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Sinai-Ruelle-Bowen Measures for Piecewise Hyperbolic Transformations

Medidas de Sinai-Ruelle-Bowen para Transformaciones Hiperbólicas a Trozos

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Abstract

In this work we give sufficient conditions for the existence of an ergodic Sinai-Ruelle-Bowen measure preserved by transformations with infinitely many hyperbolic branches.
Key words and phrases: invariant measures, SRB measures, equilibrium states, fractal dimensions, horseshoes, piecewise hyperbolic transformations.

Resumen

En este trabajo damos condiciones suficientes para que exista una medida de Sinai-Bowen-Ruelle ergódica preservada por transformaciones con infinitas ramas hiperbólicas.
Palabras y frases clave: medidas invariantes, medidas SRB, estados de equilibrio , dimensión fractal, herraduras, transformaciones hiperbólicas a trozos.

1 Introduction

We say that a Borel probability measure μ is a Sinai-Ruelle-Bowen (SRB) measure if it is smooth along unstable leaves. An SRB measure preserved by a C^2 diffeomorphism of a compact riemannian manifold is *physically observable* since it reflects the asymptotic behaviour of a set of positive volume, that

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is, for every continuous function ϕ it holds $\lim_{n\to+\infty} 1\,/\,n\sum_{k=0}^{n-1}\phi(T^k(x)) = \int\phi(z)d\mu(z)$ for a set of points x of positive volume. It is a most interesting fact that the relevant observable measures of mappings which are regular on their expanding directions are smooth along unstable leaves and they are extremals of certain variational principle. Cf. [8] and [12].

The aim of this paper is to prove the existence of SRB measures for certain generalized baker's transformations defined by infinitely many hyperbolic branches.

Theorem A *Let* $\mathsf{R} = B^u \times B^s$ *be a rectangle in* $\mathbb{R}^m$ $(m = s + u)$ *and* $T : \Omega \circlearrowleft$ *be a* C^2 *horseshoe constructed in* R *defined by* C^2 *hyperbolic branches* $T_i : S_i \longrightarrow U_i$ *where* $\{S_i\}$ *is a countable (possibly infinite) collection of non overlapping stable cylinders which cover* R *up to a subset of zero Lebesgue measure. Suppose in addition that the non-linear expansion along invariant manifolds is bounded from below by some* $\lambda > 1$ *and that the following* a priori **bounded distortion condition** *holds:*

$$\sup_{i>0} \frac{\sup_{z\in S_i}\sup_{\xi,\eta\in\mathsf{K}^u(w),\|\xi\|\le 1}\|D^2T_i(z)(\xi,\eta)\|}{(\inf_{\xi\in\mathsf{K}^u(w),\|\xi\|=1}\|DT_i(w)\xi\|)^2} < +\infty \tag{1}$$

for $i > 0$*, where* $\mathsf{K}^u(w)$ *is the unstable cone at* w*. Then* T *preserves a unique ergodic SRB measure* $\mu = \mu_\Omega$ *supported on* Ω *whose ergodic basin covers* R *up to a Lebesgue measure zero set.*

We refer to next Section for definitions. Here, as for Markov piecewise expanding endomorphisms, there is an essential difference between finite and infinite hyperbolic branches since derivatives grow with i and *a priori* relations between the first and second derivatives are key if we want to get some bounded distortion estimates.

To the best of my knowledge these type of problems were considered for the first time by Jakobson and Newhouse in [3]. However, the present approach not only provides a higher dimensional generalization of [3, Theorem 1] but it also gives a new proof of that result in dimension two.

One possible source of interest in these models comes from a program outlined at [4] aimed at describing Hénon attractors by an inducing approach similar to that used in one dimensional dynamics (cf. [2, Chapter 5]). In fact, generalized baker's transformations as described in Theorem A are natural higher dimensional generalizations of piecewise expanding interval transformations. However, as pointed to me by Marcelo Viana, in the case of Hénon maps we should not expect a reduction to a model like one described in Theorem A. Instead, we would expect to induce some sort of generalized baker's transformations having a maximal invariant compact set with hyperbolic product structure and non trivial unstable Cantor sets of positive Lebesgue measure. This is due to the presence of homoclinic tangencies. Unfortunely, as far as I know, no complete exposition of these constructions seems to be yet available. However, we can see an outline in [15].

Indeed, the objects that we shall consider are similar to the generalized horseshoes introduced in Young's paper [15]. Actually, under the *a priori* bounded distortion condition (1) we can prove that they support an SRB measure iff their unstable Cantor sets have positive Lebesgue measure. That gives an independent proof and a sort of converse to [15, Theorem 1.1].

Condition (1) seems to be a natural higher dimensional analogous of bounded distortion condition (D1) in [3] and permits to treat generalized horseshoes with non trivial unstable Cantor sets as well, improving Jakobson-Newhouse's results, even in dimension two.

Horseshoes with infinitely many branches and bounded distortion also appear when inducing hyperbolicity in non uniformly hyperbolic systems. Actually, we proved in [14], using ideas and methods of Pesin theory as exposed in [6], that given a point p in the support of an ergodic hyperbolic measure μ with positive entropy and $0 < \delta < 1$, we can find a regular neighborhood R of p and a subset $\Omega \subset \mathsf{R}$ with hyperbolic product structure, such that $\mu(\Omega) \geq (1-\delta)\mu(\mathsf{R})$. Ω has the same structure of the generalized horseshoes introduced in the present research. This result seems to sustain our claim that the study of these models can give us a better understanding of some geometrical and statistical properties of chaotical dynamical systems.

2 Generalized horseshoes: definitions and statement of results

We shall first recall some definitions and terminology needed to state our main results.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^m$ be a compact subset which is invariant by a piecewise smooth invertible transformation $T : D \longrightarrow \mathbb{R}^m$ from a domain $D \subset \mathbb{R}^m$. We say that Ω has a *hyperbolic product structure* if there are two continuous laminations by discs of complementary dimension $\mathcal{F}^s$ and $\mathcal{F}^u$ such that:

1. each $W \in \mathcal{F}^s$ is an stable invariant manifold, i.e., distances of positive iterates of points $x, y \in W$ contract exponentially; similarly for negative iterates of points on leaves of $\mathcal{F}^u$;

2. any pair of invariant submanifolds $W^s \in \mathcal{F}^s$ and $W^u \in \mathcal{F}^u$ intersect transversally at an angle bounded from below;

3. $\Omega = \bigcup \mathcal{F}^s \cap \bigcup \mathcal{F}^u$.

A generalized baker's transformation defined by countable (possibly infinitely many) hyperbolic branches defines a maximal invariant subset with hyperbolic product structure. We shall refer to these sets as *generalized horseshoes*.

For this we let $\mathsf{R} = B^s \times B^u \subseteq \mathbb{R}^m$, where $B^s \subseteq \mathbb{R}^s$ and $B^u \subseteq \mathbb{R}^u$ are unit closed balls. We shall suppose that R is endowed with two cone fields $\mathsf{K}^u = \mathsf{K}^u(x)$ and $\mathsf{K}^s = \mathsf{K}^s(x)$ defined at every point in $x \in \mathsf{R}$. We also suppose that these cones families extend continuously to a slightly larger neighborhood $\hat{\mathsf{R}}$ containing R. We associate with these cone fields *admissible stable and unstable submanifolds* $\Gamma^s(\mathsf{R}) = \{\gamma^s\}$ and $\Gamma^u(\mathsf{R}) = \{\gamma^u\}$. Each $\gamma^s \in \Gamma^s(\mathsf{R})$ is the graph of a C^1 map $\phi : B^s \longrightarrow B^u$ such that $T_xW \subseteq \mathsf{K}^s(x)$ for every $x \in \gamma^s$. Likewise for admissible unstable submanifolds. Also *stable and unstable cylinders* can be defined. A compact connected subset $S \subseteq \mathsf{R}$ is an admissible stable cylinder if it admits a foliation by admissible stable submanifolds and if its unstable sections $S \cap \gamma^u$ are convex sets for every $\gamma^u \in \Gamma^u(\mathsf{R})$. Unstable cylinders are defined similarly.

Admissible manifolds $\gamma^u \in \Gamma^u$ and $\gamma^s \in \Gamma^s$ intersect transversally with angle $\angle(T_x\gamma^u, T_x\gamma^s)$ bounded from below. *Admissible manifolds have bounded geometry*, that is, we can find a constant $C = C(\Gamma) > 1$ depending only on γ and the diameter of R such that Vol (γ^u) and diam (γ^u) are bounded in $[\,C^{-1}, C\,]$. Further, $C^{-1} \leq \mathrm{dist}_{\gamma^u}(x,y)\,/\,\|x-y\| \leq C$ for every x, y in an unstable admissible submanifold $\gamma^u \in \Gamma^u$ where dist_{γ^u} (resp. diam_{γ^u}) is the distance (resp. diameter) on the submanifold γ^u defined with the intrinsic riemannian metric. Similarly for Vol_{γ^u}. Likewise for the admissible s-submanifolds. This fact will be used throughout.

The set Γ^u (resp. Γ^s) is endowed with an structure of Banach space given by the identification with $C^1(B^u, B^s)$ equipped with the C^1 norm. In particular it is a complete metric space. Moreover, there is a well defined *graph transform* $\Gamma_T : \Gamma^u \longrightarrow \Gamma^u$ defined by $\Gamma_T(\gamma^u) = T(\gamma^u) \cap \mathsf{R}$. The following result, due to Aleeksev and Moser, will be used elsewhere without further comments: *The graph transform is a contraction, i.e., there exists* $0 < \theta < 1$ *such that* $dist_{C^1}(\Gamma_T(\gamma_1^u), \Gamma_T(\gamma_2^u)) \leq \theta\, dist_{C^1}(\gamma_1^u, \gamma_2^u)$. In addition, $(\gamma^u, \gamma^s) \longmapsto \gamma^u \cap \gamma^s$ is a Lipschitz map from $\Gamma^u \times \Gamma^s$ to R.

We will consider maps $\{T_i : S_i \longrightarrow U_i\}$ where $\{S_i\}$ (resp.$\{U_i\}$) are countable collections of non overlapping stable (resp. unstable) cylinders. Each map has a C^2 extension $\hat{T}_i : \hat{S}_i \longrightarrow \hat{U}_i$ to neighborhoods of S_i and U_i

which are stable and unstable cylinders in $\hat{\mathsf{R}}$ and such that $\hat{T}_i$ maps hyperbolically $\hat{S}_i$ onto $\hat{U}_i$, that is, *it preserves strictly the cone families* K^s *and* K^u: $\mathsf{K}^s(T_i(x)) \subseteq \operatorname{int} DT_i(x)\mathsf{K}^s(x)$ and $DT_i(x)\mathsf{K}^u(x) \subseteq \operatorname{int} \mathsf{K}^u(T_i(x))$. Each $T_i : S_i \longrightarrow U_i$ will be called an *hyperbolic branch*.

We can use the hyperbolic branches T_i to define a piecewise hyperbolic map $T : \bigcup_i \operatorname{int} S_i \longrightarrow \bigcup_i \operatorname{int} U_i$, by setting $T \mid \operatorname{int} S_i = T_i$ and extends T to some well defined measurable transformation $\hat{T} : \bigcup_i S_i \longrightarrow \bigcup_i U_i$, which is C^2 smooth in a dense subset of its domain and preserves the admissible manifolds. This extensions are non unique. However, as long as their singular set is negligible for all the purposes of ergodic theory, we can omit this arbitrariness, so we will continue denoting extensions $\hat{T}$ by T to avoid messy notations.

Now, given a family of hyperbolic branches $T_i : S_i \longrightarrow U_i$ we define nested sequences of stable and unstable cylinders converging to two laminations of stable and unstable admissible manifolds $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively. In fact, given a sequence $(i_0, \cdots, i_n, \cdots) = \mathbf{i} \in \mathbb{N}^{\mathbb{N}}$ we define

$$U_{i_0 \cdots i_{n-1}} = \bigcap_{k=0}^{n-1} T_{i_{n-1}} \circ \cdots \circ T_{i_0} U_{i_0}, \quad S_{i_0 \cdots i_{n-1}} = \bigcap_{k=0}^{n-1} T^{-1}_{i_{n-1}} \circ \cdots \circ T^{-1}_{i_0} S_{i_0}$$

where, by abuse of language, we omit the domain of compositions. We will call these *stable and unstable cylinders of level* n. The sequence $\{S_{i_0 \cdots i_{n-1}}\}_{n \geq 0}$ is a nested sequence of stable cylinders. Similarly so $\{U_{i_0 \cdots i_{n-1}}\}_{n \geq 0}$. In fact, graph transform contraction properties imply that there is a unique admissible manifold $\gamma^s = \gamma^s(\mathbf{i})$ such that $d_{C^1}(S_{i_0 \cdots i_{n-1}}, \gamma^s)$ converges to zero as $n \longrightarrow +\infty$ and likewise for the unstable cylinders.

We shall suppose that non linear expansion along unstable admissible manifolds is bounded from below in the following sense: there is a constant $C > 1$ such that, for every pair of points x and y contained in $\gamma^0 = \gamma^u \cap P$ (the intersection of an admissible unstable manifold γ^u and an stable cylinder of level n, $P \in \wp_n$), $d_{\gamma^k}(T^n(x), T^n(y)) \geq C\lambda^{n-k} d_{\gamma^0}(T^k(x), T^k(y))$ holds for $k = 0, \cdots, n-1$, uniformly in x and y, where $\gamma^k = T^k(\gamma^0)$. Due to the bounded geometry of admissible submanifolds these conditions are equivalent to $\|T^n(x) - T^n(y)\| \geq C\lambda^{n-k}\|T^k(x) - T^k(y)\|$, for a suitable constant $C > 0$ depending only on the bounded geometry of Γ^u . A similar statement holds true for the inverse T^{-1}.

Now we define $\mathcal{F}^s = \{\gamma^s(\mathbf{i}) : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$ (resp. $\mathcal{F}^u = \{\gamma^u(\mathbf{i}) : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$). These laminations are clearly T-invariant. Moreover, due to non-linear expansion properties along admissible manifolds, we conclude that $W^s(x, \mathsf{R}) = \gamma^s(\mathbf{i})$ and $W^u(x, \mathsf{R}) = \gamma^u(\mathbf{j})$ are the local stable manifold of x where $\{x\} = \gamma^s(\mathbf{i}) \cap \gamma^u(\mathbf{j})$. Indeed, $W^s(x, \mathsf{R}) = \{y \in \mathsf{R} : \|T^n(x) - T^n(y)\| \leq C\lambda^{-n}, \quad \forall n \geq 0\}$ and similarly so for the unstable local manifold for backward orbits.

We also recall for further use that there are two continuous subbundles E^s (resp. E^u) of stable (resp. unstable) subspaces which are the tangent spaces to invariant leaves. These families of subspaces satisfies a Hölder condition. This is standard. See for example [5].

Let $\Omega = \bigcup \mathcal{F}^s \cap \bigcup \mathcal{F}^u$. Ω is a compact, perfect subset of $\mathbb{R}^m$ contained in the cube R. Topologically it might be a Cantor set times an interval or a product of two Cantor sets or even it might fill R up to a measure zero set. Ω is the maximal invariant subset of $\hat{T}$. Therefore, Ω *is endowed with a hyperbolic product structure.*

Definition 2.2. A set Ω with a hyperbolic product structure and a dynamics $T : \Omega \circlearrowleft$ given by a collection of C^2 hyperbolic branches $T_i : S_i \longrightarrow U_i$ as defined above shall be called a *horseshoe with infinitely many branches* or, shortly, a *generalized horseshoe.*

Ordinary horsehoes are simply those defined by finitely many disjoint hyperbolic branches. The following is our first main result

Theorem B *Let $T : \Omega \circlearrowleft$ be a C^2 generalized horseshoe with bounded distortion and expansion coefficient bounded from below. Then there exists a unique ergodic measure $\mu = \mu_\Omega$ such that:*

1. $\dim_u(\Omega) = h_\mu(T)/\int \log J^u T(x)\, d\mu(x)$, *where $\dim_u(\Omega)$ denotes the unstable dynamical dimension of Ω;*

2. *$\mu_{\mathcal{F}^u(x)}$, the projection of μ onto the unstable leave $\mathcal{F}^u(x)$, is equivalent to the dynamical measure $\mathcal{D}_{\alpha,\mathcal{F}^u(x)}$ in dimension $\alpha = \dim_u(\Omega)$. Furthermore, there is a constant $C > 1$ such that*

$$C^{-1} \leq \frac{\mu_{\mathcal{F}^u(x)}(P)}{(Vol_{\mathcal{F}^u(x)} P)^{\dim_u(\Omega)}} \leq C \quad \textit{for every} \;\; P \in \wp, \tag{2}$$

   where $Vol_{\mathcal{F}^u(x)}$ is the volume defined by the intrinsic riemannian metric of the unstable submanifold $\mathcal{F}^u(x)$; in particular, there is a constant $C > 1$ such that

$$C^{-1} \leq \frac{\mu(P)}{Vol(P)^{\dim_u(\Omega)0}} \leq C$$

   for every geometrical cylinder;

3. *the stable lamination $\mathcal{F}^s$ is absolutely continuous with respect to the dynamical measure classes $\{\mathcal{D}_{\alpha,\gamma^u}\}_{\gamma^u \in \Gamma^u}$ and it has a continuous and*

bounded Jacobian:

$$\frac{d\,H^*\mathcal{D}_{\alpha,\overline{\gamma}^u}}{d\,\mathcal{D}_{\alpha,\gamma^u}}(x) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))}\right]^{\alpha}. \tag{3}$$

Here $H^*\mathcal{D}_{\alpha,\overline{\gamma}^u}$ is the pullback measure under the holonomy map H.

Notation: *throughout this paper we will adopt the following convention: given two positive functions f and g we will write $f \asymp g$ if there is a constant $C > 1$ such that $C^{-1} \leq f/g \leq C$ uniformly in their domain.*

Here $H_{\gamma^u,\overline{\gamma}^u}$ is the holonomy map of the local stable lamination $\mathcal{F}^s$ defined by admissible unstable sections $\gamma^u, \overline{\gamma}^u \in \Gamma^u(\mathsf{R})$: $H_{\gamma^u,\overline{\gamma}^u}(x) = \mathcal{F}^s(x) \cap \overline{\gamma}^u$, for every $x \in \gamma^u \cap \bigcup_{x\in\Omega} \mathcal{F}^s(x)$. $J(T \mid \gamma^u)$ will denote throughout this work the Jacobian of T restricted to γ^u with respect to the intrinsic riemannian volume. $J^uT(x) = J(T \mid \mathcal{F}^s(x))(x)$ is the unstable Jacobian of T with respect to to the intrinsic riemannian volume of the local unstable manifold $\mathcal{F}^s(x)$.

The measure $\mu_{\mathcal{F}^u(x)}$ is defined on the unstable Cantor sets cut by a local unstable manifold of Ω:

$$\mu_{\mathcal{F}^u(x)}(A) = \mu\left(\bigcup_{z\in A} \mathcal{F}^s(z)\right), \quad \text{for every Borel subset } A \subset \mathcal{F}^u(x) \cap \Omega.$$

We proved the above result for ordinary horseshoes. Cf. [13].

To recall what is the dynamical measure we first introduce the dynamically defined generating net of stable cylinders. Namely, let $\wp_n = \{S_{i_0\cdots i_{n-1}} : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$ be the stable cylinders of level n and $\wp_0 = \{S_i\}_{i>0}$. Clearly $\wp_n = T^{-n}\wp_0$. Further $\bigvee_{n\in\mathbb{Z}} T^{-n}\wp_0$ generates the σ-algebra of Borelian sets of Ω and $\bigvee_{n\geq 0} T^{-n}\wp_0$ generates the σ-algebra $\mathcal{B}^s$ of Borel subsets which are a reunion of stable leaves, that is, if $B \in \mathcal{B}^s$ then $\mathcal{F}^s(x) \cap B = \mathcal{F}^s(x)$, for every $x \in B$. We denote by $\wp$ the reunion of all cylinders in $\wp_n$ for $n \geq 0$.

Let $\gamma^u \in \Gamma^u$ and define a generating net of subsets on γ^u taking the intersections of γ^u with cylinders in $\wp$:

$$\wp(\gamma^u) = \{P \cap \gamma^u : P \in \wp_n \text{ is an stable n-cylinder for some } n \geq 0\}.$$

Then we define an outer measure in γ^u, using the Carathéodory construction, by taking coverings by $\wp(\gamma^u)$-sets and using a riemannian volume $\mathrm{Vol}^{\,u}_{\gamma}$ as set function setting $\mathcal{D}_{a,\wp,\gamma^u}(X) = \lim_{\delta\to 0^+} \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \mathrm{Vol}_{\gamma^u}(U_i)^a \; X \subseteq \gamma^u$, where the infimum is taken over all the δ-coverings by $\wp(\gamma^u)$-sets.

A main point in the proof of Theorem B is to show that this construction defines a non-trivial measure class which shall be denoted $\mathcal{D}_{\alpha,W}$. Associated to this fractal measure there is a Carathéodory's dimensional characteristic, the *dynamical dimension*: $\dim_{\mathcal{D},W}(X) = \inf\{a > 0 : \mathcal{D}_{a,W}(X) = 0\}$ Cf. [13] and [11]. Absolute continuity implies that there is a well defined unstable dimension, $\dim_u(\Omega) = \dim_{\mathcal{D},W}(\Omega)$, which does not depend on the particular local unstable manifold $W \in \mathcal{F}^u$.

The first statement in Theorem B follows from inequalities (2) using Billingsley's [1, Theorem 14.1] and bounded distortion estimates. Indeed, (2) implies that $\mu(\wp_n(x)) \asymp \mathrm{Vol}\,(\wp_n(x))^{\dim_u(\Omega)}$, by the definition of transversal measure and using bounded distortion estimates (see next Section). So, it holds that

$$\dim_u(\Omega) = \lim_{n\to+\infty} \frac{\ln\, \mu(\wp_n(x)}{\ln\, \mathrm{Vol}\,(\wp_n(x))}.$$

Now bounded distortion of the volume implies that $\mathrm{Vol}\,(\wp_n(x)) \asymp [J^u T^n(x)]^{-1}$, uniformly bounded by some universal constant. Then, using the Pointwise Ergodic Theorem and Shannon-McMillan-Breiman's property we get the claimed identity. This is exactly what Billingsley did in [1] in a simpler scenario.

For horseshoes in the plane we have the following result, which generalizes [3, Theorem 1] for bidimensional generalized baker's transformations producing horseshoes with non trivial unstable Cantor sets.

Theorem C *Let $T : \Omega \circlearrowleft$ be a C^2 generalized horseshoe with bounded distortion and μ the equilibrium state given by Theorem A. If $\dim_{\mathcal{H}}$ denotes the dimension of a set or of a measure, then :*

1. *the unstable dimension of Ω is the Hausdorff dimension of its unstable Cantor sets, that is, $\dim_u(\Omega) = \dim_{\mathcal{H}}(\Omega \cap \mathcal{F}^u(x))$ for every $x \in \Omega$;*

2. *the stable lamination $\mathcal{F}^s$ of Ω is Lipschitz;*

3. *transversal measures $\mu_{\mathcal{F}^u(x)}$ are equivalent to the Hausdorff measure bounded by uniform constants; indeed, there is a constant $C > 1$ such that*
$$\mu_{\mathcal{F}^u(x)}(B(z,r)) \asymp r^{\dim_u(\Omega)} \tag{4}$$
for every $x \in \Omega$ and $z \in \Omega \cap \mathcal{F}^u(x)$, bounded in $[C^{-1}, C]$ In particular, the transversal measures are dimensionally exact.

This is a consequence of the equivalence between the dynamical measure and Hausdorff measure for conformal dynamically defined Cantor sets and it generalizes [3, Theorem 1.1]. Compare [10, Chapter 4, Proposition 3, pp. 72].

Theorem D *Let $T : \Omega \circlearrowleft$ be a C^2 generalized horseshoe satisfying the hypotheses in Theorem C. The following statements are equivalent:*

1. *μ satisfies the Pesin entropy formula $h_\mu(T) = \int \log J^u T(x) d\mu(x)$;*
2. *$Vol_W(\Omega \cap W) > 0$ for some local unstable manifold $W \in \mathcal{F}^u$;*
3. *the volume $Vol_W(\Omega)$ of the unstable Cantor sets $\Omega \cap W^u(x, \mathsf{R})$ is uniformly bounded away from zero and $\mathcal{F}^s$ is absolutely continuous with respect to Lebesgue measure with uniformly bounded Jacobians;*
4. *μ_Ω is absolutely continuous with respect to the riemannian volume along the local unstable manifolds.*

In addition, if any of the above conditions hold the stable invariant lamination $\mathcal{F}^s$ is absolutely continuous with respect to Lebesgue measure and it has a bounded Jacobian, namely

$$\frac{d\, H^* \mathcal{L}_{\overline{\gamma}^u}}{d\, \mathcal{L}_{\gamma^u}}(x) = \prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))} \tag{5}$$

for every pair of admissible unstable manifolds $\gamma^u, \overline{\gamma}^u \in \Gamma^u$. Here $\mathcal{L}_{\gamma^u}$ is the Lebesgue measure class of γ^u. Therefore, the ergodic basin of the asymptotic measure contains a set of positive volume, so μ_Ω is physically observable.

Theorem A at the Introduction is simply a particular case of Theorem D.

As we can see, arguments in [13] extend straighforwardly to the present set up once we check that the dynamical measure class is non trivial and that $\tau : \Omega/\mathcal{F}^s \longrightarrow \Omega/\mathcal{F}^s$ satisfies an standard bounded distortion condition.

3 Bounded distortion and bounded geometry estimates

Let $T : \Omega \circlearrowleft$ be a C^2 hyperbolic horseshoe defined by countably (possibly infinite) many hyperbolic branches $T_i : S_i \longrightarrow U_i$. We suppose in addition that the collection T_i have non-linear expansion bounded from below and bounded distortion. We introduce for further use the following

Definition 3.1. We define the unstable infimum norm as

$$m^u(DT(z)) = \inf_{\xi \in \mathsf{K}^u(z),\, \|\xi\|=1} \|DT(z)\xi\|.$$

Lemma 3.1. *There is constant $C > 1$, only depending on the distortion, the expansion coefficient and the bounded geometry of the admissible manifolds, such that for every $\gamma^u \in \Gamma^u$ and $n \geq 0$*

$$\frac{J(T^n \mid \gamma^u)(x)}{J(T^n \mid \gamma^u)(y)} \leq \exp(C d_u(T^n(x), T^n(y))) \tag{6}$$

holds whenever $\wp_n(x) = \wp_n(y)$ and $x, y \in \gamma^u$.

In particular, we can find $C > 0$ such that for every $n > 0$

$$\left| \frac{J^u T^n(x)}{J^u T^n(y)} - 1 \right| \leq C \cdot d_u(T^n(x), T^n(y)),$$

when $y \in \wp_n(x) \cap W^u(x, \mathsf{R})$.

Proof. Using bounded distortion condition (1) we prove that for every $\gamma^u \in \Gamma^u$ and $i > 0$ the following estimate holds:

$$\frac{\sup_{z \in \gamma^u \cap S_i} \|\nabla \log J(T_i \mid \gamma^u)(z)\|}{\inf_{w \in \gamma^u \cap S_i} m^u(DT_i(w))} \leq \Delta < +\infty. \tag{7}$$

This is a straightforward computation. Now, we use (7) and a standard argument to get (6). Let $\gamma^u \in \Gamma^u$ an unstable admissible submanifold and denote $\gamma^0 = \gamma^u \cap \wp_n(x)$, $\gamma^i = T^i(\gamma^0)$:

$$\begin{aligned} \log \frac{J(T^n \mid \gamma^u)(x)}{J(T^n \mid \gamma^u)(y)} &\leq \sum_{i=0}^{n-1} |\log J(T \mid \gamma^i)(T^i(x)) - \log J(T \mid \gamma^i)(T^i(y))| \\ &\leq \sum_{i=0}^{n-1} \sup_{z \in \gamma^i} \|\nabla \log J(T \mid \gamma^i)(z)\| \cdot \|T^i(x) - T^i(y)\|. \end{aligned}$$

Now,

$$\begin{aligned} \text{lenght}\,(T(\gamma)) &\geq \inf_{\gamma'(t)} \|DT(\gamma(t))\gamma'(t)\| \,\text{lenght}\,(\gamma) \\ &\geq \inf_{t \in [0,1]} m^u(DT(\gamma(t)))\,\text{lenght}(\gamma) \end{aligned}$$

for every C^1 smooth curve $\gamma = \gamma(t)$, $t \in [0, 1]$, contained inside an unstable admissible submanifold, in particular,

$$d_u(T(x), T(y)) \geq \inf_{w \in \gamma^i} m^u(DT(w))\, d_u(x, y), \tag{8}$$

so $\|T(x) - T(y)\| \geq C^{-2} \inf_{w \in \gamma^i} m^u(DT(w)) \|x - y\|$, for every $x, y \in \gamma^i$. Therefore

$$\begin{aligned}
\log\, & J(T^n \mid \gamma^u)(x)/J(T^n \mid \gamma^u)(y) \\
&\leq C^2 \sum_{i=0}^{n-1} \frac{\sup_{z \in \gamma^i} \|\nabla \log\, J(T \mid \gamma^i)(z)\| \cdot \|T^n(x) - T^n(y)\|}{\inf_{w \in \gamma^i} m^u(DT^{n-i}(w))} \\
&\leq C^2 \sum_{i=0}^{n-1} \frac{\sup_{z \in \gamma^i} \|\nabla \log\, J(T \mid \gamma^i)(z)\| \cdot \|T^n(x) - T^n(y)\|}{\inf_{w \in \gamma^{i+1}} m^u(DT^{n-i-1}(w)) \inf_{w \in \gamma^i} m^u(DT(w)\xi)} \\
&\leq C^2 \Delta \sum_{i=0}^{n-1} \lambda^{-(n-i-1)} \cdot \|T^n(x) - T^n(y)\| \\
&\leq C^2 \Delta \sum_{n=0}^{+\infty} \lambda^{-n} \cdot \|T^n(x - T^n(y)\|,
\end{aligned}$$

which is bounded by $C^3 \Delta (1 - \lambda^{-1}) d_u(T^n(x), T^n(y))$, using again the bounded geometry of admissible manifolds and condition (7). □

Corollary 3.1. *There is a constant $C = C(\Delta, \lambda, \Gamma) > 1$ such that, for every admissible unstable manifold $\gamma^u \in \Gamma^u$ and every $n > 0$ it holds*

$$C^{-1} \inf_{\gamma^u \in \Gamma^u} Vol(\gamma^u) \leq J(T^n \mid \gamma^u)\, Vol_{\gamma^u}(\wp_n(x)) \leq C \sup_{\gamma^u \in \Gamma^u} Vol(\gamma^u).$$

Lemma 3.2. *Let $\gamma^u = \gamma^u(x)$ and $\overline{\gamma}^u = \gamma^u(y)$ be two admissible unstable manifolds passing by x and y in Ω, respectively. Suppose that $y \in W^s(x, \mathsf{R})$ and denote*

$$h(x, y) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(y))} \right]^\alpha .$$

Then $h(x, y) \leq \exp(C d_s(x, y))$ for a constant $C = C(\Delta, \lambda, \Gamma) > 1$ where $d_s(x, y) = d_{W^s(x,\mathsf{R})}(x, y)$. We have also

$$\frac{h(x, y)}{h(x', y')} \leq \max \{ e^{C d_u(x, x')} , e^{C d_u(y, y')} \},$$

for every $x' \in \gamma^u(x)$ and $y' \in \gamma^u(y)$ with $y' \in W^s(x', \mathsf{R})$, where $d_u(x, x') = d_{\gamma^u(x)}(x, x')$ and likewise $d_u(y, y')$.

Proof. We can find a C^2 foliation of R by admissible unstable leaves, say $\gamma^u = \gamma^u(z)$ passing by $\gamma^u(x)$ and $\gamma^u(y)$. Using bounded distortion condition we get

$$\frac{\sup_{w\in\gamma^s(z)} \|\nabla \log J(T_i \mid \gamma^u(w))(w)\|}{\inf_{w\in\gamma^s(z)} m^u(DT_i(w))} \leq \Delta < +\infty, \tag{9}$$

for every leave $\gamma^s(z)$ contained in S_i, every $z \in \mathsf{R}$ and $i > 0$. Let us denote $J_i(z) = J(T_i \mid \gamma^u(z))(z)$. Arguing similarly as we did before we get

$$\begin{aligned} \log \frac{J_i(T^j(x))}{J_i(T^j(y))} &\leq \sup_{w\in W^s(T^j(x),\mathsf{R})} \|\nabla \log J_i(w)\| \cdot \|T^j(x) - T^j(y)\| \\ &\leq C^2 \frac{\sup_{w\in W^s(T^j(x),\mathsf{R})} \|\nabla \log J_i(w)\|}{\inf_{w\in W^s(T^j(x),\mathsf{R})} m^u(DT_i(w))} \|T^{j+1}(x) - T^{j+1}(y)\|. \end{aligned}$$

Thus $\log(J_i(T^j(x)) \,/\, J_i(T^j(y))) \leq \Delta\, C^4 \lambda^{-(j+1)} d_s(x,y)$ and then

$$\log h(x,y) \leq \sum_{j=1}^{+\infty} \Delta\, C^4 \lambda^{-(j+1)} d_s(x,y) = \Delta\, C^4 (1-\lambda^{-1})\lambda^{-2} d_s(x,y).$$

Further, $h(x,y) \,/\, h(x',y') \leq \exp\left(C(d_s(x,y) - d_s(x',y')\right)$. Therefore,

$$\frac{h(x,y)}{h(x',y')} \leq \max\left\{\exp(C d_u(x,x')), \exp(C d_u(y,y'))\right\},$$

for some $C = C(\Delta,\lambda,\Gamma) > 1$ as claimed. □

Corollary 3.2. *There is a constant $C = C(\Delta,\lambda,\Gamma) > 1$ such that, for any two admissible manifolds γ^u and $\overline{\gamma}^u$ in Γ^u it holds $Vol_{\gamma^u}(P) \asymp Vol_{\overline{\gamma}^u}(P)$ for every $P \in \wp$.*

In particular, the volume of the cylinder $\mathrm{Vol}\,(P)$ is comparable with the volume of its unstable sections $\gamma^u \cap P$, i.e., $\mathrm{Vol}_{\gamma^u}(P) \asymp \mathrm{Vol}\,(P)$, bounded by uniform constants which do not depend on p neither $\gamma^u \in \Gamma^u$.

Proof. It follows from Corollary 3.1 and Lemma 3.2 that for every $n > 0$

$$\frac{\mathrm{Vol}_{\gamma^u}(\wp_n(x))}{\mathrm{Vol}_{\overline{\gamma}^u}(\wp_n(x))} \leq C\, \frac{\sup_{\gamma^u\in\Gamma^u} \mathrm{Vol}\,(\gamma^u)}{\inf_{\gamma^u\in\Gamma^u} \mathrm{Vol}\,(\gamma^u)},$$

and similarly for the lower bound. □

Corollary 3.3. *The holonomy of the stable lamination is absolutely continuous with respect to the dynamical class* $\{\mathcal{D}_{\alpha,\gamma^u}\}_{\gamma^u\in\Gamma^u}$ *and it has a bounded Jacobian.*

This follows from $\mathcal{D}_{\alpha,\gamma^u}(B) \asymp \mathcal{D}_{\alpha,\overline{\gamma}^u}(H(B))$, which is bounded by constants $C = C(\Delta, \lambda, \Gamma) > 1$, for every γ^u and $\overline{\gamma}^u$, where $H = H_{\gamma^u,\overline{\gamma}^u}$ is the holonomy transformation defined by these unstable admissible manifolds.

We shall prove later that $h = h_{\gamma^u,\overline{\gamma}^u}$ below is the Jacobian of H with respect to the dynamical measure class:

$$h(x) = \left[\prod_{i=1}^{+\infty} \frac{J(T \mid \gamma^u)(T^i(x))}{J(T \mid \overline{\gamma}^u)(T^i(H(x)))}\right]^{\alpha}.$$

The following result is a straightforward consequence of Lemma 3.2.

Lemma 3.3. *For every* γ^u *and* $\overline{\gamma}^u$ *in* Γ^u *it holds*

$$h(x) \le \exp(C d_s(x, H(x))) \tag{10}$$

for a constant $C = C(\Delta, \lambda, \Gamma) > 1$*. Also, for every* x *and* y *in* γ^u

$$\frac{h(x)}{h(y)} \le \max\{e^{C d_u(x,y))}, e^{C d_u(H(x),H(y)))}\}. \tag{11}$$

As a consequence of the previous discussion we get a constant $C > 1$, depending only on the distortion and the bounded geometry of admissible manifolds, such that:

1. $C^{-1} \le \mathrm{Vol}_u(P \cap \gamma^u) J(T^n \mid \gamma^u)(x) \le C$ for every $x \in P \cap \gamma^u$;

2. $C^{-1} \le \mathrm{Vol}_u(P \cap \overline{\gamma}^u) / \mathrm{Vol}_u(P \cap \gamma^u) \le C$;

3. $C^{-1} \le \mathrm{Vol}(P) / \mathrm{Vol}_u(P \cap \gamma^u) \le C$ and

4. $C^{-1} \le \mathrm{Vol}(P) J(T^n \mid \gamma^u)(x) \le C$, for every $x \in P \cap \gamma^u$,

for any pair of admissible unstable sections $\gamma^u, \overline{\gamma}^u \in \Gamma^u$, $n > 0$ and every stable cylinder $P \in \wp$. We shall refer to all these properties as the *volume lemma*.

4 Proofs of the main results

Lemma 4.1. *The dynamical measure class is non trivial.*

Proof. Let $T : \Omega \circlearrowleft$ be a C^2 hyperbolic horseshoe with infinitely many branches and bounded distortion. We claim that there is a sequence $\Omega_n \subset \Omega$ of ordinary horseshoes with finitely many branches and ergodic measures μ_n such that

1. $\Omega = \overline{\bigcup_n \Omega_n}$;
2. there is a constant $C > 1$ such that, for every local unstable manifolds $W = W^u(x, \mathsf{R})$ and for every $P \in \wp(\Omega_n)$, the family of stable cylinders generating the stable subsets of Ω_n, it holds that $\mu_{n,W}(P) \asymp \mathrm{Vol}_W(P)^{\dim_u(\Omega_n)}$ bounded by C. Here $\mu_{n,W}$ denotes the projection of μ_n the natural equilibrium state of μ_n onto W along the stable lamination.

Indeed, let μ_α the equilibrium state of the potential $-\alpha \ln J^u T$ of an ordinary C^2 horseshoe and μ_W the projection of μ_α onto $W = W^u(x, \mathsf{R})$ along the stable leaves. We proved in [13, Theorem 2.4] that μ_W is equivalent to the dynamical measure $\mathcal{D}_{\alpha,W}$. Moreover, we found a constant $C > 1$ only depending on bounds of the non linear distortion of the volume along unstable leaves, the expansion coefficient of Ω and the bounded geometry of admissible manifolds such that

$$C^{-1} \inf_{W \in \mathcal{F}^u} \mathrm{Vol}\,(W) \leq \frac{\mu_W(P)}{(\mathrm{Vol}_W P)^{\dim_u(\Omega)}} \leq C \sup_{W \in \mathcal{F}^u} \mathrm{Vol}\,(W), \tag{12}$$

for every stable cylinder $P \in \wp$.

Now, let Ω_n be the horseshoe generated by $T_i : S_i \longrightarrow U_i$, for $i = 1, \cdots, n$. Ω is the topological limit of these Ω_n, that is $\Omega = \overline{\bigcup_n \Omega_n}$. By distortion estimates in Corollary 3.1 and Lemma 3.3 we can see that the bounds for the non linear volume distortion of Ω_n are independent of $n > 0$. In particular we can find $d > 0$ such that, for every $m > 0$ and $n > 0$ it holds that

$$\frac{J^u T^m(x)}{J^u T^m(y)} \in [e^{-d}, e^d] \quad \text{whenever} \quad \wp^n_m(x) = \wp^n_m(y),$$

where $wp^n_m = wp_m(\Omega_n)$ are the generating stable cylinders of order $m > 0$ of the horseshoe Ω_n.

Let μ_n denotes the Gibbs measure of $\phi = -\dim_u(\Omega_n) \ln J^u T$. Then it is the equilibrium state which maximizes the unstable dimension for Ω_n and

$\mu_{n,W}(P) \asymp \mathrm{Vol}_W(P)^{\dim_u(\Omega_n)}$ for every $P \in \wp(\Omega_n)$, bounded in $[C^{-1}, C]$ by some constant $C = C(\Delta, \lambda, \Gamma) > 1$. Notice also that $\wp(\Omega_n) \subset \wp(\Omega_{n+1})$.

Now let $\alpha = \lim_{n\to+\infty} \dim_{\mathcal{D},W}(\Omega_n)$. This limit exists since $\dim_{\mathcal{D},W}(\Omega_n)$ is monotone and bounded. Let μ^* be a limit point of μ_n. μ^* is ergodic since it is a limit of ergodic measures and for every $P \in \wp$ it holds

$$\mu^*(P) \geq \limsup_{n\to+\infty} \mu_n(P) \geq C^{-1} \limsup_{n\to+\infty} \mathrm{Vol}_W(P)^{\dim_{\mathcal{D},W}(\Omega_n)}$$

since $P \in \wp$ is closed. Thus $\mu^*(P) \geq C^{-1}\mathrm{Vol}_W(P)^\alpha$ for every $P \in \wp$, concluding that $\mathcal{D}_{\alpha,W}(\Omega) < +\infty$, using Frostman's lemma argument. Furthermore, $\mu^*(B) \geq C^{-1} \cdot \mathcal{D}_{\alpha,W}(B)$ for every Borel subset $B \subseteq \Omega$ so it is absolutely continuous respect to the dynamical measure class. Similarly so,

$$\mu^*(\mathrm{int}\, P) \leq \liminf_{n\to+\infty} \mu_n(\mathrm{int}\, P) \leq C \liminf_{n\to+\infty} \mathrm{Vol}_W(P)^{\dim_{\mathcal{D},W}(\Omega_n)},$$

since $\mathrm{Vol}(\partial P) = 0$ for every $P \in \wp$. Then, $\mu^*(\mathrm{int}\, P) \leq C\, \mathrm{Vol}_W(P)^\alpha$ and this imply $\mathcal{D}_{\alpha,W}(\Omega) > 0$. □

Now, we recall that $\tau(x) = H_{W(T(x)),W} \circ T(x)$, is the projection of T along the stable leaves. Here $W = W^u(x, \mathsf{R})$ and $W(T(x)) = W^u(T(x), \mathsf{R})$. Now, τ has a Jacobian with respect to $\mathcal{D}_{\alpha,W}$. For this we use that the dynamical measure class is conformal and the absolute continuity of the stable lamination. Indeed, using conformality we check easily that

$$J_\alpha \tau(x) = \frac{d\, H^* \mathcal{D}_{\alpha,W}}{d\, \mathcal{D}_{\alpha,W(T(x))}}(T(x))[J^u T(x)]^\alpha,$$

for every local unstable manifold $W = W^u(x, \mathsf{R})$.

Lemma 4.2. *There exists a constant $C = C(\Delta, \lambda, \Gamma) > 1$ such that*

$$\left| \frac{J_\alpha \tau^n(x)}{J_\alpha \tau^n(y)} - 1 \right| \leq C\, d^*(\tau^n(x), \tau^n(y)),$$

whenever $\wp(n, W)(x) = \wp(n, W)(y)$, where $\wp(n, W)(x)$ denotes the connected component of the intersection $\wp_n(x) \cap W$ in $W = W^u(x, \mathsf{R})$ containing x.

Proof. First notice that there is a constant $C > 1$, depending only on the bounded geometry of the admissible manifolds, such that

$$\ln \left[\prod_{k=0}^{n-1} \frac{J^u(\tau^k(x))}{J^u(\tau^k(y))} \right] \leq C d^*(\tau^n(x), \tau^n(y))^\theta.$$

This follows from the Bounded Distortion Property (7) and $\|\tau^k(x)-\tau^k(y)\| \asymp d^*(\tau^k(x),\tau^k(y)) = d^*(T^k(x),T^k(y))$.

Also, using the Bounded Distortion Property (11) we conclude that

$$\ln \frac{h(T(\tau^k(x)))}{h(T(\tau^k(y)))} \leq C \max\{d_u(T(\tau^k(x)),T(\tau^k(y)))^\theta \,,\, d_u(\tau^{k+1}(x),\tau^{k+1}(y))^\theta\},$$

which is in turn no greater than $2Cd^*(\tau^{k+1}(x),\tau^{k+1}(y)^\theta)$, for some constant $C>1$, depending only on the bounded geometry, since d_u is comparable with d^*. Thus, for every $n>0$ we have

$$\begin{aligned}\ln\left[\prod_{k=0}^{n-1}\frac{h(T(\tau^k(x)))}{h(T(\tau^k(y)))}\right] &\leq 2C\sum_{k=0}^{n-1}\lambda^{-n-k-1}d^*(\tau^n(x),\tau^n(y))^\theta\\ &\leq C(1-\lambda^{-1})d^*(\tau^n(x),\tau^n(y))^\theta,\end{aligned}$$

absorbing the various constants into some $C=C(\Gamma)$ we get

$$\frac{J_\alpha\tau^n(x)}{J_\alpha\tau^n(y)} \leq \exp(Cd^*(\tau^n(x),\tau^n(y))^\theta).$$

Then, an easy argument concludes the proof. □

Lemma 4.2 above shows that the endomorphism τ of the unstable Cantor set $\Lambda = W\cap\Omega$ endowed with the Borel measure $\mathcal{D}_{\alpha,W}$ satisfies the hypotheses of [9, Chapter III, Theorem 1.3]. Therefore, there is a unique ergodic measure $\mu_{\mathcal{F}^u(x)}$ which is equivalent to $\mathcal{D}_{\alpha,\mathcal{F}^u(x)}$ and which maximizes dimension. Compare also [2, Chapter V, Theorem 2.2], [7, Chapter 6] and [16].

This defines an ergodic Borel probability $\tilde{\mu}$ defined over the stable Borel subsets $\mathcal{B}^s$ simply by setting

$$\mu_W(B) = \tilde{\mu}\left(\bigcup_{z\in B}\mathcal{F}^s(z)\right) \qquad \text{for every Borel subset} \quad B\subset\mathcal{F}^u(x)\cap\Omega.$$

By the absolute continuity of the stable lamination this measure $\tilde{\mu}$ does not depend on the unstable leave $W = W^u(x,\mathsf{R})$ choosen. Now, $\bigvee_{n\geq 0}T^n\mathcal{B}^s$ generates $\mathcal{B}(\Omega)$, the Borel subsets of Ω. This permits to extend $\tilde{\mu}$ to a Borel probability $\mu=\mu_\Omega$ defined over $\mathcal{B}(\Omega)$. μ_W is precisely the projection of μ_Ω onto W. This concludes the proof of Theorem A.

On the other hand it can be proved, using bounded distortion estimates, that $\mathcal{D}_{\alpha,W}(P)\asymp(\mathrm{Vol}_W P)^\alpha$ for every local unstable manifold $W\in\mathcal{F}^u$, where

$\alpha = \dim_u(\Omega)$. Compare for example [10, Chapter IV]. As the transversal measure μ_W is equivalent to the dynamical measure, we conclude that $\mu_W(P) \asymp (\mathrm{Vol}_W P)^{\dim_u(\Omega)}$. This concludes the proof of Theorem B.

Now we are ready to prove Theorems C and D.

Proof of Theorem C

Using the Bounded Distortion Lemma and volume estimates we show that for every $n \geq 0$ and $P \in \wp^*$ such that $T^n(P)$ is an unstable cylinder (i.e., such that $T^n(P)$ has full width) diam $(P \cap W^u(x, \mathsf{R})) \asymp \|J^u T^n(x)\|^{-1}$ bounded by a constant not depending on x, P or n. Now, given $r > 0$ and $x \in \Omega^*$ we define $n = n(x, r) > 0$ as the minimum positive integer satisfying $(J^u T^n(x))^{-1} \leq r$ and $(J^u T^{n-1} f(x))^{-1} > r$. Therefore,

$$B(x, C^{-1} r) \cap \gamma^u \subseteq S_n(x) \cap \gamma^u \cap \gamma^u \subseteq B(x, C\, r) \cap \gamma^u \tag{13}$$

for every $0 < r < 1$ and $x \in \Omega$ and some universal constant $C > 1$, depending only on the distortion and the bounded geometry of admissible manifolds, where $W^s(x, \mathsf{R}) = \bigcap_{n \geq 0} S_n(x)$ is a nested sequence of stable cylinders converging to the local stable manifold, $n = n(x, r)$ and $T^n(S_n(x) \cap \gamma^u)$ is an admissible unstable manifold.

We use this to define Moran's covers $\wp_r$ for every $0 < r < 1$ associated to $\wp$ according to Cf. [11, Chapter 7]. Moran's covers satisfy the following finite multiplicity property: there exists a universal constant $M > 1$ such that $B(x, r) \cap \gamma^u$ intersects at most M atoms in the family $\wp_r(\gamma^u)$, for every $x \in \Lambda$, $0 < r < 1$ and admissible unstable manifold γ^u. M should depend in principle on γ^u, however, by geometry of admissible manifolds shows that this dependence can be dropped out.

By Theorem C, $\mu_{\mathcal{F}^u(x)}(P) \asymp \mathrm{diam}\,(P \cap W^u(x, \mathsf{R}))^{\dim_u(\Omega)}$. So for every $P \in \wp$, hence $\mu_{\mathcal{F}^u(x)}(B(x, r)) \asymp \mathrm{diam}\,(B(x, r) \cap \gamma^u))^{\dim_u(\Omega)}$ which is comparable with $r^{\dim_u(\Omega)}$ for every $x \in \Omega$, $0 < r < 1$, using the bounded geometry of admissible manifolds. So, $\dim_{\mathcal{H}}(\Omega \cap W^u(x, \mathsf{R}) = \dim_u(\Omega)$, and the dynamical measure, the Hausdorff measure and the transversal measure are in the same measure class and the transversal measure $\mu_{\mathcal{F}^u(x)}$ has the strong uniform distribution property $\mu_{\mathcal{F}^u(x)}(B(x, r)) \asymp r^{\dim_u(\Omega)}$, for every $x \in \Omega$ and $0 < r < 1$, bounded by a universal constant.

In particular, the holonomies have a bounded Jacobian respect to the Hausdorff measure. Finally, by Frostman's uniform distribution property and the existence of a bounded Jacobian with respect to the Hausdorff measure for the holonomies of the stable lamination $\mathcal{F}^s$ we see that

$$\|x - y\|^\alpha \asymp \mathcal{H}_{\alpha, \gamma^u}([x, y]) \asymp \mathcal{H}_{\alpha, \overline{\gamma}^u}([H(x), H(y)]) \asymp \|H(x) - H(y)\|^\alpha,$$

where $\alpha = \dim_{\mathcal{H}}(\Omega \cap W)$, $\mathcal{H}_{\alpha,\gamma^u}$ is the restriction of the Hausdorff measure to the admissible unstable manifold γ^u and $H = H_{\gamma^u,\overline{\gamma}^u}$ is the holonomy map defined by γ^u and $\overline{\gamma}^u$. Constants are uniform, by previous remarks and the bounded geometry of admissible manifolds. Hence,

$$\frac{\|H(x) - H(y)\|}{\|x - y\|} \asymp 1, \quad \forall\, x, y \in \gamma^u \cap \bigcup_{z\in\Omega} W^s(z, \mathsf{R}),$$

meaning that $C^{-1}\|x - y\| \leq \|H(x) - H(y)\| \leq C\|x - y\|$, $\mathcal{F}^s$ is Lipschitz. We are done.

Proof of Theorem D
We notice that $\mathcal{D}_{1,\gamma^u}$ represents the Lebesgue measure class of γ^u. This is a straightforward consequence of the definitions and the fact that $\wp$ generates the stable lamination. In particular, $\mathrm{Vol}_{\gamma^u}(X) > 0$ implies $\dim_u(X) = 1$. Thus, $\dim_u(\Omega) = 1$ implies that μ_W, the projection of μ along $\mathcal{F}^s$ onto the local unstable manifolds $W = W^u(x, \mathsf{R})$, is equivalent to the restriction of the Lebesgue measure class $\mathcal{L}_W$ to $W \cap \Omega$, bounded by constants not depending on $x \in \Omega$ neither on $W \in \mathcal{F}^u$. In particular, the volume $\mathrm{Vol}_W(\Omega)$ of the unstable Cantor sets $\Omega \cap W^u(x, \mathsf{R})$ is uniformly bounded away from zero, by the absolute continuity of the lamination and since its Jacobians are uniformly bounded. As $\mathrm{Vol}_W(\Omega) > 0$ for some local unstable manifold $W = W^u(x, \mathsf{R})$ implies $\dim_u(\Omega) = 1$, we see that all these conditions are equivalent. To finish, we notice that the ergodic basin of $\mu = \mu_\Omega$, defined as

$$W^s(\mu) = \{x \in \mathsf{R} : \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k(x)) = \int \phi(z) d\mu(z) \;\; \forall\, \phi \in C^0(M)\},$$

contains the stable set of Ω, $W^s(\Omega) = \bigcup_{x\in\Omega} \mathcal{F}^s(x)$. Now, $W^s(\Omega)$ has positive volume, in view of the absolute continuity of the stable lamination with respect to the Lebesgue measure, using a Fubini's theorem argument, hence $\mathrm{Vol}\,(W^s(\mu)) > 0$ and we are done.

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References

[1] Billingsley, P. *Ergodic Theory and Information*, John Wiley & Sons, New York, 1965.

[2] De Melo, W., Van Strien *One Dimensional Dynamics*, Springer-Verlag, 1993.

[3] Jakobson, M. V., Newhouse, S. *A Two Dimensional Version of the Folklore Theorem.* Amer. Math. Soc. Transl. (2), vol. 171, 1996, 89–105.

[4] Jakobson, M. V., Newhouse, S. *On the Structure of non Hyperbolic Attractors*, Preprint. `http://www.mth.msu.edu/~sen/Papers/`

[5] Katok, A., Hasselblat, B. *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Math. and its Applications, Vol. 54, Cambridge University Press, 1995.

[6] Katok, A., Mendoza, L. *Dynamical Systems with Non-uniformly Hyperbolic Behavior*, Supplement to *Introduction to the Modern Theory of Dynamical Systems*, by A. Katok and B. Hasselblat.

[7] Keller, G. *Equilibrium States in Ergodic Theory*, London Mathematical Society Student Texts 42, Cambridge University Press, 1998.

[8] Ledrappier, F., Young, L. S. *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) **122**(3) (1985), 509–539; *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (3) (1985), 540–574.

[9] Mañé, R. *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, 1987.

[10] Palis, J., Takens, F. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1993.

[11] Pesin, Ya. *Dimension Theory and Dynamical Systems*, Chicago Lectures on Mathematics, Chicago, 1997.

[12] Pugh, C., Shub, M. *Ergodic Attractors*, Transactions of the AMS Vol. 312, No. 1, 1989.

[13] Sánchez-Salas, F. J. *Some Geometric Properties of Ergodic Attractors*, Informes de Matemática, Série F-112, IMPA, 1999.

[14] Sánchez-Salas, F. J. *Horseshoes with Infinitely Many Branches and a Characterization of Sinai-Ruelle-Bowen Measures*, Preprint, Math. Phys. Archive # 99–422.

[15] Young, L. S. *Statistical Properties of Dynamical Systems with some Hyperbolicity*, Annals of Math. 147 (1998), 585-650.

[16] Walters, P. *Invariant Measures and Equilibrium States for Some Mappings which Expand Distances*, Trans. American Math. Soc., Volume 236, 1978.