element 0_Γ. Γ is an *ordered group* if there exists a set $\Gamma_+ \subset \Gamma$ such that:

$$\Gamma_+ + \Gamma_+ = \Gamma_+, \qquad \Gamma_+ \bigcap (-\Gamma_+) = \{0_\Gamma\}, \qquad \Gamma_+ \bigcup (-\Gamma_+) = \Gamma.$$

In this case if $x, y \in \Gamma$ we write $x \leq y$ if $\mathrm{y - x} \in \Gamma_+$, we also write $x < y$ if $x \leq y$ and $x \neq y$, so $\Gamma_+ = \{\gamma \in \Gamma : \gamma \geq 0_\Gamma\}$. If there is not possibility of confusion, we will use 0 instead of 0_Γ. When Γ is a topological group it is supposed that Γ_+ is closed.

If $a, b \in \Gamma$ and $a < b$,

$$(a, b) = \{x \in \Gamma : a < x < b\}, \qquad [a, b] = \{x \in \Gamma : a \leq x \leq b\}, \qquad \text{etc.}$$

Definition 2.1. Let Γ be an ordered group, a nonempty set J contained in Γ is a *generalized interval* if J has the following property: $a, b \in J$, $a < b$ implies $(a, b) \subset J$.

We say that J is *non trivial* if J has more than one point.

Remark 2.2. It is clear that every interval is a generalized interval, but there are generalized intervals which are not intervals.

For example consider $\mathbb{Z}^2$ with the lexicographic order:

$$(m_1, n_1) < (m_2, n_2) \text{ if and only if } n_1 < n_2 \text{ or } n_1 = n_2 \text{ and } m_1 < m_2.$$

If N is a positive integer then the set

$$\{(m, n) \in \mathbb{Z}^2 : \ |n| \leq N \}$$

is a generalized interval, but it is not an interval.

3 Main result.

Theorem 3.1. *Let $(\Gamma, +)$ be an abelian ordered group and let $\Delta \subset \Gamma$ be a non trivial symmetric generalized interval.*

Let $f : \Delta \to \mathbb{C}$ be a positive definite function. Then

(a) f has a positive definite extension to the whole group Γ.

(b) If Γ is a topological group and f is continuous then any positive definite extension of f is continuous.

(c) If Γ is a locally compact group and f is measurable then any positive definite extension of f is measurable.

(d) If Γ is a locally compact group and f is measurable then there exist two positive definite functions $f^c : \Delta \to \mathbb{C}$ and $f^0 : \Delta \to \mathbb{C}$ such that

(i) $f = f^c + f^0$.

(ii) f^c is continuous.

(iii) f^0 is zero locally almost everywhere.

We need some auxiliary results before starting the proof of the theorem.

Let N be a natural number, according to the terminology of [2] a set Q contained in $\{(k,l) \in \{1,\dots,N\} \times \{1,\dots,N\} : k \leq l\}$ is called a quasitriangle if

$$l_k = \max\{l : k \leq l \leq N, (k,l) \in Q\} \geq k$$

for each $1 \leq k \leq N$ and for every (k',l') with $k \leq k' \leq l' \leq l_k$, $(k',l') \in Q$.

In the following Γ, Δ and f are as in Theorem 3.1.

Proposition 3.2. *Let $\gamma_1, \dots, \gamma_n \in \Gamma$ be such that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$. Then there exists a positive matrix*

$$A = (A_{kl})_{k,l=1}^{n} \in \mathbb{C}^{n \times n}$$

such that

$$A_{kl} = f(\gamma_l - \gamma_k) \quad \text{if} \quad \gamma_l - \gamma_k \in \Delta.$$

Proof. The proof of this proposition is based on some results given by Gr. Arsene, Zoia Ceausescu and T. Constantinescu in [2, Section 3]. Let

$$E = \{(k,l) : \ 1 \leq k \leq l \leq n \ \text{and} \ \gamma_l - \gamma_k \in \Delta\}.$$

We will show that E is a quasitriangle.

Since $0 \in \Delta$, we have that $(k,k) \in E$ for $1 \leq k \leq n$, so $l_k \geq k$.

Suppose that $k \leq k' \leq l' \leq l_k$, where $1 \leq k \leq n$, then

$$0 \leq \gamma_{l'} - \gamma_{k'} \leq \gamma_{l_k} - \gamma_k,$$

since $\gamma_{l_k} - \gamma_k \in \Delta$ and Δ is a generalized interval containing 0, we have that $\gamma_{l'} - \gamma_{k'} \in \Delta$, so $(k',l') \in E$.

Since f is positive definite, every block matrix $(f(\gamma_{l'} - \gamma_{k'}))_{k\leq k', l'\leq l_k}$ is positive, so the result follows from [2, Corollary 3.2].

□

Lemma 3.3. *Let $g : \Gamma \to \mathbb{C}$ be a positive definite function with finite support contained in Δ, then*

$$\sum_{\gamma\in\Gamma} f(\gamma)\, g(\gamma) \geq 0.$$

Proof. Consider the function $\Psi : \Gamma \to \mathbb{C}$ defined by

$$\Psi(\gamma) = \begin{cases} f(\gamma)\, g(\gamma) & \text{if } \gamma \in \Delta, \\ 0 & \text{if } \gamma \notin \Delta. \end{cases}$$

First will prove that Ψ is a positive definite function, so we have to show that for any choice of $\gamma_1, \ldots, \gamma_n \in \Gamma$, the matrix

$$(\Psi(\gamma_l - \gamma_k))_{k,l=1}^n$$

is positive.

Without loss of generality we can suppose that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$.

From Proposition 3.2 it follows that there exists a positive matrix

$$A = (A_{kl})_{k,l=1}^n \in \mathbb{C}^{n\times n}$$

such that

$$A_{kl} = f(\gamma_l - \gamma_k) \quad \textit{if} \quad \gamma_l - \gamma_k \in \Delta.$$

The Schur product $(A_{kl}\, g(\gamma_l - \gamma_k))_{k,l=1}^n$ of the matrix A and the matrix $(g(\gamma_l - \gamma_k))_{k,l=1}^n$ is positive. Since support of g is contained in Δ, we have that this product is equal to $(\Psi(\gamma_l - \gamma_k))_{k,l=1}^n$, so Ψ is positive definite.

Now consider the discrete topology on Γ, then the Haar measure is the counting measure. Let $\mathbf{1}$ be the neutral element of the dual of Γ, since Ψ is positive definite, we obtain

$$\sum_{\gamma\in\Gamma} f(\gamma)\, g(\gamma) = \widehat{\Psi}(\mathbf{1}) \geq 0.$$

□

Proof of Theorem 3.1.

(a) For this part consider Γ with the discrete topology, let G be the dual group of Γ.

Let $\mathfrak{F}(\Delta)$ be the set of the Fourier transforms $\hat{\nu}$ of the measures ν on Γ with finite support contained in Δ and let

$$\mathfrak{F}^{\,r}(\Delta) = \{p \in \mathfrak{F}(\Delta) :\ \mathrm{Rank}(p) \subset \mathbb{R}\},$$

$$\mathfrak{F}^{+}(\Delta) = \{p \in \mathfrak{F}^{\,r}(\Delta) : p \geq 0\}.$$

Let $L : \mathfrak{F}^{\,r}(\Delta) \to \mathbb{R}$ defined by

$$L(p) = \sum_{\gamma \in \Gamma} f(\gamma)\, \check{p}(\gamma).$$

We have that L is a real linear functional and, from Lemma 3.3 it follows that L is a positive functional. Then $L(p) \leq L(q)$ if $p, q \in \mathfrak{F}^{\,r}(\Delta)$ and $p \leq q$.

Let $\|\,\|$ be the uniform norm in the real linear space $\mathfrak{F}^{\,r}(\Delta)$. Without loss of generality we can suppose that $f(0) = 1$. If $p \in \mathfrak{F}^{\,r}(\Delta)$ then $-\|p\| \leq p \leq \|p\|$ and hence

$$|L(p)| \leq \|p\|\, L(1) = \|p\|.$$

Therefore L is a linear functional of norm 1 on $\mathfrak{F}^{\,r}(\Delta)$. From the Hahn-Banach theorem it follows that L can be extended to a linear functional of norm 1 on the space of the real valued continuos functions on G. From the Riesz representation theorem we have that there exists a finite measure μ on G of total variation 1 such that

$$L(p) = \int_G p(\varphi)\, d\mu(\varphi), \tag{1}$$

for $p \in \mathfrak{F}^{\,r}(\Delta)$.

Let $\gamma \in \Delta$. If we take $p(\varphi) = \varphi(\gamma) + \varphi(-\gamma)$ in equation (1) we obtain

$$f(\gamma) + f(-\gamma) = \check{\mu}(\gamma) + \check{\mu}(-\gamma),$$

and if we take $p(\varphi) = i(\varphi(\gamma) - \varphi(-\gamma))$ in equation (1) we obtain

$$f(\gamma) - f(-\gamma) = \check{\mu}(\gamma) - \check{\mu}(-\gamma),$$

therefore

$$f(\gamma) = \check{\mu}(\gamma) \quad \text{for all} \quad \gamma \in \Delta.$$

Since

$$1 = L(\mathbf{1}) = \mu(G) \leq \|\mu\| = 1,$$

we have that μ is a positive measure.

From the Herglotz-Bochner-Weil theorem ([8, Section 1.4.3]), we have that $F = \check{\mu}$ is a positive definite extension of f.

(b) Since Δ is a non trivial symmetric generalized interval it contains a set of the form $(-a, a)$, where $a > 0$. The set $(-a, a)$ is a neighborhood of 0, so the result follows from [10, Exercise 1.4.6].

(c) This part follows from [10, Theorem 3.4.4].

(d) This part follows from part (a) and the similar decomposition result for positive definite functions on the whole group given in [10, Theorem 3.1.2]. See also [3, 7] □

Corollary 3.4. *Let Λ and Γ be ordered groups and let $a \in \Gamma$, $a > 0$. If $f : \Lambda \times (-a, a) \to \mathbb{C}$ is a positive definite function, then f can be extended to a positive definite functions on the whole group $\Lambda \times \Gamma$.*

Proof. Consider the group $\Lambda \times \Gamma$ with the lexicographic order, then $\Delta = \Lambda \times (-a, a)$ is a generalized interval contained in this group, so the result follows from Theorem 3.1. □

Remark 3.5. From part (b), (c) and (d) of Theorem 3.1, it follows that similar results about continuity and measurability of the extension, also hold in this last result.

The particular case $\Gamma = \mathbb{Z}$ of this corollary was proved in [1]. Also the particular case $\Gamma = \mathbb{R}$, with the continuity assumption was proved in [5].

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