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Φ-Bounded Solutions to Nonlinear Difference Equations with Advanced Arguments

Soluciones Φ-acotadas para ecuaciones en diferencias no lineales con argumentos avanzados

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Abstract

In this work we give sufficient conditions in order that the difference equation with advanced arguments $x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + f(n, x(n), x(\sigma_2(n)))$, $\sigma_i(n) \geq n+1$ has $\Phi-$bounded solutions in the space ℓ_Φ^∞ of $\Phi-$bounded sequences, are given.
Key words and phrases: Difference equations with advanced arguments, existence, uniqueness, bounded solutions.

Resumen

En este trabajo se dan condiciones suficientes para que la ecuación en diferencias con argumentos avanzados $x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + f(n, x(n), x(\sigma_2(n)))$, $\sigma_i(n) \geq n+1$ tenga soluciones en el espacio ℓ_Φ^∞ de las sucesiones $\Phi-$acotadas.
Palabras y frases clave: Ecuaciones en diferencias con argumentos avanzados, existencia, unicidad, soluciones acotadas.

1 Introduction

Difference equations with advanced arguments is an attractive field of research in differential equations. The possibility of writing a higher difference equation

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as an equation with advanced arguments produces interesting results that have been emphasized in the papers [2, 3, 4, 5, 6, 7], devoted to the theory of equations with advanced arguments.

The study of existence of bounded solutions of equations with advanced arguments has been recognized as a first order task in this field. As a rule the existence of oscillating solutions, periodic solutions, quasi-periodic solutions, converging solutions, etc., depend on the existence of bounded solutions of these equations.

In this work we study the existence and uniqueness of bounded solutions of the nonlinear difference equation with advanced arguments

$$x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + f(n, x(n), x(\sigma_2(n))), \tag{1}$$

where the sequences $\{A(n)\}$, $\{B(n)\}$ of $r \times r$ matrices are defined on the set of natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$. The matrices $A(n)$, $n \in \mathbf{N}$, are assumed to be invertible. $\sigma_1, \sigma_2 : \mathbf{N} \to \mathbf{N}$ are sequences that move $x(n)$ to the advanced states $x(\sigma_1(n))$, $x(\sigma_2(n))$. They will satisfy the condition

$$\sigma_1(n), \sigma_2(n) \geq n + 1, \ \forall n \in \mathbf{N}. \tag{2}$$

Let Φ be the fundamental matrix [1] of the linear equation

$$x(n+1) = A(n)x(n).$$

To make easier the writing of the text we introduce the notation

$$\Lambda(n) = \Phi^{-1}(n+1),$$

which will be frequently used. The main assumption of this paper establishes a size of the sequence $\{B(n)\}$ in terms of the fundamental matrix Φ:

[C3] $$\rho := \sum_{n=0}^{\infty} |\Lambda(n)B(n)\Phi(\sigma_1(n))| < 1.$$

Hypothesis **[C3]** is inspired from Hallam's paper [9], where similar conditions have been applied to problems of asymptotic integration of ordinary differential equations. In this paper we assume **[C3]** in contrast with the hypotheses

[C1] $$\sum_{m=0}^{n-1} |\Phi(n)\Phi^{-1}(m)| \leq M, \forall n \in \mathbf{N}, \ M = \text{constant},$$

and

[C2] $$|\Phi(n)\Phi^{-1}(m)| \leq K\alpha(n)\beta(m)^{-1}, \ n \geq m,$$

used in problems of existence of bounded solutions of Eq. (1) [2, 3, 7].

Regarding equations

$$x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) \tag{3}$$

and

$$x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + g(n), \tag{4}$$

condition **[C3]** was used in the papers [2, 3, 6] to study the existence and uniqueness of bounded solutions of those equations in the sequential space ℓ_Φ^∞ of Φ-bounded sequences (see the definition of Φ-boundedness in the next section).

The main result of this paper establishes the existence of Φ-bounded solutions of Eq. (1) under condition **[C3]**. In section 4, we present an example of a block diagonal system where **[C3]** can be used. The obtained results for this example do not follow from **[C1]** and **[C2]**.

2 Preliminaries

In what follows **V** will denote the vector space $\mathbf{R}^r$ or $\mathbf{C}^r$. We will use the symbol $|x|$ to denote any convenient vector norm for $x \in \mathbf{V}$ and for a $r \times r$ matrix A, $|A|$ will denote the corresponding matrix norm.

We will define the space of the Φ-bounded sequences by

$$\ell_\Phi^\infty = \{f : \mathbf{N} \to \mathbf{V} \mid \Phi^{-1}f \in \ell^\infty\},\ |f|_\Phi^\infty = |\Phi^{-1}f|^\infty;$$

the space of Φ-summable sequences by

$$\ell_\Phi^1 = \{f : \mathbf{N} \to \mathbf{V} \mid \Phi^{-1}f \in \ell^1\},\ |f|_\Phi^1 = |\Phi^{-1}f|^1;$$

and the space

$$\ell^1_{<\Phi>_1} = \{f : \mathbf{N} \to \mathbf{V} \mid \{\Lambda(n)f(n)\} \in \ell^1\},\ |f|^1_{<\Phi>1} = |\{\Lambda(n)f(n)\}|^1.$$

We recall the notion of generating matrix of Eq. (3) [2], and the variation of constants formula of Eq. (4) [3].

Definition 1. A sequence of matrices $\{\Psi(n)\}_{n=0}^\infty$, $\Psi(0) = I$, is called a generating matrix of Eq. (3), in the sequential space ℓ_Φ^∞, iff the following holds:

(i) The sequence $\{\Psi(n)\}$ is a solution of the Eq. (3) on **N**,

(ii) $\Psi(n)$ is invertible for all values of $n \in \mathbf{N}$,

(iii) $\{\Psi_{ij}(n)\} \in \ell_\Phi^\infty$, where $\Psi(n) = (\Psi_{ij}(n))$.

In [2], the following theorem was proven.

Theorem A. *Condition* **[C3]** *implies that Eq.* (3) *has a unique generating matrix in the space* ℓ_Φ^∞ *which is denoted by* Ψ_Φ *and satisfies*

$$|\Psi_\Phi(n)\Psi_\Phi^{-1}(m)| \leq \frac{1}{1-\rho}, \ \forall n \geq m.$$

In [3] it was proven that the solution of Eq. (4), $g \in \ell^1_{<\Phi>_1}$ satisfying $x(0) = 0$, can be found by the series

$$x(n) = \Phi(n) \sum_{k=0}^{\infty} C_k[g](n),$$

where $\{C_k\}$ is a sequence of bounded and linear operators, $C_k : \ell^1_{<\Phi>_1} \to \ell_\Phi^\infty$. $C_0[g]$ is defined as the solution of the initial value problem (**IVP**)

$$\Delta C_0[g](n) = \Lambda(n)g(n), \ C_0[g](0) = 0,$$

and $C_k[g]$, for $k \geq 1$, is the solution of the **IVP**

$$\Delta C_k[g](n) = \Lambda(n)B(n)\Phi(\sigma_1(n))C_{k-1}[g](\sigma_1(n)), \ C_k[g](0) = 0.$$

Theorem B. *If* $g \in \ell^1_{<\Phi>_1}$, *then condition* **[C3]** *implies the existence of a unique solution* $\mathcal{K}_\Phi[g]$ *of Eq.* (4) *in the space* ℓ_Φ^∞, *satisfying* $x(0) = 0$, *given by*

$$\mathcal{K}_\Phi[g](n) = \Phi(n) \sum_{k=0}^{\infty} C_k[g](n), \ g \in \ell^1_{<\Phi>_1}. \tag{5}$$

This formula defines a linear operator $\mathcal{K}_\Phi : \ell^1_{<\Phi>_1} \to \ell_\Phi^\infty$, *which allows to write the unique solution* x *of Eq.* (4) *satisfying* $x(0) = \xi$, $x \in \ell_\Phi^\infty$, *in the form*

$$x(n) = \Psi_\Phi(n)\xi + \mathcal{K}_\Phi[g](n), \ \textit{if}\ g \in \ell^\infty_{<\Phi>_1}. \tag{6}$$

We recall that the operator $\mathcal{K}_\Phi$ is bounded: [3]

$$|\mathcal{K}_\Phi[g]|_\Phi^\infty \leq \theta |g|^1_{<\Phi>_1}, \ \theta := \frac{1}{1-\rho}. \tag{7}$$

3 Existence of Φ-bounded solutions

For reasons of simplicity, we will assume that $f(n,0,0)=0$, otherwise the results of this paper can be easily adapted. For a short notation in the wording, it is convenient the following substitution operator

$$\mathcal{F}[x](n) := f(n, x(n), x(\sigma_2(n))).$$

Using the operator $\mathcal{F}$, Eq. (1) can be written in the equivalent form

$$x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + \mathcal{F}[x](n).$$

The form of this last equation suggests the extension of the results of this paper to a difference equation where, in general, $\mathcal{F}$ is not a substitution operator, for example the integral operator

$$\mathcal{F}[x](n) = \sum_{s=0}^{n-1} K(n,s)x(\sigma(s)).$$

We do not investigate this kind of problems in this paper, but the most of our results will be applicable.

For a positive μ, let us define the closed ball

$$B_\Phi[0,\mu] := \{x \in \ell^\infty_\Phi : \ |x|^\infty_\Phi \le \mu\}.$$

We will assume the following Lipschitz condition:

$$|\Lambda(n)\left(\mathcal{F}[x](n) - \mathcal{F}[y](n)\right)| \le l(n,\mu)|x-y|^\infty_\Phi, \ x, y \in B_\Phi[0,\mu], \tag{8}$$

where $l(\cdot,\mu)$ is a sequence of non negative numbers. More general conditions than (8) are also possible [7], but we introduce (8) in order to present a readable paper on the subject.

We will solve the **IVP**

$$\begin{cases} x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + \mathcal{F}[x](n) \\ x(0) = \xi \end{cases} \tag{9}$$

in the space of Φ-bounded sequences. Besides **[C3]**, the existence of solutions in the space ℓ^∞_Φ rests on the forthcoming hypotheses **[H1]**-**[H2]**:

[H1] $$\mathcal{F} : \ell^\infty_\Phi \to \ell^1_{<\Phi>_1},$$

[H2] $$\Omega(\mu)=\sum_{n=0}^{\infty} l(n,\mu)<\infty,$$

According to (5), we define the operator $\mathcal{H}:\ell_\Phi^\infty\to\ell_\Phi^\infty$ by means of

$$\mathcal{H}[x]=(\mathcal{K}_\Phi o\mathcal{F})[x].$$

$\mathcal{H}$ is a Lipschitz operator: from (7) and **[H2]** we obtain for $x,y\in B_\Phi[0,\mu]$

$$|\mathcal{K}_\Phi o\mathcal{F}[x]-\mathcal{K}_\Phi o\mathcal{F}[y]|_\Phi^\infty\le\theta|\mathcal{F}[x]-\mathcal{F}[y]|_{<\Phi>_1}^1$$

$$=\theta\sum_{n=0}^{\infty}|\Lambda(n)(f(n,x(n),x(\sigma_2(n)))-f(n,y(n),y(\sigma_2(n))))|$$

$$\le\theta\sum_{n=0}^{\infty} l(n,\mu)|x-y|_\Phi^\infty=\theta\Omega(\mu)|x-y|_\Phi^\infty.$$

In particular

$$|\mathcal{H}[x]|_\Phi^\infty=|\mathcal{K}_\Phi o\mathcal{F}[x]|_\Phi^\infty\le\theta\mu\Omega(\mu),\ x\in B_\Phi[0,\mu],\tag{10}$$

implies the following

Lemma 1. *If $\theta\Omega(\mu)<1$, then $\mathcal{H}(B_\Phi[0,\mu])\subseteq B_\Phi[0,\mu]$.*

The variation of constants formula (6) allows the definition of the operator

$$\mathcal{T}[x]=\Psi_\Phi\xi+\mathcal{K}_\Phi o\mathcal{F}[x]=\Psi_\Phi\xi+\mathcal{H}[x].$$

Lemma 2. *If the initial condition $\xi\in\mathbf{V}$ satisfies the estimate*

$$|\xi|\le\mu(1-\rho-\Omega(\mu))=\mu(1-\rho)(1-\theta\Omega(\mu)),\tag{11}$$

and conditions **[C3]**, **[H1]**, **[H2]** *are satisfied, then $\mathcal{T}:B_\Phi[0,\mu]\to B_\Phi[0,\mu]$. Moreover, a sequence $x\in\ell_\Phi^\infty$ is a solution of problem (9) iff x is a fixed point of the operator $\mathcal{T}$.*

Proof. Previously, we observed that for each $x\in B_\Phi[0,\mu]$, the condition **[C3]** and (10) imply

$$|\mathcal{T}[x]|_\Phi^\infty\le|\Psi_\Phi\xi|_\Phi^\infty+|\mathcal{H}[x]|_\Phi^\infty\le\frac{|\xi|}{1-\rho}+\theta\mu\Omega(\mu).$$

Using (11) we obtain

$$|\mathcal{T}[x]|_{\Phi}^{\infty} \leq \mu[1-\theta\Omega(\mu)] + \theta\mu\Omega(\mu) = \mu.$$

This means that the operator $\mathcal{T}$ acts from the ball $B_{\Phi}[0,\mu]$ into itself. Let x be a solution of (9) belonging to ℓ_{Φ}^{∞}. Since $\mathcal{H}:\ell_{<\Phi>}^{\infty} \to \ell_{<\Phi>}^{\infty}$, it follows that, x is a solution of the **IVP**

$$x(n+1) = A(n)x(n) + B(n)x(\sigma_1(n)) + g(n),\ x(0) = \xi,$$

contained in ℓ_{Φ}^{∞}. But from formula (6) this solution is given by

$$x(n) = \Psi_{\Phi}(n)\xi + \mathcal{K}_{\Phi}[g](n) = \Psi_{\Phi}(n)\xi + \mathcal{K}_{\Phi}o\mathcal{F}[x](n).$$

This last implies $x = \mathcal{T}[x]$. On the other hand, if $x = \mathcal{T}[x]$, then

$$x(n) = \mathcal{T}[x](n) = \Psi_{\Phi}(n)\xi + \mathcal{H}[x](n) = \Psi_{\Phi}(n)\xi + \mathcal{K}_{\Phi}o\mathcal{F}[x](n).$$

The last identity proves that x is a solution of Eq. (4) with

$g(n) = \mathcal{F}[x](n) = f(n, x(n), x(\sigma_2(n))).$ □

Theorem 1. *For every initial condition $\xi \in \mathbf{V}$ satisfiying (11), the hypotheses* **[C3]**, **[H1]**, **[H2]**, $\theta\Omega(\mu) < 1$ *imply the existence of a unique Φ-bounded solution x in the ball $B_{\Phi}[0,\mu]$ of the* **IVP** (9).

Proof. Since the operator $\mathcal{T}$ gives the solutions of the problem (9), we will verify that this operator is a contraction on the ball $B_{\Phi}[0,\mu]$. From Lemma 2 we have that the operator $\mathcal{T}$ acts from the ball $B_{\Phi}[0,\mu]$ on $B_{\Phi}[0,\mu]$, and for every $x, y \in B_{\Phi}[0,\mu]$ satisfies

$$|\mathcal{T}[x] - \mathcal{T}[y]|_{\Phi}^{\infty} \leq |\mathcal{H}[x] - \mathcal{H}[y]|_{\Phi}^{\infty} \leq \theta\Omega(\mu)|x-y|_{\Phi}^{\infty}.$$

Since $\theta\Omega(\mu) < 1$ we have that $\mathcal{T}$ is a contraction acting from $B_{\Phi}[0,\mu]$ into itself. The unique fixed point x is a solution of the Eq. (9). Therefore

$$x(n) = \Psi_{\Phi}(n)\xi + \mathcal{H}[x](n),$$

and according to Lemma 2 x is the unique $\Phi-$bounded solution of the **IVP** (9) satisfying $x(0) = \xi$ in the ball $B_{\Phi}[0,\mu]$. □

4 Block diagonal systems

Let us consider the system (1), where we suppose $x \in \mathbf{R}^r$, $x = \mathrm{col}(u, v)$, $u \in \mathbf{R}^{r_1}$, $v \in \mathbf{R}^{r_2}$, $r_1 > 0$, $r_2 > 0$, $r_1 + r_2 = r$, $|x| = \max\{|u|_{r_1}, |v|_{r_2}\}$, where $|u|_{r_1}$, $|v|_{r_2}$ are norms in $\mathbf{R}^{r_1}$, $\mathbf{R}^{r_2}$, respectively. Also, we assume that

$$f(n, x, y) = \mathrm{col}(f_1(n, x, y), f_2(n, x, y)),\ f_i(n, 0, 0) = 0, i = 1, 2,$$

where

$$f_i : \mathbf{N} \times \mathbf{R}^r \times \mathbf{R}^r \to \mathbf{R}^{r_i},\ i = 1, 2.$$

Moreover, we assume

$$A(n) = \mathrm{diag}\{A_1(n), A_2(n)\};\ B(n) = \mathrm{diag}\{B_1(n), B_2(n)\}.$$

We define

$$\Phi_i(n) = \prod_{k=0}^{n-1} A_i(k); \Lambda_i(n) = \Phi_i^{-1}(n + 1), i = 1, 2; \Phi = \mathrm{diag}\{\Phi_1, \Phi_2\},$$

and the substitution operators

$$\mathcal{F}_i[x](n) = f_i(n, x(n), x(\sigma_2(n))),\ i = 1, 2.$$

Therefore condition (8) is accomplished if we impose

$$|\Lambda_i(n)(\mathcal{F}_i[x](n) - \mathcal{F}_i[y](n))|_{r_i} \le l_i(n, \mu)|x - y|_\Phi^\infty,\ i = 1, 2,$$

We will assume that

$$\rho_i = \sum_{n=0}^{\infty} |\Lambda_i(n)B_i(n)\Phi_i(\sigma_1(n))|_{r_i} < 1,\ i = 1, 2. \tag{12}$$

Condition **[C3]** is satisfied, since $\rho = \max\{\rho_1, \rho_2\} < 1$. In condition (12), we observe that the sequence $|B_i(n)|$ can be "large", but this situation is compensated by a "small" Φ_i and viceversa. According to Theorem 1, if

$$\frac{1}{1-\rho} \max\left\{\sum_{n=0}^{\infty} l_1(n, \mu), \sum_{n=0}^{\infty} l_2(n, \mu)\right\} < 1,$$

then the nonlinear system (1), with block diagonal matrices $A(n)$, $B(n)$, for every initial condition ξ satisfying

$$|\xi| \le \mu \left(1 - \rho - \max\{\sum_{n=0}^{\infty} l_1(n, \mu), \sum_{n=0}^{\infty} l_2(n, \mu)\},\right)$$

has a unique Φ-bounded solution contained in the ball $B_\Phi[0,\mu]$.

As an example, let us consider the two dimensional system

$$\begin{pmatrix} u(n+1) \\ v(n+1) \end{pmatrix} = \begin{pmatrix} 2u(n) \\ \frac{1}{2}v(n) \end{pmatrix} + \begin{pmatrix} b_1(n)u(\sigma_1(n)) \\ b_2(n)v(\sigma_1(n)) \end{pmatrix} + \begin{pmatrix} (v(\sigma_2(n))^2 \\ 0 \end{pmatrix}. \quad (13)$$

In this case the matrices Φ_i are given by

$$\Phi_1(n) = 2^n, \;\; \Phi_2(n) = 2^{-n}$$

and the substitution operators are

$$\mathcal{F}_1[x](n) = (v(\sigma_2(n)))^2, \;\; \mathcal{F}_2 = 0.$$

According to (12) we will require

$$\rho_1 = \sum_{n=0}^{\infty} |2^{-n-1}b_1(n)2^{\sigma_1(n)}| = \sum_{n=0}^{\infty} 2^{\sigma_1(n)-n-1}|b_1(n)| < 1$$

and

$$\rho_2 = \sum_{n=0}^{\infty} |2^{n+1}b_2(n)2^{-\sigma_1(n)}| = \sum_{n=0}^{\infty} 2^{n-\sigma_1(n)+1}|b_2(n)| < 1.$$

Further on

$$|\mathcal{F}[x]|^1_{<\Phi>_1} \leq \sum_{n=0}^{\infty} |\Lambda_1(n)(v(\sigma_2(n)))^2|$$

$$\leq \mu^2 \sum_{n=0}^{\infty} 2^{-2\sigma_2(n)-n-1} \leq \mu^2 \sum_{n=0}^{\infty} 2^{-3(n+1)} = \frac{\mu^2}{7}$$

implies **[H1]**. In order to verify **[H2]**, we observe that for

$$x(n) = \mathrm{col}(u_1, v_1)(n), \;\; y(n) = \mathrm{col}(u_2, v_2)(n)$$

in $B_\Phi[0,\mu]$ we have

$$|\Lambda_1(n)(\mathcal{F}_1[x](n) - \mathcal{F}_1[y](n))|$$

$$= |\Phi_1^{-1}(n+1)\Phi_2^2(\sigma_2(n))(\Phi_2^{-1}(\sigma_2(n))v_1(\sigma_2(n)))^2 - (\Phi_2^{-1}(\sigma_2(n))v_2(\sigma_2(n)))^2)|$$

$$\leq \mu 2^{-\sigma_2(n)-n}|x-y|^{\infty}_{\Phi}.$$

From where

$$l_1(n,\mu) = \mu 2^{-\sigma_2(n)-n}, \quad l_2(n,\mu) = 0.$$

The hypothesis **[H2]** is satisfied since

$$\Omega(\mu) = \sum_{n=0}^{\infty} l_1(n,\mu) = \sum_{n=0}^{\infty} \mu 2^{-\sigma_2(n)-n} \leq \frac{2\mu}{3}.$$

Computing θ, we have

$$\theta = \frac{1}{1-\max\{\rho_1,\rho_2\}}.$$

If $\theta\Omega(\mu) = \frac{2\mu}{3(1-\max\{\rho_1,\rho_2\})} < 1$, then for every $\xi \in \mathbf{R}^2$ satisfying

$$|\xi| \leq \mu[1-\rho-\Omega(\mu)] = \mu[1-\max\{\rho_1,\rho_2\} - \frac{2\mu}{3}]$$

there exists a unique $\Phi-$bounded solution x of Eq. (13) in $B_\Phi[0,\mu]$. If $x(n) = \mathrm{col}(u(n),v(n))$, then $2^{-n}|u(n)| \leq \mu$ and $2^n|v(n)| \leq \mu$. This implies that $\{v(n)\}$ is bounded, but this is not necessarily true for the sequence $\{u(n)\}$.

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