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# Tubular Splines Using A Higher Dimensional Representation

*Splines Tubulares Usando Una Representación Multidimensional*

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**Abstract**

Tubular splines are like the usual spline curves, but have thickness, i.e. locally look like circular cones. They are constructed with pieces of indefinitely differentiable surfaces whose profiles are circles, which are joined together with a prescribed degree of smoothness. We consider splines constructed joining together pieces of canal surfaces bounded by circles. A canal surface is determined by a moving sphere of changing radius. We explore a new way to control the geometric shape of the individual sections of the spline without affecting the neighboring pieces of the spline, using the representation of spheres as points in a 4-dimensional space.
**Key words and phrases:** tube, spline, envelope, circular profiles, shape control.

**Resumen**

Los splines tubulares se construyen con secciones de envolventes de esferas de radio variable, es decir localmente son conos circulares. A estas superficies se le denomina superficies canales y se determinan por el movimiento de una esfera de radio variable en 3D. La curva de conexión entre dos segmentos de una superficie canal es un círculo y los planos tangentes de ambas secciones sobre este círculo coinciden, esto es, se conectan con tangencia continua. En este trabajo se exploran nuevas maneras de controlar la forma geométrica de cada pedazo individual del spline tubular, preservando las condiciones de suavidad de las

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conexiones con los segmentos adyacentes. Para lo anterior se usa una representación de las esferas en 3D, por medio de puntos de un espacio 4-dimensional.
**Palabras y frases clave**: tubular, splines, envolvente, asas de control.

# 1 Introduction

A tubular spline is a smooth surface (i.e. its unit normal varies continuously) which is composed of circles. A simple minded example is one half of a torus joined along a circle to a cylinder in such a way that the tangent planes to both surfaces coincide at the points of their common circle. Figure 1 illustrates a spline that is constructed with three segments: a piece of torus, a piece of cylinder and a section of Dupin cyclide.

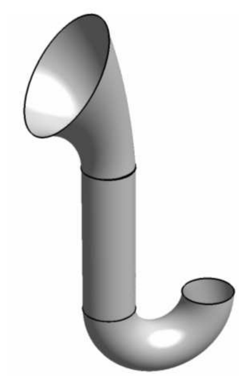

Figure 1: A tubular spline.

The main fact to be emphasized here is that the torus and the cylinder have a common tangent sphere along the circle that they share and likewise the cylinder and the cyclide on the other end. The existence of such a tangent sphere is the necessary and sufficient condition for the smoothness across the joint. We will refer to this tangent sphere and the shared circle which lies on it as a circular contact. The notion of circular contact can be carried over to join more interesting surfaces, those which are tangent to spheres along circles. A specially nice set of surfaces with the above property is the family of canal surfaces. To define a canal surface consider a 1-parameter family of spheres, i.e. a differentiable map $t \to (\mathbf{c}(t), r(t))$, where $\mathbf{c}(t)$ is the center and $r(t)$, the radius, of the sphere corresponding to $t$. See Figure 2.

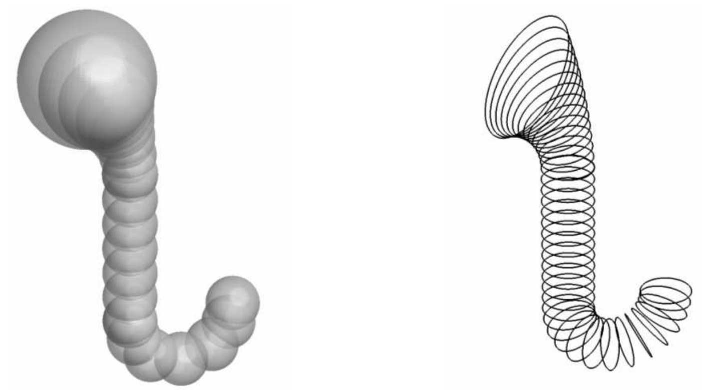

Figure 2: 1-parameter families of spheres and the composing circles of the envelope section.

By eliminating $t$ from the system

$$\begin{cases} \frac{d}{dt}(\|\mathbf{x}-\mathbf{c}(t)\|^2 - r^2(t)) = 0 \\ (\|\mathbf{x}-\mathbf{c}(t)\|^2 - r^2(t)) = 0 \end{cases}$$

the resulting expression in $\mathbf{x} = (x, y, z)$ is satisfied by the canal surface defined by the 1-parameter family. Ocasionally this expression factors; in this case the equation of the canal surface is taken as one of the factors, namely that whose surface is tangent to every sphere of the given family. Usually one refers to the latter as the envelope of the 1-parameter family. See [6], [5] and [1] for a more detailed discussion.

The envelope is composed fully of circles, since for any given $t$, the intersection circle of the tangent spheres $(\mathbf{c}(t), r(t))$ and $(\mathbf{c}(t+\Delta t), r(t+\Delta t))$ tends to a circle which lies on the envelope as $\Delta t$ tends to zero. Figure 2 illustrates the 1-parameter families corresponding to the three sections of the tubular spline of Figure 1 and their composing circles.

These circles are called characteristic circles in the classical geometry literature. An interesting situation arises when the spheres corresponding to $(\mathbf{c}(t), r(t))$ and $(\mathbf{c}(t+\Delta t), r(t+\Delta t))$ have empty intersection as $\Delta t$ tends to zero, in this case the envelope is disconnected, see Figure 3.

To construct tubular splines we consider connected pieces of canal surfaces that join circles in $3D$ space. A tubular spline is constructed by stitching together a finite number of pieces of canal surfaces, so that two neighboring

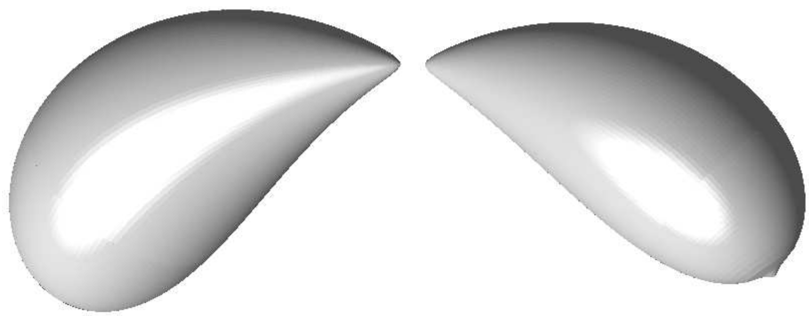

Figure 3: Disconnected cyclide.

pieces share a common circle which lies on a sphere which is tangent to both surfaces along that circle. See Figure 4. Shortly, we say that both pieces share a circular contact.

Figure 4: Two envelopes of 1-parameter families of spheres, joining smoothly.

The above guarantees that the tubular spline is a surface whose unit normal varies smoothly.

# 2 Higher dimensional representation

The set of spheres in Euclidean 3D space depends on four parameters. Following [2] we identify a sphere of center $\mathbf{c} = (X, Y, Z)$ and radius $r$, with the point in 4D $(X, Y, Z, W)$, where $W = X^2 + Y^2 + Z^2 - r^2$. So the points for which $X^2 + Y^2 + Z^2 - W > 0$ represent spheres and the points that lie on the Pedoe paraboloid: $W = X^2 + Y^2 + Z^2$, represent points in 3D, that is, spheres of zero radius.

It's easy to see that the points of the hyperplane $W = 0$ correspond to spheres that pass through the origin and those below this hyperplane contain the origin inside. Two points outside the Pedoe paraboloid, i.e. satisfying $X^2 + Y^2 + Z^2 - W > 0$, which lie on a vertical line correspond to concentric spheres. In Figure 5 the displacement of a point on the vertical line from under the hyperplane $W = 0$ up to the Pedoe paraboloid means to decrease the radius of the corresponding sphere to zero. In the positions $A$, $B$ and $C$ the corresponding sphere contains, passes and is exterior to the origin, respectively.

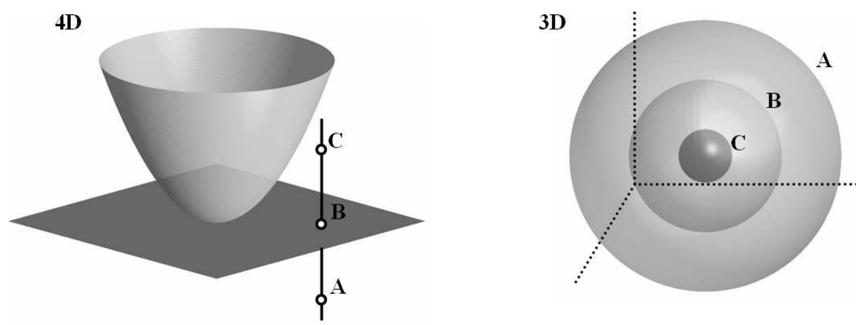


Figure 5: The Pedoe paraboloid and point lying on vertical line.

Points lying on a line which does not meet the Pedoe paraboloid correspond to intersecting spheres; the latter are tangent if and only if the line touches the Pedoe paraboloid. In fact, a non intersecting line in 4D corresponds to the set spheres which pass through some circle. Hence circles in 3D can be represented by lines in 4D which do not meet the Pedoe paraboloid, and a circular contact is a line and a point on it.

Given the line $t \to (X_1, Y_1, Z_1, W_1)t + (X_2, Y_2, Z_2, W_2)$, the corresponding circle lies on the plane $X_1x + Y_1y + Z_1z = \frac{1}{2}W_1$.

A differentiable curve which lies outside the Pedoe paraboloid, corresponds to a 1-parameter family of spheres. The envelope of this 1-parameter family is composed of circles which are represented by the tangent lines to the curve which do not intersect the Pedoe paraboloid. Moreover, a point on the curve and its tangent line correspond to a sphere tangent to the envelope and the composing circle along which it is tangent, to the surface. See Figure 6.

The simplest curves in 4D are conics, which are rational curves of degree two and may be expressed as follows:

$$\mathbf{X}(t) = \frac{\mathbf{X}_0(1-t)^2 + 2w\mathbf{X}_1(1-t)t + \mathbf{X}_2t^2}{(1-t)^2 + 2w(1-t)t + t^2}$$

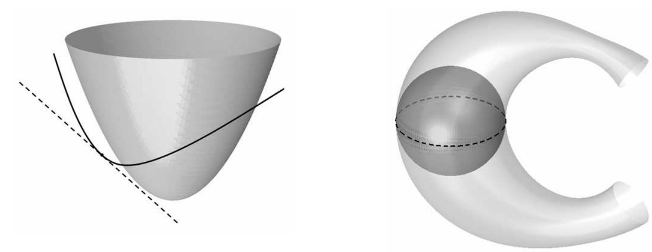

Figure 6: A 4D curve lying outside the Pedoe paraboloid and its tangent line at a given point and its corresponding envelope with its circular contact.

where $\mathbf{X}_0$, $\mathbf{X}_1$ and $\mathbf{X}_2$ are the control points and $w$ is the weight. The expression above is called a rational Bézier curve. It's easy to show that $\mathbf{X}(t)$ passes through $\mathbf{X}_0$ for $t = 0$ and that $\mathbf{X}(1) = \mathbf{X}_2$, and also that the tangent lines at these points pass through $\mathbf{X}_0$ and $\mathbf{X}_1$, and $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Let's assume that for $t$ in $[0, 1]$ the tangents to $\mathbf{X}(t)$ do not meet the Pedoe paraboloid. Then it follows that the part of the envelope of the 1-parameter family of spheres corresponding to $\mathbf{X}(t)$ for t in $[0, 1]$ is a tube that starts at the circle corresponding to the line through $\mathbf{X}_0$ and $\mathbf{X}_1$ and it is tangent to the sphere given by $\mathbf{X}_0$. Likewise it ends at the circle corresponding to the line through $\mathbf{X}_2$ and $\mathbf{X}_1$ and it is tangent to the sphere given by $\mathbf{X}_2$. See Figure 7.

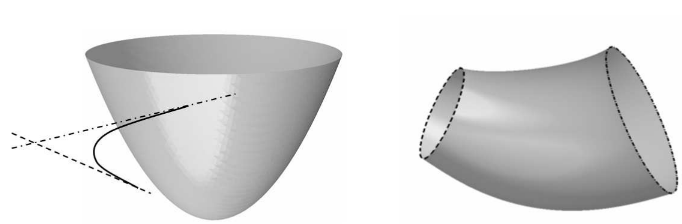

Figure 7: Envelope joining two circles that correponds to a Bézier conic and its tangents at the endpoints.

# 3 Some special cases

The set of spheres of a fixed radius $r$, correspond to the points $(X, Y, Z, W)$ that satisfy the condition $X^2+Y^2+Z^2-W=r^2$, which is also a paraboloid. We refer to it as the $r$-paraboloid. Any 2-plane section of a $r$-paraboloid determines a 1-parameter family of spheres of radius $r$ and hence the envelope is a tube of constant width. In fact any curve on an $r$-paraboloid determines a tube of constant width. See Figure 8.

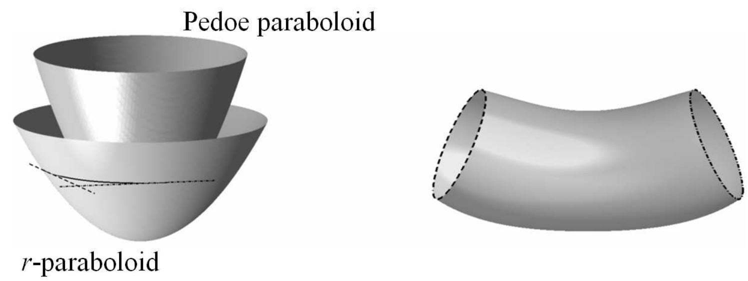


Figure 8: Envelope corresponding to a curve lying on $r$-paraboloid.

A curve contained in a 2-plane parallel to the $W$ axis determines a family of spheres with collinear centers; hence the corresponding envelope is a surface of revolution.

A curve contained in a 2-plane which is not parallel to $W=0$ determines an envelope composed of circles which lie on a coaxal pencil of planes, i.e. the set of all planes through a fixed line. See Figure 9.

# 4 Tubular splines

A tubular spline is a sequence of pieces of envelopes of 1-parameter families of spheres that are joined along circles. This is executed so that for each circle the two adjacent envelopes share a circular contact. In terms of the higher dimensional representation in 4D this means that we have a sequence of curve segments that share common tangents at the joints. In other words, a smooth spline in 4D represents a tubular spline in 3D. To guarantee that each of the tubular spline pieces is connected it is necessary and sufficient that the tangent lines of each segment of the spline in 4D do not meet the Pedoe paraboloid. See [3] for a detailed explanation of how does the tubular spline get disconnected

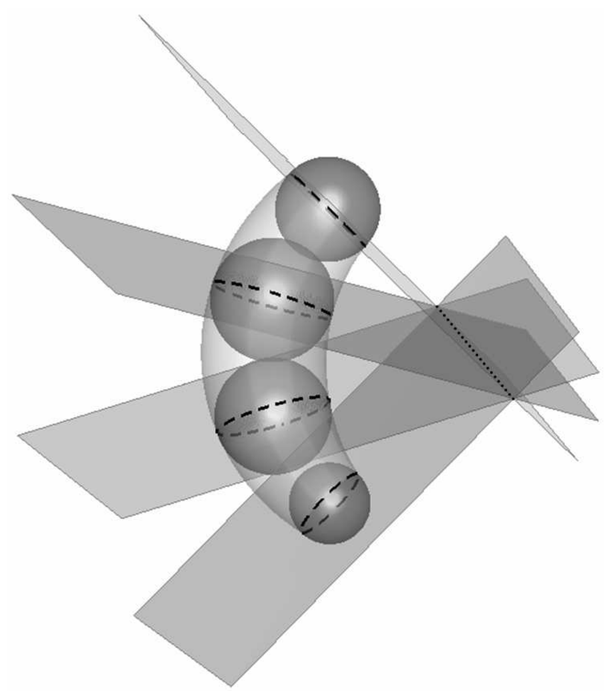

Figure 9: Coaxal pencil of planes containing the circles of the envelope.

in terms of the representing curve in higher dimensional space. The simplest smooth spline in 4D is quadratic, i.e. it consists of segments of conics.

The envelope of a conic is an algebraic surface of degree four; in fact the implicit equation of the surface may be given in terms of the Bézier points and the weight, it is $E_0E_2 - w^2E_1^2 = 0$, where $E_i$ is the equation of the sphere given by control point $\mathbf{X}_i$. The actual expression is long, but can be retrieved with a symbolic algebra package. Hence it follows that the above tubular spline is a smooth piecewise algebraic surface of degree four. If the degree of the spline is higher then the algebraic degree of the tubular spline also goes up. See [4].

For the local control of the tubular spline one is usually interested in modifying any one particular segment preserving interpolations and/or smoothness at one or both extremes of the piece. For the free form construction of a spline curve it is usual to extend the curve by adding the next point to be interpolated. We use the same idea for the free form construction of a tubular spline. Consider a tubular spline we wish to extend from its final circular contact towards a sphere, as illustrated in Figure 10. And consider the corresponding features in 4D. Namely, the line $a$ and a point **A** on it (that corresponds to the circular contact) and the point **B** (corresponding to the sphere **B**), and look at the family of conics through **B** which are tangent to the line $a$ at **A**.

Note that we use the same label to denote a point in 4D and its corresponding sphere; and likewise the line and its corresponding circle. Figure 11 illustrates three conic segments joining **A** to **B**.

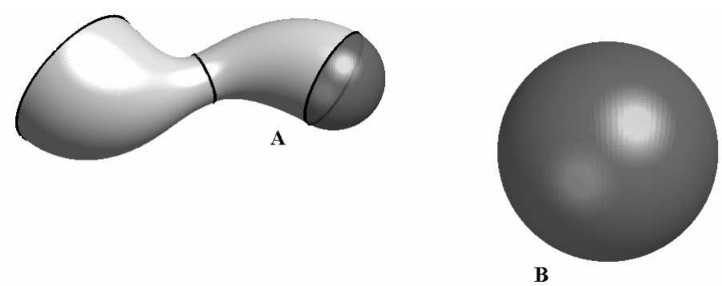


Figure 10: Extending a tubular spline from a circular contact on sphere **A** towards a sphere **B**.

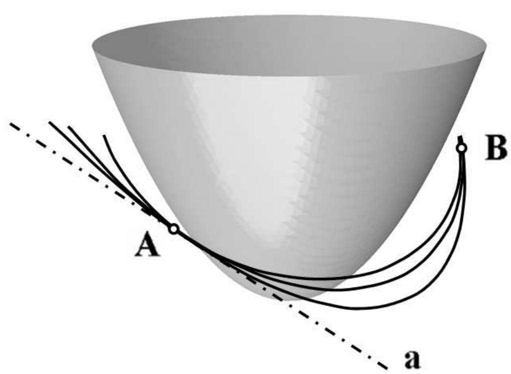


Figure 11: The Pedoe paraboloid and the family of conics through **A** and **B** and tangent to line $a$.

If the sphere **B** does not intersect the circle $a$, then in the 4D framework it is always possible to choose a conic from the above family whose tangent lines do not meet the Pedoe paraboloid. Hence we have an extension of the tubular spline to touch the sphere **B**. The envelope piece joining the circle $a$ on the sphere **A** to the sphere **B** has algebraic degree four or less. See Figure 12.

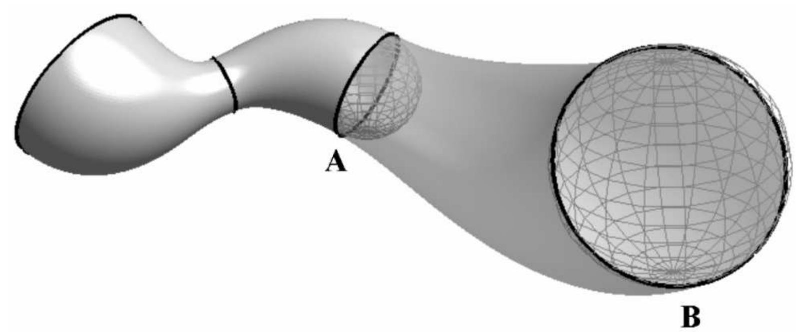


Figure 12: The tubular spline extension.

A second situation of free form construction arises when we prescribe a circle $b$ instead of a sphere and require that the tubular spline extension passes through it. In our 4D representation this corresponds to prescribing a point **A** which lies on the line $a$ and a second line $b$. We further assume that the circles $a$ and $b$ do not intersect and are not interlocked. And we have to construct a low degree polynomial curve $\mathbf{X}(t)$ such that $\mathbf{X}(0) = \mathbf{A}$, its tangent lines at $t = 0$ and $t = 1$ coincide with $a$ and $b$, respectively and also that the tangent lines for $t$ in $(0, 1)$ do not meet the Pedoe paraboloid. If the lines $a$ and $b$ intersect or are parallel (i.e. are not skew lines) then there always exist conics which are tangent to $a$ and $b$, all of whose tangent lines do not meet the Pedoe paraboloid. This allows for many envelope sections, of degree less or equal to four, to join the circular contact to circle $b$. See Figure 13.

If the lines $a$ and $b$ do not intersect or are parallel then there is no general cyclide piece joining the spline to circle $b$. Although it could exist a cubic (or a higher degree) polynomial curve segment with the required properties whose tangent lines do not meet the Pedoe paraboloid. The envelope of the latter would in fact join the circular contact at **A** and the circle $b$, but its algebraic degree would be typically higher than four. The general necessary and sufficient condition for the existence of the above cubic polynomial curve

In Eucliedean terms in 3D this means that the circles $b$ and $a$ are cospherical or coplanar.

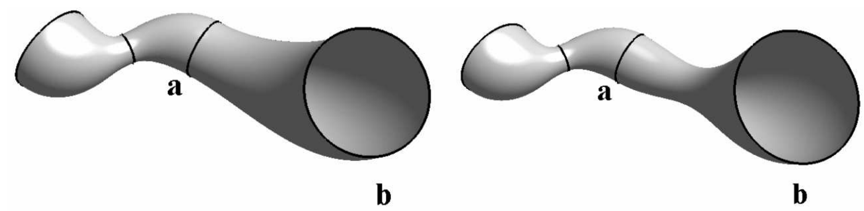


Figure 13: Two different spline extension that interpolate the circle $b$.

is an open question. Finally we consider the local modification of the tubular spline, i.e. the deformation of one of its segments preserving the circular contacts at both of its ends. In 4D this corresponds to substituting a curve segment $\mathbf{X}(t)$ by another, say $\mathbf{Y}(t)$, such that $\mathbf{Y}(0) = \mathbf{X}(0)$, $\mathbf{Y}(1) = \mathbf{X}(1)$ and the tangent lines of $\mathbf{Y}(t)$ at the end points coincide with those of $\mathbf{X}(t)$. Moreover, we need to maintain the condition that the tangents to $\mathbf{Y}(t)$, for $t$ in $(0, 1)$ do not meet the Pedoe paraboloid. If $\mathbf{X}(t)$ is a rational Bézier curve then $\mathbf{Y}(t)$ can be constructed by modifying the appropriate Bézier point and the weight.

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