Divulgaciones Matemáticas Vol. 16 No. 1(2008), pp. 107–124

Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup.

Convergencia no tangencial para el semigrupo de Ornstein-Uhlenbeck

Ebner Pineda (epineda@uicm.ucla.edu.ve)
Departamento de Matemática, Decanato de Ciencia y Tecnología, UCLA
Apartado 400 Barquisimeto 3001 Venezuela

Wilfredo Urbina R.(wurbina@euler.ciens.ucv.ve)
Departamento de Matemáticas, Facultad de Ciencias, UCV.
Apartado 47195, Los Chaguaramos, Caracas 1041-A Venezuela, and
Department of Mathematics and Statistics, University of New Mexico,
Albuquerque, NM, 87131, USA.

Abstract

In this paper we are going to get the non tangential convergence, in an appropriated parabolic "gaussian cone", of the Ornstein-Uhlenbeck semigroup in providing two proofs of this fact. One is a direct proof by using the truncated non tangential maximal function associated. The second one is obtained by using a general statement. This second proof also allows us to get a similar result for the Poisson-Hermite semigroup.
Key words and phrases: Non tangential convergence, Ornstein-Uhlenbeck semigroup, Poisson-Hermite semigroup, Hermite expansions.

Resumen

En este artículo vamos a obtener la convergencia no tangencial, en un "cono gaussiano "parabólico apropiado, del semigrupo de Ornstein-Uhlenbeck dando dos pruebas diferentes de ello. La primera es una prueba directa usando la función maximal no tangencial truncada asociada. La segunda prueba se obtiene usando principios generales. Esta última prueba nos permite obtener un resultado análogo para el semigrupo de Poisson-Hermite.
Palabras y frases claves: Convergencia no tangencial, semigrupo de Ornstein-Uhlenbeck, semigrupo de Poisson-Hermite, desarrollos de Hermite.

Received 2006/03/05. Revised 2006/07/02. Accepted 2006/07/25.
MSC (2000): Primary 42C10; Secondary 26A99.

1 Introduction

Let us consider the Gaussian measure $\gamma_d(x) = \frac{e^{-|x|^2}}{\pi^{d/2}}$ with $x \in \mathbb{R}^d$ and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2}\triangle_x - \langle x, \nabla_x \rangle . \tag{1}$$

Let $\beta = (\beta_1, ..., \beta_d) \in \mathbb{N}^d$ be a multi-index, let $\beta! = \prod_{i=1}^d \beta_i!$, $|\beta| = \sum_{i=1}^d \beta_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\beta = \partial_1^{\beta_1} ... \partial_d^{\beta_d}$.

Let us consider the normalized Hermite polynomial of order β, in d variables

$$h_\beta(x) = \frac{1}{\left(2^{|\beta|}\beta!\right)^{1/2}} \prod_{i=1}^d (-1)^{\beta_i} e^{x_i^2} \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}} (e^{-x_i^2}), \tag{2}$$

then, since the one dimensional Hermite polynomials satisfies the Hermite equation, see [7], then the the normalized Hermite polynomial h_β is an eigenfunction of L, with eigenvalue $-|\beta|$,

$$Lh_\beta(x) = -|\beta| \, h_\beta(x). \tag{3}$$

Given a function $f \in L^1(\gamma_d)$ its β-Fourier-Hermite coefficient is defined by

$$\hat{f}(\beta) =< f, h_\beta >_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_\beta(x) \gamma_d(dx).$$

Let C_n be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\{h_\beta \ : |\beta| = n\}$. By the orthogonality of the Hermite polynomials with respect to γ_d it is easy to see that $\{C_n\}$ is an orthogonal decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n$$

which is called the Wiener chaos.

Let J_n be the orthogonal projection of $L^2(\gamma_d)$ onto C_n. If f is a polynomial,

$$J_n f = \sum_{|\beta|=n} \hat{f}(\beta) h_\beta.$$

The Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is given by

$$\begin{aligned} T_t f(x) &= \frac{1}{(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy) \\ &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} f(y) dy. \end{aligned} \tag{4}$$

$\{T_t\}_{t\geq 0}$ is a strongly continuous Markov semigroup of contractions on $L^p(\gamma_d)$, with infinitesimal generator L. Also, by a change of variable we can write,

$$T_t f(x) = \int_{\mathbb{R}^d} f(\sqrt{1-e^{-2t}}u + e^{-t}x)\gamma_d(du). \tag{5}$$

Definition 1.1. The maximal function for the Ornstein-Uhlenbeck semigroup is defined as

$$\begin{aligned} T^* f(x) &= \sup_{t>0} |T_t f(x)| \\ &= \sup_{0<r<1} \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} |\int_{\mathbb{R}^d} e^{-\frac{|y-rx|^2}{1-r^2}} f(y)dy|. \end{aligned} \tag{6}$$

In [4] C. Gutiérrez and W. Urbina obtained the following inequality for the maximal function $T^* f$,

$$T^* f(x) \leq C_d M_{\gamma_d} f(x) + (2 \vee |x|)^d e^{|x|^2} ||f||_{1,\gamma_d}, \tag{7}$$

where $M_{\gamma_d} f$ is the Hardy-Littlewood maximal function of f with respect to the gaussian measure γ_d,

$$M_{\gamma_d} f(x) = \sup_{r>0} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)|\gamma_d(dy). \tag{8}$$

Unfortunately, this inequality only allows to get the weak (1,1) continuity of $T^* f$ in the one dimensional case, $d = 1$, but allows to get a pointwise convergence result. Several results of this paper, see Lemma 1.1 and Theorem 1.2, use techniques contained in that paper.

If $f \in L^1(\gamma_d), u(x,t) = T_t f(x)$ is a solution of the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = Lu(x,t) \\ u(x,0) = f(x) \end{cases}$$

where $u(x,0) = f(x)$ means that

$$\lim_{t\to 0^+} u(x,t) = f(x), \text{ a.e. } x$$

We want to prove that this convergence is also non-tangential in the following sense. Let

$$\Gamma_\gamma^p(x) = \left\{ (y,t) \in \mathbb{R}_+^{d+1} : | y - x | < t^{\frac{1}{2}} \wedge \frac{1}{| x |} \wedge 1 \right\} \tag{9}$$

be a parabolic "gaussian cone". We want to prove that

$$\lim_{(y,t)\to x,(y,t)\in\Gamma^p_\gamma(x)} T_t f(y) = f(x), \text{ a.e. } x$$

Using the Bochner subordination formula (see [6]),

$$e^{-\lambda} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\lambda^2/4u}du,$$

we define the Poisson-Hermite semigroup $\{P_t\}_{t\geq 0}$ as

$$P_t f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)du. \tag{10}$$

$\{P_t\}_{t\geq 0}$ is also a strongly continuous semigroup on $L^p(\gamma_d)$, with infinitesimal generator $-(-L)^{1/2}$. From (4) we obtain, after the change of variable $r = e^{-t^2/4u}$,

$$P_t f(x) = \frac{1}{2\pi^{(d+1)/2}}\int_{\mathbb{R}^d}\int_0^1 t\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r} f(y)dy. \tag{11}$$

Definition 1.2. The maximal function for the Poisson-Hermite semigroup is defined as

$$P^* f(x) = \sup_{t>0} |P_t f(x)|. \tag{12}$$

If $f \in L^1(\gamma_d), u(x,t) = P_t f(x)$ is solution of the initial value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = -Lu(x,t) \\ u(x,0) = f(x) \end{cases}$$

where $u(x,0) = f(x)$ means that

$$\lim_{t\to 0^+} u(x,t) = f(x), \text{ a.e. } x$$

We want to prove that this convergence, for the Poisson-Hermite semigroup, is also non-tangential in the following sense. Let

$$\Gamma_\gamma(x) = \left\{(y,t)\in\mathbb{R}^{d+1}_+ : \ |y-x| < t \wedge \frac{1}{|x|} \wedge 1\right\}, \tag{13}$$

be a "gaussian cone". Also we want to prove that

$$\lim_{(y,t)\to x,(y,t)\in\Gamma_\gamma(x)} P_t f(y) = f(x), \text{ a.e.}\, x$$

In order to study the non-tangential convergence for the Ornstein-Uhlenbeck semigroup we are going to consider the following maximal function, that was defined by L. Forzani and E. Fabes [3].

Definition 1.3. The non tangential maximal function associated to the Ornstein-Uhlenbeck semigroup is defined as

$$\mathcal{T}_\gamma^* f(x) = \sup_{(y,t)\in\Gamma_\gamma^p(x)} \mid T_t f(y) \mid . \tag{14}$$

Using an inequality for a generalized maximal function, obtained by L. Forzani in [2] (for more details see [8] pag 65–73 and 88–92), it can be proved that $\mathcal{T}_\gamma^* f$ is weak $(1,1)$ and strong (p,p) for $1 < p < \infty$, with respect to the Gaussian measure.

Actually for the non-tangential convergence for the Ornstein-Uhlenbeck semigroup it is enough to consider a "truncated" maximal function. Let

$$\Gamma^p(x) = \left\{(y,t) \in \mathbb{R}_+^{d+1} : \;\; |y-x| < t^{\frac{1}{2}}, 0 < t < \frac{1}{|x|^2} \wedge \frac{1}{4}\right\}, \tag{15}$$

be a truncated parabolic "gaussian cone".

Definition 1.4. The truncated non-tangencial maximal function associated to the Ornstein-Uhlenbeck semigroup is defined as

$$\mathcal{T}^* f(x) = \sup_{(y,t)\in\Gamma^p(x)} \mid T_t f(y) \mid . \tag{16}$$

In the next lemma we are going to get a inequality better than (8) for the truncated non tangential maximal function $\mathcal{T}^* f$, which implies, immediately, that $\mathcal{T}^* f$ is weak $(1,1)$ and strong (p,p) for $1 < p < \infty$, with respect to the gaussian measure.

Lemma 1.1.

$$\mathcal{T}^* f(x) \le C_d M_{\gamma_d} f(x), \tag{17}$$

for all $x \in \mathbb{R}^d$

Proof. Let us take $u(y,t) = T_t f(y)$ and without loss of generality let us assume $f \geq 0$.

Let $a_o = 0$ and $a_j = \sqrt{j}, \quad j \in \mathbb{N}$, then $a_j < a_{j+1} \quad \forall j \in \mathbb{N}$, and let us denote

$$A_j(y,t) = \{u \in \mathbb{R}^d : a_{j-1}(1-e^{-2t})^{\frac{1}{2}} \leq \mid e^{-t}y - u \mid < a_j(1-e^{-2t})^{\frac{1}{2}}\},$$

the annulus with center $e^{-t}y$. Now consider for each $j \in \mathbb{N}$ the ball with center $e^{-t}y$, and radius $a_j(1-e^{-2t})^{\frac{1}{2}}$ and let us denote it by $B_j(y,t) = B(e^{-t}y, a_j(1-e^{-2t})^{\frac{1}{2}})$, then

$$A_j(y,t) = B_j(y,t) \setminus B_{j-1}(y,t)$$

$$\begin{aligned}
u(y,t) &= \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\dfrac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} f(u)du \\
&= \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \sum_{j=1}^{\infty} \int_{A_j(y,t)} e^{\dfrac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} f(u)du
\end{aligned}$$

Now if $(y,t) \in \Gamma^p(x)$ and $\mid e^{-t}y - u \mid < a_j(1-e^{-2t})^{\frac{1}{2}}$ then,

$$\begin{aligned}
\mid e^{-t}x - u \mid &= \mid e^{-t}x - e^{-t}y + e^{-t}y - u \mid \\
&\leq \mid e^{-t}(x-y) \mid + \mid e^{-t}y - u \mid \\
&< e^{-t}t^{\frac{1}{2}} + a_j(1-e^{-2t})^{\frac{1}{2}} \\
&< t^{\frac{1}{2}} + a_j(1-e^{-2t})^{\frac{1}{2}} \\
&< (1+a_j)(1-e^{-2t})^{\frac{1}{2}},
\end{aligned}$$

since $t < 1-e^{-2t}$ if $t < 0.8$

Considering $C_j(x,t) = B(e^{-t}x, (1+a_j)(1-e^{-2t})^{\frac{1}{2}})$, we have

$$u(y,t) \leq \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \sum_{j=1}^{\infty} e^{-a_{j-1}^2} \int_{C_j(x,t)} f(u)du$$

Now,

$$\int_{C_j(x,t)} f(u)du = \int_{C_j(x,t)} f(u)e^{|u|^2}e^{-|u|^2}du$$

$$= \int_{C_j(x,t)} f(u)e^{|u-e^{-t}x|^2+2e^{-t}x.(u-e^{-t}x)+|e^{-t}x|^2}e^{-|u|^2}du$$

$$\leq e^{(1+a_j)^2(1-e^{-2t})+2(1+a_j)(1-e^{-2t})^{\frac{1}{2}}|e^{-t}x|+|e^{-t}x|^2}\int_{C_j(x,t)} f(u)e^{-|u|^2}du$$

but, $| e^{-t}x - u |< (1+a_j)(1-e^{-2t})^{\frac{1}{2}}$ and therefore,

$$| x-u |=| x-e^{-t}x+e^{-t}x-u |< (1-e^{-t}) | x | +(1+a_j)(1-e^{-2t})^{\frac{1}{2}}.$$

Taking

$$D_j(x,t) = B(x,(1-e^{-t}) | x | +(1+a_j)(1-e^{-2t})^{\frac{1}{2}}),$$

we get

$$\int_{C_j(x,t)} f(u)e^{-|u|^2}du \leq \int_{D_j(x,t)} f(u)e^{-|u|^2}du$$

$$\leq M_{\gamma_d}f(x)\int_{D_j(x,t)} e^{-|u|^2}du = M_{\gamma_d}f(x)\int_{D_j(x,t)} e^{-|u-x|^2+2x(x-u)-|x|^2}du$$

$$\leq M_{\gamma_d}f(x)e^{-|x|^2}\int_{D_j(x,t)} e^{-|u-x|^2+2|x||x-u|}du$$

$$\leq M_{\gamma_d}f(x)e^{-|x|^2+2|x|((1-e^{-t})|x|+(1+a_j)(1-e^{-2t})^{\frac{1}{2}})}\int_{D_j(x,t)} e^{-|u-x|^2}du$$

$$= M_{\gamma_d}f(x)e^{-|x|^2+2|x|((1-e^{-t})|x|+(1+a_j)(1-e^{-2t})^{\frac{1}{2}})}\int_{E_j(x,t)} e^{-|w|^2}dw$$

where $E_j(x,t) = B(0,(1-e^{-t}) | x | +(1+a_j)(1-e^{-2t})^{\frac{1}{2}})$.

Since γ_d is a $d-$dimensional measure, and using that $t < \dfrac{1}{| x |^2} \wedge \frac{1}{4}$, we get

$$\begin{aligned}\int_{C_j(x,t)} f(u)e^{-|u|^2}du &\leq C_d M_{\gamma_d} f(x) e^{-|x|^2+2|x|((1-e^{-t})|x|+(1+a_j)(1-e^{-2t})^{\frac{1}{2}})} \\ &\qquad \times((1-e^{-t}) \mid x \mid +(1+a_j)(1-e^{-2t})^{\frac{1}{2}})^d \\ &= C_d M_{\gamma_d} f(x) e^{-|x|^2+2|x|((1-e^{-t})|x|+(1+a_j)(1-e^{-2t})^{\frac{1}{2}}} \\ &\qquad \times(1-e^{-t})^{\frac{d}{2}}((1-e^{-t})^{\frac{1}{2}} \mid x \mid +(1+a_j)(1+e^{-t})^{\frac{1}{2}})^d \\ &\leq C_d M_{\gamma_d} f(x) e^{-|x|^2+2\frac{(1-e^{-t})}{t}+2(1+a_j)\frac{(1-e^{-2t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}} \\ &\qquad \times(1-e^{-t})^{\frac{d}{2}}\left(\frac{(1-e^{-t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}+(1+a_j)(1+e^{-t})^{\frac{1}{2}}\right)^d .\end{aligned}$$

Therefore

$$\begin{aligned}&\int\limits_{C_j(x,t)} f(u)\,du \leq e^{(1+a_j)^2(1-e^{-2t})+2(1+a_j)\frac{(1-e^{-2t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}+e^{-2t}|x|^2} \int\limits_{C_j(x,t)} f(u)e^{-|u|^2}du \\ &\leq e^{(1+a_j)^2(1-e^{-2t})+2(1+a_j)\frac{(1-e^{-2t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}+|x|^2} C_d M_{\gamma_d} f(x) e^{-|x|^2+2\frac{(1-e^{-t})}{t}+2(1+a_j)\frac{(1-e^{-2t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}} \\ &\qquad \times(1-e^{-t})^{\frac{d}{2}}(\frac{(1-e^{-t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}+(1+a_j)(1+e^{-t})^{\frac{1}{2}})^d \\ &\leq e^{(1+a_j)^2(1-e^{-\frac{1}{2}})+4(1+a_j)\frac{(1-e^{-2t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}+\frac{2(1-e^{-t})}{t}}(1-e^{-t})^{\frac{d}{2}} \\ &\qquad \times\left(\frac{(1-e^{-2t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}+(1+a_j)(1+e^{-t})^{\frac{1}{2}}\right)^d C_d M_{\gamma_d} f(x) \\ &\leq e^{(1+a_j)^2(1-e^{-\frac{1}{2}})+4(1+a_j)\sqrt{2}+2}.(1-e^{-t})^{\frac{d}{2}}(1+(1+a_j)\sqrt{2})^d C_d M_{\gamma_d} f(x),\end{aligned}$$

since $0 < t < \frac{1}{4}$ and $\dfrac{1-e^{-t}}{t} < 1, 1+e^{-t} < 2$, if $t > 0$.

Thus,

$$\begin{aligned} u(y,t) \leq & \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \sum_{j=1}^{\infty} e^{-a_{j-1}^2} \int_{C_j(x,t)} f(u)du \\ & \leq C_d M_{\gamma_d} f(x) \frac{1}{\pi^{\frac{d}{2}}(1+e^{-t})^{\frac{d}{2}}(1-e^{-t})^{\frac{d}{2}}} \\ & \quad \times \sum_{j=1}^{\infty} e^{-a_{j-1}^2} e^{(1+a_j)^2(1-e^{-\frac{1}{2}})+4(1+a_j)\sqrt{2}+2}(1-e^{-t})^{\frac{d}{2}}(1+(1+a_j)\sqrt{2})^d \\ & \leq C_d M_{\gamma_d} f(x) \frac{1}{\pi^{\frac{1}{2}}} \sum_{j=1}^{\infty} e^{-a_{j-1}^2+(1+a_j)^2(1-e^{-\frac{1}{2}})+4(1+a_j)\sqrt{2}+2}(1+(1+a_j)\sqrt{2})^d, \end{aligned}$$

since $1+e^{-t} \geq 1$. Now it is easy to see that

$$\begin{aligned} & -a_{j-1}^2 + (1+a_j)^2(1-e^{-\frac{1}{2}}) + 4(1+a_j)\sqrt{2} + 2 \\ & \quad = 4 + 4\sqrt{2} - e^{-\frac{1}{2}} - [-(2(1-e^{-\frac{1}{2}}) + 4\sqrt{2}) + e^{-\frac{1}{2}}\sqrt{j}]\sqrt{j}, \end{aligned}$$

which is negative for j sufficiently big, then

$$\sum_{j=1}^{\infty} e^{-a_{j-1}^2+(1+a_j)^2(1-e^{-\frac{1}{2}})+4(1+a_j)\sqrt{2}+2}.(1+(1+a_j).\sqrt{2})^d < \infty.$$

Thus $u(y,t) \leq C_d M_{\gamma_d} f(x)$ and since $(y,t) \in \Gamma^p(x)$ is arbitrary

$$\mathcal{T}^* f(x) = \sup_{(y,t)\in\Gamma^p(x)} u(y,t) \leq C_d M_{\gamma_d} f(x).$$

□

Now we are ready to establish the convergence result for the Ornstein-Uhlenbeck semigroup.

Theorem 1.2. *The Ornstein-Uhlenbeck semigroup $\{T_t f\}$ converges in $L^1(\gamma_d)$ a.e if $t \to 0^+$, for any function $f \in L^1(\gamma_d)$,*

$$\lim_{t\to 0^+} u(x,t) = f(x), \ \ a.e.\, x \tag{18}$$

Moreover, if $u(y,t) = T_t f(y)$ then $u(y,t)$ tends to $f(x)$ non tangentially ,i.e.

$$\lim_{(y,t)\to x,(y,t)\in\Gamma_\gamma^p(x)} T_t f(y) = f(x), \ \ a.e.\, x. \tag{19}$$

Proof. We have,

$$u(y,t) = \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\dfrac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} f(u)du,$$

considering

$$\Omega f(x) = \lim_{\alpha \to 0^+} \left[\sup_{(y,s)\in\Gamma_\gamma^p(x), 0<s<\alpha} \mid u(y,s) - f(x) \mid \right],$$

and let us set $f(x) = f(x)\chi_{(0,k)} + f(x)(I - \chi_{(0,k)}) = f_1(x) + f_2(x)$, for $k \in \mathbb{N}$ fix.

Let us prove that

$$\Omega f(x) \leq C_d M_{\gamma_d} f_2(x), \text{a.e},$$

for $\mid x \mid \leq k-1$.

Let us consider x a Lebesgue's point for $f \in L^1(\gamma_d)$, i.e. x verifies

$$\lim_{r\to 0^+} \frac{1}{\gamma_d(B(x;r))} \int_{B(x;r)} \mid f(u) - f(x) \mid \gamma_d(du) = 0$$

Then given $\epsilon > 0$ there exists $0 < \delta < 1$ such that

$$\frac{1}{\gamma_d(B(x;r))} \int_{B(x;r)} \mid f(u) - f(x) \mid \gamma_d(du) < \epsilon,$$

for $0 < r < \delta$. Let us define g as $g(u) = \begin{cases} f(u) - f(x) & if \quad \mid u - x \mid \leq \delta \\ 0 & if \quad \mid u - x \mid > \delta \end{cases}$ Thus g depends on x and $M_{\gamma_d} g(x) < \epsilon$.

On the other hand, since

$$u(y,t) - f(x) = u^1(y,t) - f_1(x) + u^2(y,t) - f_2(x)$$

where

$$u^i(y,t) = \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\dfrac{-\mid e^{-t}y - u \mid^2}{1-e^{-2t}}} f_i(u)du \quad i = 1, 2,$$

then we get,

$$u^1(y,t) - f_1(x) = \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\dfrac{-\mid e^{-t}y - u \mid^2}{1-e^{-2t}}} (f_1(u) - f_1(x))du$$

$$= \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|x-u|\leq\delta} e^{\dfrac{-\mid e^{-t}y - u \mid^2}{1-e^{-2t}}} (f_1(u) - f_1(x))du$$

$$+ \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|x-u|>\delta} e^{\dfrac{-\mid e^{-t}y - u \mid^2}{1-e^{-2t}}} (f_1(u) - f_1(x))du.$$

Now we have that if $\mid x \mid \leq k-1$ and $(y,t) \in \Gamma_\gamma^p(x)$ with $t < \frac{1}{|x|^2} \wedge \frac{1}{4}$, then $(y,t) \in \Gamma^p(x)$. Thus $\mid u - x \mid \leq \delta$ implies

$$\mid u \mid = \mid u - x + x \mid \leq \mid u - x \mid + \mid x \mid < \delta + k - 1 < 1 + k - 1 = k$$

and then, $f_1(u) = f(u) \wedge f_1(x) = f(x)$. Therefore

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\left|\int_{|x-u|\leq\delta} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}}(f_1(u)-f_1(x))du\right| \\
= &\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\left|\int_{|x-u|\leq\delta} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}}(f(u)-f(x))du\right| \\
= &\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\left|\int_{\mathbb{R}^d} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}} g(u)du\right| \\
\leq &\mathcal{T}^* g(x) \leq C_d M_{\gamma_d} g(x) \leq C_d \epsilon.
\end{aligned}$$

Now observe that if $(y,t) \in \Gamma_\gamma^p(x)$ and $t^{\frac{1}{2}} \leq \frac{\delta}{2}$ then, $|u-x| > \delta$ implies $\delta < |u-x| \leq |u-y| + |y-x|$ and thus

$$\delta < |u-y| + |y-x| < |u-y| + t^{\frac{1}{2}} \leq |u-y| + \frac{\delta}{2},$$

thus $|u-y| > \frac{\delta}{2}$. Therefore,

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\left|\int_{|u-x|>\delta} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}}(f_1(u)-f_1(x))du\right| \\
\leq &\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-x|>\delta} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}} |f_1(u)| du \\
&+\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} |f_1(x)| \int_{|u-x|>\delta} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}} du \\
\leq &\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-y|>\frac{\delta}{2}} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}} |f_1(u)| du \\
&+\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} |f_1(x)| \int_{|u-x|>\delta} e^{\frac{-|e^{-t}y-u|^2}{1-e^{-2t}}} du.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-y|>\frac{\delta}{2}} e^{\frac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} \mid f_1(u)\mid du\\
=\;&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-y|>\frac{\delta}{2},|u|<k} e^{\frac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} \mid f_1(u)\mid du\\
=\;&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-y|>\frac{\delta}{2},|u|<k} e^{\frac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} \mid f(u)\mid du\\
=\;&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-y|>\frac{\delta}{2},|u|<k} e^{\frac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} e^{|u|^2}\mid f(u)\mid e^{-|u|^2}du\\
\leq\;&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}e^{k^2}\int_{|u-y|>\frac{\delta}{2},|u|<k} e^{\frac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} \mid f(u)\mid e^{-|u|^2}du.
\end{aligned}$$

Then for $0<t<\log\left(\frac{4k+2\delta}{4k+\delta}\right)$; $\mid u-y\mid>\frac{\delta}{2}, \mid u\mid<k$ implies that

$$\begin{aligned}
\mid e^{-t}y-u\mid &= \mid e^{-t}y-e^{-t}u+e^{-t}u-u\mid=\mid e^{-t}(y-u)-(u-e^{-t}u)\mid\\
&\geq e^{-t}\mid y-u\mid-\mid u-e^{-t}u\mid= e^{-t}\mid y-u\mid-(1-e^{-t})\mid u\mid\\
&\geq e^{-t}\frac{\delta}{2}-k(1-e^{-t})=e^{-t}\left(\frac{\delta}{2}+k\right)-k,
\end{aligned}$$

but $0<t<\log\left(\frac{4k+2\delta}{4k+\delta}\right)$ and therefore $e^{-t}>\frac{4k+\delta}{4k+2\delta}$, then,

$$\begin{aligned}
e^{-t}\left(\frac{\delta}{2}+k\right)-k &> \frac{4k+\delta}{4k+2\delta}\left(\frac{\delta+2k}{2}\right)-k\\
&= \frac{4k+\delta}{4(2k+\delta)}(2k+\delta)-k\\
&= \frac{4k+\delta}{4}-k=\frac{4k+\delta-4k}{4}=\frac{\delta}{4}.
\end{aligned}$$

Therefore $\mid u-y\mid > \frac{\delta}{2}$, $\mid u \mid < k$ implies $\mid e^{-t}y-u\mid > \frac{\delta}{4}$ and thus

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|u-y|>\frac{\delta}{2}} e^{\dfrac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} \mid f_1(u)\mid du\\
\leq\;& \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} e^{k^2}\int_{|u-y|>\frac{\delta}{2},|u|<k} e^{-\frac{\delta^2}{16(1-e^{-2t})}} \mid f(u)\mid e^{-|u|^2}du\\
\leq\;& \frac{e^{-\frac{\delta^2}{16(1-e^{-2t})}+k^2}}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{\mathbb{R}^d}\mid f(u)\mid e^{-|u|^2}du = \frac{e^{-\frac{\delta^2}{16(1-e^{-2t})}+k^2}}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\|f\|_{1,\gamma_d}
\end{aligned}$$

On the other hand, taking the change of variable $s = u - e^{-t}y$, we have

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\mid f_1(x)\mid \int_{|u-x|>\delta} e^{\dfrac{-\mid e^{-t}y-u\mid^2}{1-e^{-2t}}} du\\
=\;& \frac{\mid f_1(x)\mid}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|x-s-e^{-t}y|>\delta} e^{\frac{-|s|^2}{1-e^{-2t}}}ds\\
=\;& \frac{\mid f(x)\mid}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}}\int_{|x-s-e^{-t}y|>\delta} e^{\frac{-|s|^2}{1-e^{-2t}}}ds,
\end{aligned}$$

since, $f_1(x) = f(x)$ as $\mid x\mid \leq k-1 < k$.

Thus taking $0 < t < \log\left(\frac{k-1-\delta/2}{k-1-3\delta/4}\right)$, $\mid x-s-e^{-t}y\mid > \delta$ implies

$$\begin{aligned}
\mid s\mid &= \mid s-x+e^{-t}y+x-e^{-t}y\mid = \mid s-x+e^{-t}y-(e^{-t}y-x)\mid\\
&\geq \mid s-x+e^{-t}y\mid - \mid e^{-t}y-x\mid .
\end{aligned}$$

But

$$\begin{aligned}
\mid e^{-t}y-x\mid &= \mid e^{-t}y-e^{-t}x+e^{-t}x-x\mid \leq e^{-t}\mid y-x\mid + (1-e^{-t})\mid x\mid\\
&\leq e^{-t}t^{\frac{1}{2}} + (1-e^{-t})(k-1).
\end{aligned}$$

Thus, since $t^{\frac{1}{2}} \leq \frac{\delta}{2}$,

$$\begin{aligned}
\mid s-x+e^{-t}y\mid - \mid e^{-t}y-x\mid &> \delta - e^{-t}t^{\frac{1}{2}} - (1-e^{-t})(k-1)\\
&\geq \delta - e^{-t}\frac{\delta}{2} - (k-1)(1-e^{-t})\\
&= \delta-(k-1)+(k-1-\frac{\delta}{2})e^{-t},
\end{aligned}$$

and as $0 < t < \log\left(\frac{k-1-\delta/2}{k-1-3\delta/4}\right)$, then $e^{-t} > \frac{k-1-3\delta/4}{k-1-\delta/2}$. Hence,

$$\begin{aligned}
| s | &> \delta - (k-1) + (k-1-\delta/2)e^{-t} \\
&> \delta - (k-1) + (k-1-\delta/2)\frac{k-1-3\delta/4}{k-1-\delta/2} \\
&= \delta - (k-1) + k - 1 - 3\delta/4 = \delta - 3\delta/4 = \frac{\delta}{4}.
\end{aligned}$$

Then $| x - s - e^{-t}y |> \delta$ implies $| s |> \frac{\delta}{4}$ if $0 < t < \log\left(\frac{k-1-\delta/2}{k-1-3\delta/4}\right)$. Therefore, taking $w = \frac{s}{\sqrt{1-e^{-2t}}}$,

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|u-y|>\frac{\delta}{2}} e^{\dfrac{-| e^{-t}y - u |^2}{1-e^{-2t}}} | f_1(u) | \, du \\
\leq\ & \frac{| f(x) |}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|s|>\frac{\delta}{4}} e^{\dfrac{-| s |^2}{1-e^{-2t}}} ds \\
=\ & \frac{| f(x) |}{\pi^{\frac{d}{2}}} \int_{|w|>\frac{\delta}{4\sqrt{1-e^{-2t}}}} e^{-|w|^2} dw.
\end{aligned}$$

Now since, $| x |\leq k-1 < k$, then$f_2(x) = 0$. Hence

$$| u^2(y,t) - f_2(x) |=| u^2(y,t) |\leq \mathcal{T}^* f_2(x) \leq C_d M_{\gamma_d} f_2(x)$$

for $(y,t) \in \Gamma^p(x)$. Therefore,

$$\begin{aligned}
&| u(y,t) - f(x) |\leq| u^1(y,t) - f_1(x) | + | u^2(y,t) - f_2(x) | \\
&=| u^1(y,t) - f_1(x) | + | u^2(y,t) | \\
&\leq C_d\epsilon + \frac{e^{\frac{-\delta}{16(1-e^{-2t})}+k^2}}{(1-e^{-2t})^{\frac{d}{2}}} \|f\|_{1,\gamma_d} + \frac{| f(x) |}{\pi^{\frac{d}{2}}} \int_{|w|>\frac{\delta}{4\sqrt{1-e^{-2t}}}} e^{-|w|^2} dw \\
&\qquad + C_d M_{\gamma_d} f_2(x),
\end{aligned}$$

if $(y,t) \in \Gamma^p_\gamma(x)$ and

$$0 < t < \min\left\{\log\left(\frac{4k+2\delta}{4k+\delta}\right), \log\left(\frac{k-1-\delta/2}{k-1-3\delta/4}\right), \frac{1}{|x|^2} \wedge \frac{1}{4}\right\} =: a.$$

Thus taking supremum on $(y,t) \in \Gamma^p_\gamma(x)$, $0 < t < \alpha < a$ and then taking $\alpha \to 0^+$ we obtain,

$$\Omega f(x) \leq C_d(\epsilon + M_{\gamma_d} f_2(x))$$

for all $\epsilon > 0$ and almost every x with $| x | \leq k-1$.

Given $\epsilon > 0$, let us take k sufficiently large such that

$$\|f_2\|_{1,\gamma_d} \leq C_d \epsilon^2,$$

then by the estimation of Ω and the weak continuity of M_{γ_d} we get

$$\gamma_d(\{x \in \mathbb{R}^d :| x | \leq k-1, \Omega f(x) > \epsilon\}) \leq \epsilon$$

and that implies that $\Omega f(x) = 0$ a.e. □

A similar proof for the Poisson-Hermite semigroup, using the non-tangential maximal function defined as

$$\mathcal{P}^*_\gamma f(x) = \sup_{(y,t)\in\Gamma_\gamma(x)} | P_t f(y) |, \tag{20}$$

and its analogous truncated version, should be possible but it has some technical difficulties that we have been unable to overcome so far.

Let us now prove a general statement for families of linear operators that will allow us to get a simpler proof of the non-tangential convergence, both for the Ornstein-Uhlenbeck semigroup and also for the Poisson-Hermite semigroup. It is a generalization of Theorem 2.2 of J. Duoandikoetxea's book [1].

Theorem 1.3. *Let $\{T_t\}_{t>0}$ be a family of linear operators on $L^p(\mathbb{R}^d, \mu)$ and for any $x \in \mathbb{R}^d$, let $\Gamma(x)$ be a subset of $\mathbb{R}^{d+1}_+$ such that x is in $(\Gamma(x))'$, that is to say x is an accumulation point of $\Gamma(x)$. Let us define*

$$T^* f(x) = \sup\{|T_t f(y)| : (y,t) \in \Gamma(x)\},$$

for $f \in L^p(\mathbb{R}^d, \mu)$ and $x \in \mathbb{R}^d$. If T^* is weak (p,q) then the set

$$S = \left\{ f \in L^p(\mathbb{R}^d, \mu) : \lim_{(y,t)\to x, (y,t)\in\Gamma(x)} T_t f(y) = f(x) \, a.e. \right\}$$

is closed in $L^p(\mathbb{R}^d, \mu)$.

Proof. Let us consider a sequence (f_n) in S such that $f_n \to f$ in $L^p(\mathbb{R}^d, \mu)$, then

$$|T_t f(y) - f(x)| - |T_t f_n(y) - f_n(x)| \leq |T_t(f - f_n)(y) - (f(x) - f_n(x))|,$$

this implies that for each $n \in \mathbb{N}$, for almost every x,

$$\begin{aligned}
\limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} & |T_t f(y) - f(x)| \\
\leq & \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t(f - f_n)(y) - (f(x) - f_n(x))| \\
\leq & \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t(f - f_n)(y)| \\
& \quad + \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |f(x) - f_n(x)| \\
\leq & \; T^*(f - f_n)(x) + |f(x) - f_n(x)|.
\end{aligned}$$

On the other hand, if we know that $a \leq b + c$ then $a > \lambda$ implies $b > \frac{\lambda}{2} \vee c > \frac{\lambda}{2}$.

Then, given $\lambda > 0$ and $n \in \mathbb{N}$, $\limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t f(y) - f(x)| > \lambda$ implies

$$T^*(f - f_n)(x) > \frac{\lambda}{2} \vee |f(x) - f_n(x)| > \frac{\lambda}{2} \; a.e.$$

and this implies that, given $\lambda > 0$,

$$\begin{aligned}
\mu & \left(\left\{ x : \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t f(y) - f(x)| > \lambda \right\} \right) \\
& \leq \mu \left(\left\{ x : T^*(f - f_n)(x) > \frac{\lambda}{2} \right\} \right) \\
& \quad + \mu \left(\left\{ x : |f(x) - f_n(x)| > \frac{\lambda}{2} \right\} \right) \\
& \leq \left(\frac{2C}{\lambda} \|f - f_n\|_p \right)^q + \left(\frac{2}{\lambda} \|f - f_n\|_p \right)^p,
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\mu \left(\left\{ x : \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t f(y) - f(x)| > \lambda \right\} \right) = 0$$

and since this is true for all $\lambda > 0$, we get that

$$\mu \left(\left\{ x : \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t f(y) - f(x)| > 0 \right\} \right) = 0,$$

as

$$\begin{aligned}
& \left\{ x : \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t f(y) - f(x)| > 0 \right\} \\
& \qquad = \bigcup_{n=1}^{\infty} \left\{ x : \limsup_{(y,t)\to x,(y,t)\in\Gamma(x)} |T_t f(y) - f(x)| > \frac{1}{n} \right\}.
\end{aligned}$$

Thus

$$\lim_{(y,t)\to x,(y,t)\in\Gamma(x)} T_t f(y) = f(x)\, a.e.$$

and then $f \in S$. Therefore S is a closed set in $L^p(\mathbb{R}^d, \mu)$. □

Finally, as a consequence of this result, we get the non-tangential convergence for the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t>0}$ and the Poisson-Hermite semigroup $\{P_t\}_{t>0}$.

Corollary 1.4. *The Ornstein-Uhlenbeck semigroup* $\{T_t\}_{t>0}$ *and the Poisson-Hermite semigroup* $\{P_t\}_{t>0}$ *verify*

$$\lim_{(y,t)\to x,(y,t)\in\Gamma^p_\gamma(x)} T_t f(y) = f(x)\, a.e.\, x,$$

$$\lim_{(y,t)\to x,(y,t)\in\Gamma_\gamma(x)} P_t f(y) = f(x)\, a.e.\, x.$$

Proof. Let us discuss the proof for the the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t>0}$. The proof for the Poisson-Hermite semigroup $\{P_t\}_{t>0}$ is totally similar.

It is immediate that for any given polynomial $f(x) = \sum_{k=0}^n J_k f(x)$, since $T_t f(y) = T_t\big(\sum_{k=0}^n J_k f(y)\big) = \sum_{k=0}^n e^{-tk} J_k f(y)$, we have the non-tangential convergence,

$$\lim_{(y,t)\to x,(y,t)\in\Gamma^p_\gamma(x)} T_t f(y) = f(x),$$

for all $x \in \mathbb{R}^d$. Now considering the set

$$S = \left\{ f \in L^p(\gamma_d) : \lim_{(y,t)\to x,(y,t)\in\Gamma^p_\gamma(x)} T_t f(y) = f(x)\, a.e. \right\},$$

corresponding to the Ornstein-Uhlenbeck semigroup, then the polynomials are in S. From the previous result, since non-tangential maximal function for the Ornstein-Uhlenbeck semigroup $\mathcal{T}^*_\gamma f$ is weak $(1,1)$ with respect to the Gaussian measure, we get that the set S is closed in $L^p(\gamma_d)$ and since the polynomials are dense in $L^p(\gamma_d)$ then $S = L^p(\gamma_d)$. □

Acknowledgement

We want to thank the referees for their suggestions and/or corrections that improved the presentation of this paper.

References

[1] Duoandikoetxea, J. *Fourier Analysis*, Graduated Studies in Mathematics, Volume 29, AMS, R. I., 2001.

[2] Forzani, L. *Lemas de cubrimiento de tipo Besicovitch y su aplicación al estudio del operador maximal de Ornstein-Uhlenbeck.* Tesis de Doctorado. Universidad Nacional de San Luis, Argentina, 1993.

[3] Forzani, L., Fabes, E. *Unpublished manuscript*, 1994.

[4] Gutiérrez, C., Urbina, W., *Estimates for the maximal operator of the Ornstein-Uhlenbeck semigroup.* Proc. Amer. Math. Soc. 113 (1991), 99–104.

[5] Sjögren P., *Operators associated with the Hermite Semigroup-A Survey*, J. Fourier Anal. Appl., (3)(1997), 813–823.

[6] Stein E., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.

[7] Szegö, G., *Orthogonal polynomials*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1959.

[8] Urbina W.m *Análisis Armónico Gaussiano: una visión panorámica*, Trabajo de Ascenso, Facultad de Ciencias, UCV, 1998. Available in
`http://euler.ciens.ucv.ve/~wurbina/notes.html`

[9] Zygmund, A., *Trigonometric Series*, 2nd. ed., Cambridge Univ. Press., Cambridge, 1959.