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On hypergeometric functions and Pochhammer k-symbol

Sobre funciones hipergeométricas y el k-símbolo de Pochhammer

Rafael Díaz
Departamento de Matemáticas. Universidad Central de Venezuela.
Caracas. Venezuela.

Eddy Pariguan
Departamento de Matemáticas. Universidad Central de Venezuela.
Caracas. Venezuela.

Abstract

We introduce the k-generalized gamma function Γ_k, beta function B_k and Pochhammer k-symbol $(x)_{n,k}$. We prove several identities generalizing those satisfied by the classical gamma function, beta function and Pochhammer symbol. We provide integral representation for the Γ_k and B_k functions.
Key words and phrases: hypergeometric functions, Pochhammer symbol, gamma function, beta function.

Resumen

Introducimos la función gamma k-generalizada Γ_k, la función beta B_k y el k-símbolo de Pochhammer $(x)_{n,k}$. Demostramos varias identidades que generalizan las que satisfacen las funciones gamma, beta y el símbolo de Pochhammer clásicos. Damos representaciones integrales para las funciones Γ_k y B_k.
Palabras y frases clave: funciones hipergeométricas, símbolo de Pochhammer, funcón gamma, funcón beta.

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1 Introduction

The main goal of this paper is to introduce the k-gamma function Γ_k which is a one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. Our motivation to introduce Γ_k comes from the repeated appearance of expressions of the form

$$x(x+k)(x+2k)\dots(x+(n-1)k) \tag{1}$$

in a variety of contexts, such as, the combinatorics of creation and annihilation operators [5], [6] and the perturbative computation of Feynman integrals, see [3]. The function of variable x given by formula (1) will be denoted by $(x)_{n,k}$, and will be called the Pochhammer k-symbol. Setting $k = 1$ one obtains the usual Pochhammer symbol $(x)_n$, also known as the raising factorial [9], [10]. It is in principle possible to study the Pochhammer k-symbol using the gamma function, just as it is done for the case $k = 1$, however one of the main purposes of this paper is to show that it is most natural to relate the Pochhammer k-symbol to the k-gamma function Γ_k to be introduce in section 2. Γ_k is given by the formula

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \qquad k > 0, \quad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

The function Γ_k restricted to $(0, \infty)$ is characterized by the following properties 1) $\Gamma_k(x+k) = x\Gamma_k(x)$, 2) $\Gamma_k(k) = 1$ and 3) $\Gamma_k(x)$ is logaritmically convex. Notice that the characterization above is indeed a generalization of the Bohr-Mollerup theorem [2]. Just as for the usual Γ the function Γ_k admits an infinite product expression given by

$$\frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}}\right). \tag{2}$$

For $\mathrm{Re}(x) > 0$, the function Γ_k is given by the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt.$$

We deduce from the steepest descent theorem a k-generalization of the famous Stirling's formula

$$\Gamma_k(x+1) = (2\pi)^{\frac{1}{2}} (kx)^{-\frac{1}{2}} x^{\frac{x+1}{k}} e^{-\frac{x}{k}} + O\left(\frac{1}{x}\right), \qquad \text{for} \quad x \in \mathbb{R}^+.$$

It is an interesting problem to understand how the function Γ_k changes as the parameter k varies. Theorem 11 on section 2 shows that the function $\psi(k,x) = \log \Gamma_k(x)$ is a solution of the non-linear partial differential equation

$$-kx^2\partial_x^2\psi + k^3\partial_k^2\psi + 2k^2\partial_k\psi = -x(k+1).$$

In the last section of this article we study hypergeometric functions from the point of view of the Pochhammer k-symbol. We k-generalize some well-known identities for hypergeometric functions such as: for any $a \in \mathbb{C}^p$, $k \in (\mathbb{R}^+)^p$, $s \in (\mathbb{R}^+)^q$, $b = (b_1, \ldots, b_q) \in \mathbb{C}^q$ such that $b_i \in \mathbb{C} \setminus s_i\mathbb{Z}^-$ the following identity holds

$$F(a,k,b,s)(x) = \prod_{j=1}^{p+1} \frac{1}{\Gamma_{k_j}(a_j)} \int_{(\mathbb{R}^+)^{p+1}} \prod_{j=1}^{p+1} e^{-\frac{t_j^{k_j}}{k_j}} t_j^{a_j-1} \left(\sum_{n=0}^{\infty} \frac{1}{(b)_{n,s}} \frac{(xt_1^{k_1} \ldots t_{p+1}^{k_{p+1}})^n}{n!} \right) dt, \tag{3}$$

where $(b)_{n,s} = (b_1)_{n,s_1} \ldots (b_q)_{n,s_q}$, $dt = dt_1 \ldots dt_{p+1}$, $p \leq q$, $\mathrm{Re}(a_j) > 0$ for all $1 \leq j \leq p+1$, and term-by-term integration is permitted. Our final result Theorem 25 provides combinatorial interpretation in terms of planar forest for the coefficients of hypergeometric functions.

2 Pochhammer k-symbol and k-gamma function

In this section we present the definition of the Pochhammer k-symbol and introduce the k-analogue of the gamma function. We provided representations for the Γ_k function in term of limits, integrals, recursive formulae, and infinite products, as well as a generalization of the Stirling's formula.

Definition 1. *Let $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}^+$, the Pochhammer k-symbol is given by*

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k).$$

Given $s, n \in \mathbb{N}$ with $0 \leq s \leq n$, the s-th elementary symmetric function $\sum_{1 \leq i_1 < \ldots < i_s \leq n} x_{i_1} \ldots x_{i_s}$ on variables $x_1, \ldots, x_n$ is denoted by $e_s^n(x_1, \ldots, x_n)$. Part (1) of the next proposition provides a formula for the Pochhammer k-symbol in terms of the elementary symmetric functions.

Proposition 2. *The following identities hold*

1. $(x)_{n,k} = \sum_{s=0}^{n-1} e_s^{n-1}(1,2,\ldots,n-1)k^s x^{n-s}$.

2. $\frac{\partial}{\partial k}(x)_{n,k} = \sum_{s=1}^{n-1} s(x)_{s,k}(x+(s+1)k)_{n-1-s,k}$.

Proof. Part (1) follows by induction on n, using the well-known identity for elementary symmetric functions

$$e_s^{n-1}(x_1,\dots,x_{n-1}) + ne_{s-1}^{n-1}(x_1,\dots,x_{n-1}) = e_s^n(x_1,\dots,x_n).$$

Part (2) follows using the logarithmic derivative. □

Definition 3. *For $k > 0$, the k-gamma function Γ_k is given by*

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

Proposition 4. *Given $x \in \mathbb{C} \setminus k\mathbb{Z}^-$, $k, s > 0$ and $n \in \mathbb{N}^+$, the following identity holds*

1. $(x)_{n,s} = \left(\frac{s}{k}\right)^n \left(\frac{kx}{s}\right)_{n,k}$.

2. $\Gamma_s(x) = \left(\frac{s}{k}\right)^{\frac{x}{s}-1} \Gamma_k\left(\frac{kx}{s}\right)$.

Proposition 5. *For $x \in \mathbb{C}$, $\mathrm{Re}(x) > 0$, we have $\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt$.*

Proof. By Definition 3

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt = \lim_{n\to\infty}\int_0^{(nk)^{\frac{1}{k}}} \left(1-\frac{t^k}{nk}\right)^n t^{x-1}dt.$$

Let $A_{n,i}(x)$, $i = 0,\dots,n$, be given by $A_{n,i}(x) = \int_0^{(nk)^{\frac{1}{k}}} \left(1-\frac{t^k}{nk}\right)^i t^{x-1}dt$.

The following recursive formula is proven using integration by parts

$$A_{n,i}(x) = \frac{i}{nx}A_{n,i-1}(x+k).$$

Also,

$$A_{n,0}(x) = \int_0^{(nk)^{\frac{1}{k}}} t^{x-1}dt = \frac{(nk)^{\frac{x}{k}}}{x}.$$

Therefore,

$$A_{n,n}(x) = \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}\left(1+\frac{x}{nk}\right)},$$

and

$$\Gamma_k(x) = \lim_{n\to\infty} A_{n,n}(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}.$$

□

Notice that the case $k = 2$ is of particular interest since

$$\Gamma_2(x) = \int_0^\infty t^{x-1}e^{-\frac{t^2}{2}}\,dt$$

is the Gaussian integral.

Proposition 6. *The k-gamma function $\Gamma_k(x)$ satisfies the following properties*

1. $\Gamma_k(x+k) = x\Gamma_k(x)$.
2. $(x)_{n,k} = \dfrac{\Gamma_k(x+nk)}{\Gamma_k(x)}$.
3. $\Gamma_k(k) = 1$.
4. $\Gamma_k(x)$ *is logarithmically convex, for $x \in \mathbb{R}$.*
5. $\Gamma_k(x) = a^{\frac{x}{k}} \displaystyle\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}a}\,dt,$ *for* $a \in \mathbb{R}$.
6. $\dfrac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}}e^{\frac{x}{k}\gamma}\displaystyle\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{nk}\right)e^{-\frac{x}{nk}}\right)$, *where* $\gamma = \displaystyle\lim_{n\to\infty}(1+\cdots+\frac{1}{n}-\log(n))$.
7. $\Gamma_k(x)\Gamma_k(k-x) = \dfrac{\pi}{\sin\left(\frac{\pi x}{k}\right)}$.

Proof. Properties 2), 3) and 5) follow directly from definition. Property 4) is Corollary 12 below. 1), 6) and 7) follows from $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$. □

Our next result is a generalization of the Bohr-Mollerup theorem.

Theorem 7. *Let $f(x)$ be a positive valued function defined on $(0,\infty)$. Assume that $f(k)=1$, $f(x+k)=xf(x)$ and f is logarithmically convex, then $f(x)=\Gamma_k(x)$, for all $x\in(0,\infty)$.*

Proof. Identity $f(x)=\Gamma_k(x)$ holds if and only if $\lim\limits_{n\to\infty}\frac{(x)_{n,k}f(x)}{n!k^n(nk)^{\frac{x}{k}-1}}=1$. Equivalently,

$$\lim_{n\to\infty}\log\left(\frac{(x)_{n,k}}{n!k^n(nk)^{\frac{x}{k}-1}}\right)+\log(f(x))=0.$$

Since f is logarithmically convex the following inequality holds

$$\frac{1}{k}\log\left(\frac{f(nk+k)}{f(nk)}\right)\leq\frac{1}{x}\log\left(\frac{f(nk+k+x)}{f(nk+k)}\right)\leq\frac{1}{k}\log\left(\frac{f(nk+2k)}{f(nk+k)}\right).$$

As $f(x+k)=xf(x)$, we have

$$\frac{x}{k}\log(nk)\leq\log\left(\frac{(x+nk)(x+(n-1)k)\dots xf(x)}{n!k^n}\right)\leq\frac{x}{k}\log((n+1)k)$$

$$\log(nk)^{\frac{x}{k}}\leq\log\left(\frac{(x+nk)(x+(n-1)k)\dots xf(x)}{n!k^n}\right)\leq\log((n+1)k)^{\frac{x}{k}}$$

$$0\leq\log\left(\frac{(x+nk)(x+(n-1)k)\dots xf(x)}{(nk)^{\frac{x}{k}}n!k^n}\right)\leq\log\left(\frac{(n+1)k}{nk}\right)^{\frac{x}{k}}$$

$$0\leq\lim_{n\to\infty}\log\left(\frac{(x+nk)(x+(n-1)k)\dots xf(x)}{(nk)^{\frac{x}{k}}n!k^n}\right)\leq\lim_{n\to\infty}\log\left(\frac{(n+1)k}{nk}\right)^{\frac{x}{k}}.$$

Since

$$\lim_{n\to\infty}\log\left(\frac{(n+1)k}{nk}\right)^{\frac{x}{k}}=\frac{x}{k}\log(1)=0,$$

we get

$$0\leq\lim_{n\to\infty}\log\left(\frac{(x+nk)(x+(n-1)k)\dots x}{(nk)^{\frac{x}{k}}n!k^n}\right)+\log(f(x))\leq 0.$$

Therefore, $f(x)=\Gamma_k(x)$. □

A proof of Theorem 8 below may be found in [7].

Theorem 8. *Assume that* $f : (a, b) \longrightarrow \mathbb{R}$*, with* $a, b \in [0, \infty)$ *attains a global minimum at a unique point* $c \in (a, b)$*, such that* $f''(c) > 0$*. Then one has*

$$\int_a^b g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} \sqrt{2\pi} \frac{g(c)}{\sqrt{f''(c)}} + O(\hbar).$$

As promised in the introduction, we now provide an analogue of the Stirling's formula for Γ_k.

Theorem 9. *For* $\mathrm{Re}(x) > 0$*, the following identity holds*

$$\Gamma_k(x+1) = (2\pi)^{\frac{1}{2}} (kx)^{-\frac{1}{2}} x^{\frac{x+1}{k}} e^{-\frac{x}{k}} + O\left(\frac{1}{x}\right). \tag{4}$$

Proof. Recall that $\Gamma_k(x+1) = \int_0^\infty t^x e^{-\frac{t^k}{k}} dt$. Consider the following change of variables $t = x^{\frac{1}{k}} v$, we get

$$\frac{\Gamma_k(x+1)}{x^{\frac{x+1}{k}}} = \int_0^\infty v^x e^{-\frac{(xv)^k}{k}} dv = \int_0^\infty e^{-x(\frac{v^k}{k} - \log v)} dv.$$

Let $f(s) = \frac{s^k}{k} - \log(s)$. Clearly $f'(s) = 0$ if and only if $s = 1$. Also $f''(1) = k$. Using Theorem 8, we have

$$\int_0^\infty v^x e^{-\frac{(xv)^k}{k}} dv = \frac{(2\pi)^{\frac{1}{2}}}{(kx)^{\frac{1}{2}}} e^{-\frac{x}{k}} + O\left(\frac{1}{x}\right),$$

thus

$$\Gamma_k(x+1) = \frac{(2\pi)^{\frac{1}{2}}}{(kx)^{\frac{1}{2}}} x^{\frac{x+1}{k}} e^{-\frac{x}{k}} + O\left(\frac{1}{x}\right).$$

□

Proposition 10 and Theorem 11 bellow provide information on the dependence of Γ_k on the parameter k.

Proposition 10. *For* $\mathrm{Re}(x) > 0$*, the following identity holds*

$$\partial_k \Gamma_k(x+1) = \frac{1}{k^2} \Gamma_k(x+k+1) - \frac{1}{k} \int_0^\infty t^{x+k} \log(t) e^{-\frac{t^k}{k}} dt.$$

Proof. Follows from formula

$$\Gamma_k(x+1) = \int_0^\infty t^x e^{-\frac{t^k}{k}} dt.$$

□

Theorem 11. *For $x > 0$, the function $\psi(k,x) = \log \Gamma_k(x)$ is a solution of the non-linear partial differential equation*

$$-kx^2\partial_x^2\psi + k^3\partial_k^2\psi + 2k^2\partial_k\psi = -x(k+1).$$

Proof. Starting from

$$\frac{1}{\Gamma_k(x)} = xk^{\frac{-x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{nk}\right) e^{\frac{-x}{nk}}\right).$$

The following equations can be proven easily.

$$\begin{aligned}
\psi(k,x) &= -\log(x) + \frac{x}{k}\log(k) - \frac{x}{k}\gamma - \sum_{n=1}^{\infty}\left(\log\left(1+\frac{x}{nk}\right) - \frac{x}{nk}\right).\\
\partial_x\psi(k,x) &= -\frac{1}{x} + \frac{\log(k)-\gamma}{k} - \sum_{n=1}^{\infty}\left(\frac{1}{x+nk} - \frac{1}{nk}\right).\\
\partial_x^2\psi(k,x) &= \sum_{n=0}^{\infty}\frac{1}{(x+nk)^2}.\\
\partial_k\psi(k,x) &= \frac{x}{k^2}\left((1-\log k+\gamma) + \sum_{n=1}^{\infty}\left(\frac{k}{x+nk} - \frac{1}{n}\right)\right).\\
\partial_k(k^2\partial_k\psi(k,x)) &= -\frac{x}{k} + \sum_{n=1}^{\infty}\frac{x^2}{(x+nk)^2}.
\end{aligned}$$

□

The third equation above shows

Corollary 12. *The k-gamma function Γ_k is logarithmically convex on $(0,\infty)$.*

We remark that the q-analogues of the k-gamma and k-beta functions has been introduced in [4].

3 k-beta and k-zeta functions

In this section, we introduce the k-beta function B_k and the k-zeta function ζ_k. We provide explicit formulae that relate the k-beta B_k and k-gamma Γ_k, in similar fashion to the classical case.

Definition 13. *The k-beta function $B_k(x,y)$ is given by the formula*

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \qquad \mathrm{Re}(x) > 0, \quad \mathrm{Re}(y) > 0.$$

Proposition 14. *The k-beta function satisfies the following identities*

1. $B_k(x,y) = \int_0^\infty t^{x-1}(1+t^k)^{-\frac{x+y}{k}}\,dt.$
2. $B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt.$
3. $B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right).$
4. $B_k(x,y) = \frac{(x+y)}{xy}\prod_{n=0}^{\infty}\frac{nk(nk+x+y)}{(nk+x)(nk+y)}.$

Definition 15. *The k-zeta function is given by $\zeta_k(x,s) = \sum_{n=0}^{\infty}\frac{1}{(x+nk)^s}$, for $k, x > 0$ and $s > 1$.*

Theorem 16. *The k-zeta function satisfies the following identities*

1. $\zeta_k(x,2) = \partial_x^2(\log\Gamma_k(x)).$
2. $\partial_x^2(\partial_s\zeta_k)\Big|_{s=0} = -\partial_x^2(\log\Gamma_k(x)).$
3. $\partial_k^m\zeta_k(x,s) = -x(s)_m\sum_{n=0}^{\infty}\frac{n^m}{(x+nk)^{m+s}}.$

Proof. Follows from equations

$$\begin{aligned}
\partial_s\zeta_k(x,s)\Big|_{s=0} &= \sum_{n=0}^{\infty}\log(x+nk).\\
\partial_x(\partial_s\zeta_k(x,s))\Big|_{s=0} &= \sum_{n=0}^{\infty}\frac{1}{(x+nk)}.\\
\partial_x^2(\partial_s\zeta_k(x,s))\Big|_{s=0} &= -\sum_{n=0}^{\infty}\frac{1}{(x+nk)^2}.
\end{aligned}$$

□

4 Hypergeometric Functions

In this section we strongly follow the ideas and notations of [1]. We study hypergeometric functions, see [1] and [8] for an introduction, from the point of view of the Pochhammer k-symbol.

Definition 17. *Given $a \in \mathbb{C}^p$, $k \in (\mathbb{R}^+)^p$, $s \in (\mathbb{R}^+)^q$, $b = (b_1, \ldots, b_q) \in \mathbb{C}^q$ such that $b_i \in \mathbb{C} \setminus s_i\mathbb{Z}^-$. The hypergeometric function $F(a,k,b,s)$ is given by the formal power series*

$$F(a,k,b,s)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\ldots(a_p)_{n,k_p}}{(b_1)_{n,s_1}(b_2)_{n,s_2}\ldots(b_q)_{n,s_q}} \frac{x^n}{n!}. \tag{5}$$

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $\overline{x} = x_1 \ldots x_n$. Using the radio test one can show that the series (5) converges for all x if $p \leq q$. If $p > q+1$ the series diverges, and if $p = q+1$, it converges for all x such that $|x| < \dfrac{s_1 \ldots s_q}{k_1 \ldots k_p}$. Also it is easy to check that the hypergeometric function $y(x) = F(a,k,b,s)(x)$ solves the equation

$$D(s_1D + b_1 - s_1)\ldots(s_qD + b_q - s_q)(y) = x(k_1D + a_1)\ldots(k_pD + a_p)(y),$$

where $D = x\partial_x$.
Notice that hypergeometric function $F(a,1,b,1)$ is given by

$$F(a,1,b,1)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!},$$

and thus agrees with the classical expression for hypergeometric functions. We now show how to transfer from the classical notation for hypergeometric functions to our notation using the Pochhammer k-symbol.

Proposition 18. *Given $a \in \mathbb{C}^p$, $k \in (\mathbb{R}^+)^p$, $s \in (\mathbb{R}^+)^q$, $b = (b_1, \ldots, b_q) \in \mathbb{C}^q$ such that $b_i \in \mathbb{C} \setminus s_i\mathbb{Z}^-$, the following identity holds*

$$F(a,k,b,s)(x) = F\left(\frac{a}{k}, 1, \frac{b}{s}, 1\right)\left(\frac{x\overline{k}}{\overline{s}}\right),$$

where $\dfrac{a}{k} = \left(\dfrac{a_1}{k_1}, \ldots, \dfrac{a_p}{k_p}\right)$, $\dfrac{b}{s} = \left(\dfrac{b_1}{s_1}, \ldots, \dfrac{b_q}{s_q}\right)$ *and* $1 = (1, \ldots, 1)$.

Proof.

$$F(a,k,b,s)(x) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(b)_{n,s}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{k})_n}{(\frac{b}{s})_n} \left(\frac{xk_1 \dots k_p}{s_1 \dots s_q} \right)^n \frac{1}{n!} = F\left(\frac{a}{k}, 1, \frac{b}{s}, 1 \right) \left(\frac{x\overline{k}}{\overline{s}} \right).$$

□

Example 19. *For any $a \in \mathbb{C}$, $k > 0$ and $|x| < \frac{1}{k}$, the following identity holds*

$$\sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} x^n = (1 - kx)^{-\frac{a}{k}}. \tag{6}$$

We next provide an integral representation for the hypergeometric function $F(a,k,b,s)$. Let us first prove a proposition that we will be needed to obtain the integral representation. Given $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ we denote $x_{\leq i} = (x_1, \dots, x_i)$.

Proposition 20. *Let a, k, b, s be as in Definition 17. The following identity holds*

$$F(a,k,b,s)(x) = \frac{1}{\Gamma_{k_{p+1}}(a_{p+1})} \int_0^{\infty} e^{-\frac{t^{k_{p+1}}}{k_{p+1}}} t^{a_{p+1}-1} F(a_{\leq p}, k_{\leq p}, b, s)(xt^{k_{p+1}}) dt \tag{7}$$

when $p \leq q$, $\mathrm{Re}(a_{p+1}) > 0$, and term-by-term integration is permitted.

Proof. $\displaystyle\int_0^{\infty} e^{-\frac{t^{k_{p+1}}}{k_{p+1}}} t^{a_{p+1}-1} F(a_{\leq p}, k_{\leq p}, b, s)(xt^{k_{p+1}}) dt =$

$$F(a_{\leq p}, k_{\leq p}, b, s)(x) \int_0^{\infty} e^{-\frac{t^{k_{p+1}}}{k_{p+1}}} t^{a_{p+1}+nk_{p+1}-1} dt = \Gamma_{k_{p+1}}(a_{p+1}) F(a,k,b,s)(x)$$

□

Theorem 21. *For any a, k, b, s be as in Definition 17. The following formula holds*

$$F(a,k,b,s)(x) = \prod_{j=1}^{p+1} \frac{1}{\Gamma_{k_j}(a_j)} \int_{(\mathbb{R}^+)^{p+1}} \prod_{j=1}^{p+1} e^{-\frac{t_j^{k_j}}{k_j}} t_j^{a_j-1} \left(\sum_{n=0}^{\infty} \frac{1}{(b)_{n,s}} \frac{(xt_1^{k_1} \dots t_{p+1}^{k_{p+1}})^n}{n!} \right) dt, \tag{8}$$

where $(b)_{n,s} = (b_1)_{n,s_1} \dots (b_q)_{n,s_q}$, $dt = dt_1 \dots dt_{p+1}$, $p \leq q$, $\mathrm{Re}(a_j) > 0$ for all $1 \leq j \leq p+1$, and term-by-term integration is permitted.

Proof. Use equation (7) and induction on p. □

Example 22. *For $k = (2, \dots, 2)$, the hypergeometric function $F(a, 2, b, s)(x)$ is given by*

$$F(a,2,b,s)=\prod_{j=1}^{p+1}\frac{1}{\Gamma_2(a_j)}\int_{(\mathbb{R}^+)^{p+1}}\prod_{j=1}^{p+1}e^{-\frac{t_j^2}{2}}t_j^{a_j-1}\left(\sum_{n=0}^{\infty}\frac{1}{(b)_{n,s}}\frac{(xt_1^2\dots t_{p+1}^2)^n}{n!}\right)dt,$$

where $dt = dt_1 \dots dt_n$, $(b)_{n,s} = (b_1)_{n,s_1} \dots (b_q)_{n,s_q}$, $\mathrm{Re}(a_j) > 0$ for all $1 \leq j \leq p+1$ and term-by-term integration is permitted

We now proceed to study the combinatorial interpretation of the coefficient of hypergeometric functions.

Definition 23. *A planar forest F consist of the following data:*

1. *A finite totally order set $V_r(F) = \{r_1 < \dots < r_m\}$ whose elements are called roots.*
2. *A finite totally order set $V_i(F) = \{v_1 < \dots < v_n\}$ whose elements are called internal vertices.*
3. *A finite set $V_t(F)$ whose elements are called tail vertices.*
4. *A map $N : V(T) \to V(T)$.*
5. *Total order on $N^{-1}(v)$ for each $v \in V(F) := V_r(F) \sqcup V_i(F) \sqcup V_t(F)$.*

These data satisfies the following properties:

- *$N(r_j) = r_j$, for all $j = 1, \dots, m$ and $N^k(v) = r_j$ for some $j = 1, \dots, m$ and any $k >> 1$.*
- *$N(V(F)) \cap V_t(F) = \emptyset$.*
- *For any $r_j \in V_r(F)$, there is an unique $v \in V(F)$, $v \neq r_j$ such that $N(v) = r_j$.*

Definition 24. *a) For any $a, k \in \mathbb{N}^+$, $G^a_{n,k}$ denotes the set of isomorphisms classes of planar forest F such that*

1. *$V_r(F) = \{r_1 < \dots < r_a\}$.*
2. *$V_i(F) = \{v_1 < \dots < v_n\}$.*
3. *$|N^{-1}(v_i)| = k + 1$ for all $v_i \in V_i(F)$.*
4. *If $N(v_i) = v_j$, then $i < j$.*

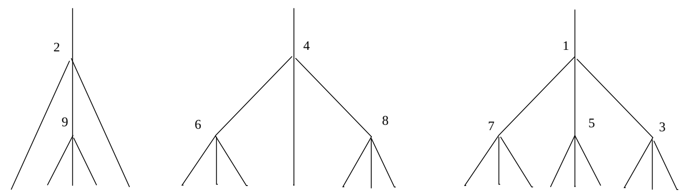

Figure 1: Example of a forest in $G^3_{9,2}$.

b) For any $a, k \in (\mathbb{N}^+)^p$*, we set* $G^a_{n,k} = G^{a_1}_{n,k_1} \times \cdots \times G^{a_p}_{n,k_p}$.

Figure 1 provides an example of an element of $G^3_{9,2}$

Theorem 25. *Given* $a, k \in (\mathbb{N}^+)^p$*,* $b, s \in (\mathbb{N}^+)^q$ *and* $n \in \mathbb{N}^+$*, we have*

$$\frac{\partial^n}{\partial x^n} F(a,k,b,s)(x)\Big|_{x=0} = \frac{|G^n_{a,k}|}{|G^n_{b,s}|}.$$

Proof. It enough to show that $(a)_{n,k} = |G^a_{n,k}|$, for any $a, k, n \in \mathbb{N}^+$. We use induction on n. Since $(a)_{1,k} = a$ and $(a)_{n+1,k} = (a)_{n,k}(a+nk)$, we have to check that $|G^a_{1,k}| = a$, which is obvious from Figure 2, and $|G^a_{n+1,k}| = |G^a_{n,k}|(a+nk)$. It should be clear the any forest in $G^a_{n+1,k}$ is obtained from a forest F in $G^a_{n,k}$, by attaching a new vertex v_{n+1} to a tail of F, see Figure 3. One can prove easily that $|V_t(F)| = a + nk$, for all $F \in G^a_{n,k}$. Therefore $|G^a_{n+1,k}| = |G^a_{n,k}|(a+nk)$. □

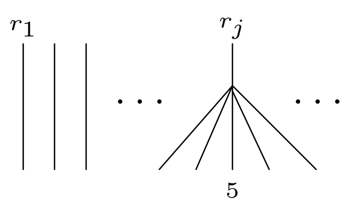

Figure 2: Example of a forest in $G^a_{1,4}$.

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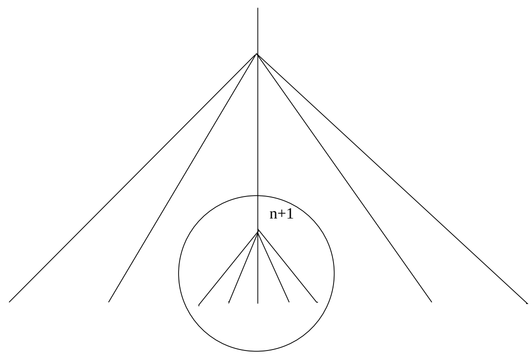

Figure 3: Attaching vertex v_{n+1} to a forest in $G^a_{n,k}$
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