Do Global Attractors Depend on Boundary Conditions?

Bernold Fiedler

Received: February 14, 1996

Communicated by Alfred K. Louis

Abstract. We consider global attractors of infinite dimensional dynamical systems given by dissipative partial differential equations

$$u_t = u_{xx} + f(x, u, u_x)$$

on the unit interval $0 < x < 1$ under separated, linear, dissipative boundary conditions. Global attractors are called orbit equivalent, if there exists a homeomorphism between them which maps orbits to orbits. The global attractor class is the set of all equivalence classes of global attractors arising for dissipative nonlinearities f. We show that the global attractor class does not depend on the choice of boundary conditions. In particular, Dirichlet and Neumann boundary conditions yield the same global attractor class.

The results are based on joint work with Carlos Rocha.

1 Attractor classes

Parabolic partial differential equations modelling reaction, diffusion, and drift are an important class of nonlinear infinite dimensional dynamical systems. Aside from applied motivation, much of the mathematical interest has centered on the dynamics of their finite dimensional global attractors. See for example [Hal88], [Lad91], [BV89], [Tem88], and the references there. The influence of boundary conditions has mainly been investigated in connection with stability of equilibria and shape of the underlying spatial domain, see for example [MM83], [Mat84].

Equations in one-dimensional domains have been studied in much detail, see for example [Cha74], [CI74], [Mat79], [Mat82], [Mat88], [Hen81], [Hen85], [Ang86], [Ang88], [BF88], [BF89], [AF88], [FMP89], [Nad90], [FP90]; mostly under Dirichlet or under Neumann boundary conditions, separately. In the present paper, we follow an approach developed more recently by [FR91], [Fie94], [FiRo94]. There the emphasis is on Neumann boundary conditions. Here, we indicate the necessary adaptations for general separated, linear, dissipative boundary conditions. Although the global attractors for a given nonlinearity will in general depend on our choice of boundary conditions, [HR87], we will show that the set of their orbit equivalence classes does not.

To be specific, we consider scalar equations

$$u_t = u_{xx} + f(x, u, u_x) \tag{1.1}$$

on the unit interval $0 \le x \le 1$. Fixing $0 \le \tau_0, \tau_1 \le 1$, we impose boundary conditions

$$(1 - \tau_\iota)u + \tau_\iota \partial_\nu u = 0 \tag{1.2}$$

at $x = 0, 1$. Here $\iota = 0,\ 1$; $\partial_\nu u = \pm u_x$ indicates the outward "normal" derivative with $+$ at $x = 1$, $-$ at $x = 0$, and subscripts t, x indicate partial derivatives of solutions $u = u(t, x)$. For the nonlinearities $f \in C^2$ we impose dissipation conditions

$$f(x, u, 0) \cdot u < 0, \tag{1.3}$$

for $|u| \ge C_1$ and, with continuous functions a, b as well as an exponent $\gamma < 2$

$$|f(x, u, p)| \le a(u) + b(u)|p|^\gamma \tag{1.4}$$

at all arguments (x, u, p) of f. The estimators C_1, a, b, γ are allowed to depend on f. This setting is fixed throughout this paper.

The dissipation conditions (1.3), (1.4) guarantee the local semiflow of x-profiles of solutions $u(t, \cdot) \in X^\tau,\ t \ge 0$, to be globally defined and dissipative: any solution eventually remains in a fixed large ball $B \subseteq X^\tau$. See [Ama85], theorem 5.3 for a reference. In fact, we can choose B such that $|u| < C_1$ and $|p| < C_2$ on B. The state space X^τ is the Sobolev space H^2 intersected with boundary conditions (1.2), $\tau = (\tau_0, \tau_1)$.

By dissipativeness, equations (1.1), (1.2) possess a *global attractor* $\mathcal{A}_f^\tau \subseteq X^\tau$. This is the maximal compact invariant subset of X^τ or, here equivalently, the set of bounded solutions $u(t, \cdot),\ t \in \mathbb{R}$. Yes, including negative t. This global attractor is our principal object of study here. We call a global attractor $\mathcal{A}_f^\tau$ *orbit equivalent* to $\mathcal{A}_g^\sigma$,

$$\mathcal{A}_f^\tau \cong \mathcal{A}_g^\sigma, \tag{1.5}$$

if there exists a homeomorphism $H : \mathcal{A}_f^\tau \to \mathcal{A}_g^\sigma$ which maps orbits $\{u(t, \cdot) \,|\, t \in \mathbb{R}\}$ on $\mathcal{A}_f^\tau$ onto orbits in $\mathcal{A}_g^\sigma$. Obviously, $\cong$ is an equivalence relation and defines orbit equivalence classes of global attractors.

Let $\mathcal{E}_f^\tau$ denote the set of equilibrium solutions $u_t = 0$ of (1.1), (1.2). Clearly, $\mathcal{E}_f^\tau \subseteq \mathcal{A}_f^\tau$. We assume all equilibria to be hyperbolic: all eigenvalues of corresponding Sturm-Liouville eigenvalue problem are nonzero, for linearizations at equilibria. This is a generic nondegeneracy condition on f, for any given τ.

For given boundary conditions τ, we define the *attractor class* $\mathcal{A}(\tau)$ as the set of orbit equivalence classes of global attractors $\mathcal{A}_f^\tau$. Here $f \in C^2$ are assumed to be dissipative, as in (1.3), (1.4), with only hyperbolic equilibria.

Theorem 1.1 *In the above setting, the global attractor class $\mathcal{A}(\tau)$ does not depend on the boundary conditions (1.2) given by $\tau = (\tau_0, \tau_1) \in Q := [0,1]^2$. In other words, let $\tau, \sigma \in Q$. Then*

$$\mathcal{A}(\tau) = \mathcal{A}(\sigma). \tag{1.6}$$

Specifically, for any dissipative $f \in C^2$ with hyperbolic equilibria $\mathcal{E}_f^\tau$ there exists a dissipative $g \in C^2$, also with hyperbolic equilibria $\mathcal{E}_g^\sigma$, such that the respective global attractors $\mathcal{A}_f^\tau, \mathcal{A}_g^\sigma$ are orbit equivalent

$$\mathcal{A}_f^\tau \cong \mathcal{A}_g^\sigma, \tag{1.7}$$

in the sense of definition (1.5).

In section 2 we prove theorem 1.1. We conclude with a discussion of our result, in section 3.

For $\sigma, \tau \in (0,1]^2$, excluding the Dirichlet cases, the theorem is very easy to prove. We use a rescaling argument by Rafael Ortega. Let

$$u(x) = A(x)\, v(x) \tag{1.8}$$

with some smooth amplitude function $A > 0$ satisfying

$$\begin{array}{rcl} \tau_\iota A(\iota) & = & \sigma_\iota \\ (1-\tau_\iota)A(\iota) + \tau_\iota \delta_\nu A(\iota) & = & 1 - \sigma_\iota \end{array} \tag{1.9}$$

at $\iota = 0, 1$. Then the transformation (1.8) defines a linear isomorphism between the state spaces $u \in X^\tau$ and $v \in X^\sigma$ associated to boundary conditions τ and σ. Also, v satisfies an equation (1.1) with an appropriately rescaled dissipative nonlinearity g instead of f. Therefore $A_f^\tau \cong \mathcal{A}_g^\sigma$, by (1.8), (1.9), and $\mathcal{A}(\tau) = \mathcal{A}(\sigma)$ in the non-Dirichlet cases. (Strictly speaking, though, the transformation does not preserve the precise form (1.3) of our dissipation condition.) Our slightly more involved proof, given in section 2, will include even the Dirichlet case. In particular, the Neumann and the Dirichlet attractor classes will be shown to coincide. Note that all spaces X^τ, including the Dirichlet cases, are closed linear subspaces of $X = H^2$ depending continuously on the parameters τ; in particular all these spaces are isomorphic from an abstract view point.

We briefly outline the Morse-Smale structure behind our proof of theorem 1.1, in the remainder of the present section. Following [FiRo96], we first normalize f, for simplicity, such that

$$f(x,u,p) = -u \tag{1.10}$$

for $|x| \geq C_1$ or $|p| \geq C_2$. Such a normalization can be achieved without changing $\mathcal{A}_f^\tau$ or the flow on it, by dissipation conditions (1.3), (1.4). For $u \in X^\tau$ consider functionals

$$V(u) := \int_0^1 F(x,u,u_x)dx. \tag{1.11}$$

Following [Mat88] we observe that

$$\frac{d}{dt}V(u(t,\cdot)) = -\int_0^1 F_{pp} \cdot u_t^2 dx \tag{1.12}$$

along solutions $u(t,x)$ of (1.1), (1.2), if F satisfies

$$pF_{ppu} - fF_{ppp} + F_{ppx} = f_pF_{pp}, \tag{1.13}$$

for all (x,u,p) and obeys the boundary condition

$$F_p \cdot u_t = 0 \tag{1.14}$$

at $x = 0, 1$. By the standard method of characteristics, [Joh82], it is easy to find a solution $w = w(x,u,p)$ of the first order equation

$$pw_u - fw_p + w_x = f_p; \tag{1.15}$$

see also (2.1). In fact, normalization condition (1.10), guarantees global solvability of the characteristic equation of (1.15) which is studied in more detail in section 2 below. Solving then

$$F_{pp} = \exp(w) \tag{1.16}$$

we have solved (1.13). The boundary conditions (1.14) hold trivially in the Dirichlet case. We require $F_p = 0$, as an initial condition for (1.16) with respect to p, along the lines in (x,u,p)-space given by the boundary conditions (1.2), in all other cases.

By this construction, $F_{pp} = \exp(w)$ is positive. In particular the functional V becomes a Lyapunov functional on X^τ which decreases strictly along non-equilibrium orbits. With respect to the Riemannian metric on X^τ defined by F_{pp}, the semiflow (1.1), (1.2) is in fact *gradient*, or *Morse* with respect to V.

The functional V reveals that the global attractor $\mathcal{A}_f^\tau$ consists entirely of equilibria $\mathcal{E}_f^\tau$ and *heteroclinic* or *connecting* orbits. These orbits, by definition, limit onto (different) equilibria $\tilde{u}, u$ for $t \to +\infty$, $\quad t \to -\infty$, respectively. They can be viewed as intersections of unstable and stable manifolds $W^u(u) \cap W^s(\tilde{u})$. Note that

$$\mathcal{A}_f^\tau = \mathcal{E}_f^\tau \cup \bigcup_{u \in \mathcal{E}_f^\tau} W^u(u). \tag{1.17}$$

Although this will not be very visible below, we emphasize the importance of *nodal properties* in our proof of theorem 1.1. Based on observations for linear equations, they imply that

$$t \mapsto z(u^1(t,\cdot) - u^2(t,\cdot)) \tag{1.18}$$

is nonincreasing along solutions $u^1(t,\cdot)$, $u^2(t,\cdot)$ of (1.1), (1.2). Here z, the zero number, denotes the number of strict sign changes of x-profiles. The zero number in (1.18) drops strictly whenever a multiple zero of the x-profile is encountered. Historically, the use of nodal properties dates back as far as [Stu36]. In [Mat82], their importance for infinite dimensional nonlinear dynamics was first realized. A comprehensive modern account of zero numbers is given in [Ang88].

The most striking consequence of nodal properties for our global attractors $\mathcal{A}_f^\tau$ is the *Morse-Smale property.* The Morse structure is generated by the Lyapunov functional V, as discussed above. By [Hen85], [Ang86], the intersections between stable and unstable manifolds which make up the global attractors $\mathcal{A}_f^\tau$ are automatically transverse, without further genericity assumption on f or τ:

$$W^u(u) \pitchfork W^s(\tilde{u}), \tag{1.19}$$

without any nondegeneracy assumptions except hyperbolicity of the equilibria $\mathcal{E}_f^\tau$.

Structural stability is the most important consequence of this Morse-Smale property. In fact, let g be C^2-near f and satisfy dissipation conditions (1.3), (1.4). Let also $\sigma \in Q = [0,1]^2$ be near τ. Then

$$\mathcal{A}_g^\sigma \cong \mathcal{A}_f^\tau, \tag{1.20}$$

as claimed in (1.7). For reference see [P69], [PS70], [PdM82], and for the infinite dimensional case [Oli92]. Since the argument is local in f, τ, this does not prove our theorem, of course.

2 Proof of theorem 1.1

If all equilibria $\mathcal{E}_f^\tau$ are hyperbolic, then the global attractor is Morse-Smale and therefore structurally stable, as we have seen at the end of the previous section. We give a geometric criterion for hyperbolicity of $\mathcal{E}_f^\tau$, in lemma 2.1. The criterion is based on a shooting approach to equilibria. In lemma 2.2, we relate global attractors for different boundary conditions, by an augmentation argument. Piecing these elements together, we finally prove theorem 1.1 by a homotopy argument which uses the Morse-Smale property.

Our geometric criterion for hyperbolicity is a slight adaptation of an argument in [Roc91]. Equilibria $u \in \mathcal{E}_f^\tau$ are solutions of

$$\begin{array}{rcl} \dot{u} &=& p \\ \dot{p} &=& -f(x,u,p) \\ \dot{x} &=& 1 \end{array} \tag{2.1}$$

which in addition satisfy the boundary conditions

$$\begin{array}{lll} l_0: & (1-\tau_0)u - \tau_0 p = 0, & \text{at } x = 0, \\ l_1: & (1-\tau_1)u + \tau_1 p = 0, & \text{at } x = 1. \end{array} \tag{2.2}$$

In passing we note that (2.1), together with $\dot{w} = f_p(x,u,p)$, are the equations of the characteristics of (1.15). For any real a, let $u(x,a)$, $p(x,a)$ denote the solution of (2.1) with initial condition

$$\begin{array}{rcl} u(0,a) &:=& \tau_0 a \\ p(0,a) &:=& (1-\tau_0)a \end{array} \tag{2.3}$$

at $x = 0$. By normalization (1.10), these solutions are globally defined. Define the *shooting surface* $S_f^\tau \subseteq [0,1] \times \mathbb{R}^2$ as

$$S_f^\tau := \{(x,u,p) \mid u = u(x,a), p = p(x,a), a \in \mathbb{R}\}. \tag{2.4}$$

The sections $S_f^{\tau,x} \subseteq \mathbb{R}^2$ of S_f^τ for given x are called *shooting curves* . The shooting curves are planar C^1 Jordan curves, parametrized by the shooting parameter $a \in \mathbb{R}$. The set $\mathcal{E}_f^\tau$ of equilibria is given by precisely those values $a \in \mathbb{R}$ where the shooting curve $S_f^{\tau,x}$ at $x = 1$ intersects the line l_1 of boundary conditions at $x = 1$.

LEMMA 2.1 *An equilibrium in $\mathcal{E}_f^\tau$ given by the shooting parameter a is hyperbolic if, and only if, the shooting curve $S_f^{\tau,x=1}$ intersects the target line l_1 transversely, at the intersection value a.*

PROOF: Consider the equilibrium $u(x,a)$ corresponding to the intersection value a, or vice versa. The partial derivative $(u_a(x,a), p_a(x,a))$ is *the* nontrivial solution of the linearized equation which satisfies the homogeneous linear boundary condition l_0. (Well, we could take constant multiples instead.) Clearly, u is nonhyperbolic if, and only if, this partial derivative also satisfies the other boundary condition l_1, at $x = 1$. Reinterpreting geometrically, nonhyperbolicity is then equivalent to the tangent vector of the shooting curve $S_f^{\tau,x=1}$: $a \mapsto (u(1,a), p(1,a))$ being parallel to the line l_1. This is exactly nontransversality of intersection, and the lemma is proved. □

Augmentation works as follows. We append new segments

$$\begin{array}{lcl} I_0 & := & [-\xi_0, 0) \\ I_1 & := & (1, 1+\xi_1] \end{array} \tag{2.5}$$

$\xi_0, \xi_1 > 0$, to the original x-interval $x \in [0,1]$. In the appended intervals, we define f by

$$f(x,u,p) := -\lambda^2(x)u, \tag{2.6}$$

where $\lambda^2(x) := \lambda_\iota^2 > 0$ is constant for $x \in I_\iota$. We will specify $\lambda = (\lambda_0, \lambda_1)$ below. In I_ι the shooting equation (2.1) becomes the hyperbolic linear equation

$$\begin{array}{lcl} \dot{u} & = & p \\ \dot{p} & = & \lambda_\iota^2 u \end{array} \tag{2.7}$$

In (u,p)-space, this linear equation induces a flow on the lines of boundary conditions

$$l(\tau_\iota): \quad (1-\tau_\iota)u \pm \tau_\iota p = 0. \tag{2.8}$$

Here $\iota = 1$ carries the plus-sign, whereas $\iota = 0$ requires a minus. The boundary condition parameters $\tau_\iota(x)$ are now considered to depend on $x \in I_\iota$, with their values at $x = \iota$ taken from the original boundary conditions (1.2). The flow induced by (2.7), (2.8) on τ_ι is

$$\dot{\tau}_\iota = \pm(\lambda_\iota^2 \tau_\iota^2 - (1-\tau_\iota)^2), \tag{2.9}$$

by direct calculation. This equation plays a central role in the proof of the following lemma.

For abbreviation, let $\mathcal{F}$ denote the set of dissipative nonlinearities $f, g \in C^2$ with hyperbolic equilibria as specified in theorem 1.1.

LEMMA 2.2 *Let $f \in \mathcal{F}$ and consider arbitrary boundary conditions $\tau \in (0,1)^2$, in the interior of the closed unit square $Q = [0,1]^2$, and $\sigma \in Q$. Then there exists $g \in \mathcal{F}$ such that the global attractors $\mathcal{A}_f^\tau, \mathcal{A}_g^\sigma$ are orbit equivalent,*

$$\mathcal{A}_f^\tau \cong \mathcal{A}_g^\sigma \tag{2.10}$$

Proof: We will use augmentation (2.5)–(2.9) to construct g on an interval $x \in [-\xi_0, 1+\xi_1]$. Rescaling x by a scaling factor $s := 1/(1+\xi_0+\xi_1)$ and adjusting σ accordingly, first, we will not lose generality. As a main step, we will then construct a dissipative homotopy from (f,τ) to (g,σ), by augmentation, such that hyperbolicity of equilibria is preserved throughout the homotopy. In a final, third step we address the issue of C^2-regularization of our piecewise defined nonlinearities, by smoothing. By Morse-Smale structural stability, the homotopy which preserves hyperbolicity of equilibria then proves the lemma.

Rescaling x to $\tilde{x}$ by $x = s(\tilde{x}+\xi)$ transforms the x-interval $[0,1]$ to an $\tilde{x}$-interval $[-\xi, 1+\xi]$, if we choose $s = (1+2\xi)^{-1} \in (0,1)$. Simultaneously, boundary conditions $\sigma = (\sigma_0, \sigma_1)$ at $x = 0,1$ get transformed to boundary conditions $\tilde{\sigma} = (\tilde{\sigma}_0, \tilde{\sigma}_1)$, for $\tilde{u}(t,\tilde{x}) := u(t,x)$, which are given explicitly by

$$\tilde{\sigma}_\iota = \frac{s\sigma_\iota}{1-\sigma_\iota+s\sigma_\iota}. \tag{2.11}$$

Note that $\tilde{\sigma}_\iota = 0,1$ for $\sigma_\iota = 0,1$, respectively.

We consider the case $0 \le \sigma_\iota < 1$ first. Fix $\xi = \bar{\xi} > 0$ large enough or, equivalently, $s = (1+2\bar{\xi})^{-1}$ small enough, such that in particular

$$0 \le \tilde{\sigma}_\iota < \tau_\iota < 1. \tag{2.12}$$

Now consider the τ_ι flow (2.9) in an equation which is augmented according to (2.6). We choose $\lambda_\iota > 0, \iota = 0,1$, such that the time which the τ_ι flow (2.9) takes from τ_ι to $\tilde{\sigma}_\iota$ coincides with the large prescribed value $\bar{\xi}$:

$$\tilde{\sigma}_\iota = \tau_\iota(\pm\bar{\xi}) \tag{2.13}$$

for the initial values $\tau_\iota(0) = \tau_\iota$ at $x = \iota$. Indeed this can be achieved by choosing $\lambda_\iota > 0$ such that the unique equilibrium

$$\tau_\iota^* = (1+\lambda_\iota)^{-1} \tag{2.14}$$

of (2.9) in $(0,1)$ is slightly above $\tau_\iota < 1$.

In the remaining Neumann case $\tilde{\sigma}_\iota = \sigma_\iota = 1$, we simply choose $\tau_\iota^* > 0$ slightly below $\tau_\iota > 0$, and (2.13) remains valid.

We describe our homotopy of attractors in terms of changing the boundaries $x = -\xi, 1+\xi$, simultaneously, from their original value $\xi = 0$ to their final values $\xi = \bar{\xi}$. On these larger x-intervals the nonlinearity f is augmented to f^ξ by (2.6). The boundary conditions $\tau = \tau(\xi)$ are adjusted, according to (2.9), in parallel with the homotopy parameter ξ. Note that by a rescaling of x with factor $s = 1/(1+2\xi)$, this induces a homotopy of global attractors for rescaled nonlinearities in the class $\mathcal{F}$. Clearly, dissipativeness is preserved. In view of Morse-Smale structural stability, it therefore only remains to prove that hyperbolicity of equilibria is preserved throughout the homotopy.

Hyperbolicity of equilibria follows from lemma 2.1. Indeed, transversality of the shooting curve $S_{f^\xi}^{\tau(\xi),x}$, at $x = 1+\xi$, to the line $l(\tau_1(\xi))$ follows in three steps, using (2.5)–(2.9). First, in $I_0 = [-\xi, 0)$, the initial line $l(\tilde{\sigma}_0) = l(\tau_0(\xi)) = S_{f^\xi}^{\tau(\xi),-\xi}$ gets mapped diffeomorphically to the line $l(\tau_0(0)) = l(\tau_0) = l_0$. Second, in $x \in [0,1]$, we obtain the original f-shooting curve

$$S_f^{\tau,1} \pitchfork l(\tau_1). \tag{2.15}$$

Here we use hyperbolicity of $\mathcal{E}_f^\tau$ and lemma 2.1. Third, in $I_1 = (1, 1+\xi]$, the line $l_1 = l(\tau_1) = l(\tau_1(0))$ and the f-shooting curve $S_f^{\tau,1}$ get mapped onto

$$S_{f^\xi}^{\tau(\xi),1+\xi} \quad \pitchfork l(\tilde{\sigma}_1), \tag{2.16}$$

by the shooting diffeomorphism. Transversality is inherited from (2.15). A final application of lemma 2.1 proves that hyperbolicity of equilibria is preserved during our homotopy $0 \le \xi \le \bar{\xi}$. Of course, rescaling of x does not affect hyperbolicity.

Smoothing the discontinuities of our augmentation of f, at $x = 0, 1$, we obtain a C^2-augmentation. Making the x-intervals, where smoothing acts, small enough, we can guarantee transversality (2.16) to hold throughout our homotopy $0 \le \xi \le \bar{\xi}$. In particular, all f^ξ are Morse-Smale. Defining g as (the rescaled version of) $f^{\bar{\xi}}$, structural stability of Morse-Smale systems finally implies

$$\mathcal{A}_f^\tau \cong \mathcal{A}_g^\sigma. \tag{2.17}$$

In (2.17) we have used that rescaling does not change the orbit type of the global attractor and, simultaneously, transforms $\tilde{\sigma} = \tau(\bar{\xi})$ to the boundary condition σ by (2.11). This proves the lemma. □

In the previous lemma we have shown $\mathcal{A}(\tau) = \mathcal{A}(\sigma)$, for attractor classes with $\tau, \sigma \in (0,1)^2$. (The transformation (1.8) would even allow for $\tau, \sigma \in (0,1]^2$.) To complete the proof of theorem 1.1, anyway, it remains to address the case of τ or σ in the boundary ∂Q of the square $Q = [0,1]$. If $g \in \mathcal{F}, \sigma \in \partial Q$, then local structural stability of Morse Smale systems shows that for $f := g$ and any $\tau \in (0,1)^2$ close to σ we have orbit equivalence $\mathcal{A}_g^\sigma \cong \mathcal{A}_f^\tau$. Therefore $\mathcal{A}(\sigma) \subseteq \mathcal{A}(\tau)$. To complete the proof of theorem 1.1 it remains to show that, conversely,

$$\mathcal{A}(\sigma) \supseteq \mathcal{A}(\tau), \tag{2.18}$$

for some $\tau \in (0,1)^2$. By lemma 2.2, claim (2.18) actually holds for all $\tau \in (0,1)^2, \sigma \in Q$. This completes our proof of theorem 1.1. □

3 DISCUSSION

We begin our discussion with remarks on x-dependent diffusion and on another attempt of simplifying our proof, by transformation of x. We then indicate why periodic boundary conditions $x \in S^1$ produce a class of Morse-Smale attractors quite different from the class $\mathcal{A}(\tau)$ of separated boundary conditions $\tau = (\tau_0, \tau_1) \in Q = [0,1]^2$. We conclude with a few comments on global attractors in the case of higher space dimension, dim $x > 1$, and the case of systems, dim $u > 1$.

Transforming x to $y = \eta(x) \in [0,1]$ in (1.1), (1.2) and denoting $v(y) := u(x)$ yields an equation

$$D(y)^{-1}\; v_t = v_{yy} + g(y, v, v_y) \tag{3.1}$$

with transformed boundary conditions

$$(1-\sigma_\iota)v \pm \sigma_\iota\; v_y = 0 \tag{3.2}$$

at $y = \iota = 0, 1$. Explicitly, we have

$$\begin{array}{rcl} D(y) & = & (\eta_x(x))^2 > 0, \\ \sigma_\iota & = & (1 + \frac{1-\tau_\iota}{\tau_\iota}\eta_x)^{-1}. \end{array} \tag{3.3}$$

Given $\tau \in (0,1)^2$ we can clearly reach all $\sigma \in (0,1)^2$ by a proper choice of the function η. The standard linear homotopy from $D(y)^{-1}v_t$ to v_t, in (3.1), is a Morse-Smale homotopy of attractors which does not change the equilibria. Indeed, the shooting surface never changes, during the homotopy, because $D(y)^{-1}$ only multiplies the time derivative. Therefore we conclude $\mathcal{A}_g^\sigma \cong \mathcal{A}_f^\tau$, as stated in lemma 2.2.

A main disadvantage of this rather simple argument is the fact that Neumann as well as Dirichlet boundary conditions $\tau_\iota = 0, 1$ remain unchanged by the transformation η; see (3.3). It is the case $\sigma \in \partial Q$, where we really seem to need the augmentation in lemma 2.2.

Of course we could have discussed orbit equivalence of attractors in the class of pairs (f, D), allowing for space–dependent diffusion from the very start. Fixing $D \equiv 1$, though, provides a stronger statement in theorem 1.1. Parenthetically we note that introducing $D > 0$ does not produce any additional global attractors, by the above arguments. As we have argued in the discussion section of [FiRo96], we do not expect additional global attractors to arise, even in fully nonlinear, uniformly parabolic, dissipative cases.

Passing to higher-dimensional domains $x \in \Omega \subseteq \mathbb{R}^d$, with $\partial\Omega$ smooth and bounded, we may again consider dissipative scalar equations

$$u_t = \Delta u + f(x, u, \nabla u) \tag{3.4}$$

on Ω, under mixed boundary conditions

$$(1-\tau)u + \tau\partial_\nu u = 0. \tag{3.5}$$

Now $\tau = \tau(x) \in [0,1]$ is a given function on $\partial\Omega$. A transformation $u(x) = A(x)v(x)$ is still feasible, normalizing $\tau \in (0,1]$ to become a uniform Neumann condition $\sigma \equiv 1$ for v; see (1.8), (1.9). But we have lost variational structure, nodal properties, and Morse-Smale when passing to (3.4), (3.5). Essentially arbitrary finite-dimensional flows occur in (3.4), see [Pol95]. Even if we assume the global attractor $\mathcal{A}_f^\tau$ to be structurally stable, there is no reason to believe that its orbit equivalence class is determined by the equilibria, alone.

To include the Dirichlet cases, it is tempting to try and augment Ω, by attaching a collar outside $\partial\Omega$, such that boundary conditions on the enlarged region differ from the original ones. A structurally stable attractor $\mathcal{A}_f^{\sigma,\Omega}$ should still be recovered in the enlarged region Ω'. If Ω is starshaped with respect to the origin, a homothety

$\Omega' = s\Omega$ by a scaling factor $s > 1$ comes to mind. In the annular region $A = \Omega' \setminus \Omega$ we can determine an eigenfunction for a positive eigenvalue λ of the Laplacian Δ with boundary conditions

$$\begin{aligned} (1-\tau)u - \tau\partial_{\nu'}u &= 0 \quad \text{on} \quad \partial\Omega \\ (1-\sigma)u + \sigma\partial_{\nu'}u &= 0 \quad \text{on} \quad \partial\Omega'. \end{aligned} \tag{3.6}$$

Here ν' denotes the outward normal of A; on $\partial\Omega$ we have $\nu' = -\nu$. In the one-dimensional case, this eigenfunction was the crucial shooting augmentation in the "annulus" $A = I_0 \cup I_1$. The task remains open to augment the PDE (3.4) in A in such a "singular way" that the original attractor $\mathcal{A}_f^{\tau,\Omega}$ is recovered on Ω' with new boundary conditions σ. For systems, $u \in \mathbb{R}^k$, a similar problem arises. Even in the case of one-dimensional x, though, is is not yet clear how to properly recover $\mathcal{A}_f^\tau$ on the enlarged interval Ω' then.

Jacobi systems are the spatially discrete ODE analogue to our scalar PDE (1.1), (1.2) in one space dimension; see [FO88]. Specifically, Jacobi systems have the tri-diagonal nonlinear form

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}), \tag{3.7}$$

$i = 0, \dots, n$, with strictly positive partial derivatives of the nonlinearities f_i with respect to the off-diagonal entries u_{i-1}, u_{i+1}. For convenience we impose linear boundary conditions in the following form

$$\begin{aligned} (1+\tau_0)u_{-1} - 2\tau_0 u_0 &= 0 \\ (1+\tau_1)u_{n+1} - 2\tau_1 u_n &= 0. \end{aligned} \tag{3.8}$$

System (3.7), (3.8) may, but need not, arise by finite difference semidiscretization in space of (1.1), (1.2). Then $\tau_\iota = 1$, $\iota = 0, 1$ corresponds to Neumann boundary conditions, as before, and $\tau_\iota = 0$ are Dirichlet conditions

$$u_{-1} = u_{n+1} = 0. \tag{3.9}$$

Only boundary conditions $0 \le \tau_\iota \le 1$ arise by discretization of dissipatively admissible PDE boundary conditions. Note, however, that the choice $\tau_\iota = -1$ again corresponds to Dirichlet boundary conditions

$$u_0 = u_n = 0, \tag{3.10}$$

at least formally.

For $\tau_\iota \neq -1$, the state space of our system (3.7), (3.8) is $u = (u_0, \cdots, u_n) \in X = \mathbb{R}^{n+1}$. A natural dissipation condition is

$$u_i \cdot f_i(u_i, u_i, u_i) < 0 \tag{3.11}$$

for all $i = 0, \cdots, n$, provided $|u_i| \ge C$. Here C is a large constant. Under boundary conditions (3.8) with

$$|\tau_0|, |\tau_1| \le 1, \tag{3.12}$$

condition (3.11) ensures that $\|u\| := \max |u_i|$ decreases to level C or below, eventually. If τ_0 or τ_1 violate condition (3.12), then $\max |u_i|$ may grow indefinitely on the boundary, in spite of dissipation condition (3.11). Therefore we restrict attention to the region (3.12).

For Neumann condition $\tau_\iota = 1$, it was argued in [FiRo96], theorem 8.2, that the attractor class $\mathcal{A}^{dis}(\tau)$ for Jacobi systems (3.7), (3.8) coincides with the PDE attractor class $\mathcal{A}^{con}(\tau) := \mathcal{A}(\tau)$ of our present theorem 1.1. Here $\mathcal{A}^{dis}(\tau)$ ranges over all Jacobi systems of any dimension n. For $0 \le \tau_\iota \le 1$, we expect similar arguments to provide

$$\mathcal{A}^{dis}(\tau) = \mathcal{A}^{con}(\tau) \tag{3.13}$$

to be τ-independent, by theorem 1.1.

Augmentation to $i \in \{-2, -1, \cdots, n+1, n+2\}$ also seems a viable approach to discrete attractor classes. Consider the new left boundary u_{-2}, u_{-1}, for example. Comparing a new boundary condition $\sigma = (\sigma_0, \sigma_1)$ with the old τ-condition (3.8), at the left end, we obtain

$$\begin{array}{rcl} (1+\tau_0)u_{-1} - 2\tau_0 u_0 & = & 0 \\ (1+\sigma_0)u_{-2} - 2\sigma_0 u_{-1} & = & 0. \end{array} \tag{3.14}$$

Adding the two equations with real coefficients $-\beta, \alpha$ we obtain the right hand side of an augmentation

$$\dot{u}_{-1} = \alpha(1+\sigma_0)u_{-2} - (2\alpha\sigma_0 + \beta(1+\tau_0))u_{-1} + 2\beta\tau_0 u_0. \tag{3.15}$$

This augmentation is Jacobi and dissipative, for $|\sigma_\iota|, |\tau_\iota| \le 1$, if

$$\begin{array}{rcl} \alpha & > & 0 \\ \beta\tau_0 & > & 0 \\ \alpha(1-\sigma_0) & < & \beta(1-\tau_0) \end{array} \tag{3.16}$$

Note that equilibrium shooting, $\dot{u}_{-1} \equiv 0$, maps the σ_0 boundary condition to the τ_0 condition under our choice (3.15) of augmentation.

Let $\mathcal{A}^n(\tau)$ denote the attractor class for Jacobi systems (3.7), (3.8), this time with fixed dimension $n+1$. In view of theorem 1.1 and (3.13) it seems natural to ask whether $\mathcal{A}^n(\tau)$ can be independent of τ, at least for $0 \le \tau_\iota \le 1$. More daringly: let $\mathcal{A}_n^{con}$ denote the set of attractor classes in $\mathcal{A}^{con}$ of dimension at most $n+1$. Is it true, for $0 \le \tau_\iota \le 1$ and at least for large n, that

$$\mathcal{A}^n(\tau) = \mathcal{A}_n^{con}? \tag{3.17}$$

In particular $\mathcal{A}^n(\tau)$ would not depend on τ, of course.

Transforming the boundary value at $i = -1$ by $s\tilde{u}_{-1} := u_{-1}$ requires $0 < s < 1$ to remain in the class of dissipative Jacobi systems where (3.11) holds. For $0 < |\tau_0| \le |\sigma_0| \le 1$ of equal sign we obtain an embedding

$$\mathcal{A}^n(\tau_0, \tau_1) \subseteq \mathcal{A}^n(\sigma_0, \sigma_1), \tag{3.18}$$

which does not quite answer our question. Dissipative Jacobi augmentation (3.15), (3.16) does not provide an answer, either. Some modest conclusions are

$$\begin{array}{c} \mathcal{A}^n(\tau_0, \tau_1) \subseteq \mathcal{A}^{n+1}(1, \tau_1) \\ \mathcal{A}^n(\tau_0, \tau_1) \subseteq \mathcal{A}^{n+1}(\sigma_0, \tau_1). \end{array} \tag{3.19}$$

Again $|\tau_\iota|, |\sigma_\iota| \leq 1$. In addition we require $\tau_0 \neq 0$ and, in the second equation, $\tau_0 \neq 1$. Aside from these constraints, τ and σ are arbitrary. Replacing i by $n-i$ we also observe symmetry for all τ,

$$\mathcal{A}^n(\tau_0, \tau_1) = \mathcal{A}^n(\tau_1, \tau_0). \tag{3.20}$$

For example, this implies Neumann embedding

$$\mathcal{A}^n(\tau_0, \tau_1) \subseteq \mathcal{A}^{n+2}(1,1), \tag{3.21}$$

for $\tau_0, \tau_1 \neq 1$. Similarly, for $\tau_0, \tau_1 \neq 0, 1$ and all σ we obtain

$$\mathcal{A}^n(\tau) \subseteq \mathcal{A}^{n+2}(\sigma) \tag{3.22}$$

from (3.19), (3.20).

Note that independence of $\mathcal{A}^{n+1}(\sigma)$ from σ might break down, at least for $\sigma_0 \searrow -1$. In that case, the boundary condition (3.14) collapses to $u_{-1} = 0$, formally. This is equivalent to the Dirichlet attractor class $\mathcal{A}^n(0, \sigma_1)$ of Jacobi systems in one lower dimension.

As a final remark, we emphasize that periodic boundary conditions $x \in S^1$ generate sets $\mathcal{A}^{con}(per)$, $\mathcal{A}^n(per)$ of attractor classes which are much richer than their colleagues $\mathcal{A}^{con}(\tau) = \mathcal{A}^{con}(sep)$ living in separated boundary conditions. In fact, the Neumann class can be shown to be contained in the periodic class $\tau = (1,1)$, by reflection through the boundary and smoothing:

$$\mathcal{A}^{con}(sep) \subset \mathcal{A}^{con}(per), \tag{3.23}$$

again by theorem 1.1. As remarked in [AF88], even for nonlinearities $f = f(u,p)$ independent of x, time periodic rotating waves can arise in $\mathcal{A}^{con}(per)$, which simply do not possess any counterpart in the gradient case $\mathcal{A}^{con}(sep)$. In particular, Lyapunov functionals like V fail. A similar remark applies to the spatially discrete case $\mathcal{A}^n(per)$ of cyclic Jacobi systems $i \pmod{(n+1)}$. Since reflection through the boundary for Neumann condition yields only an embedding

$$\mathcal{A}^n(1,1) \not\subseteq \mathcal{A}^{2n+1}(per),$$

the characterization of attractor classes in the case of periodic boundary conditions remains wide open.

REFERENCES

[Ama85] H. Amann. Global existence for semilinear parabolic systems. *J. Reine Angew. Math.*, 360: 47–83, (1985).

[AF88] S. Angenent and B. Fiedler. The dynamics of rotating waves in scalar reaction diffusion equations. *Trans. Amer. Math. Soc.*, 307: 545–568, (1988).

[Ang86] S. Angenent. The Morse-Smale property for a semi-linear parabolic equation. *J. Diff. Eq.*, 62: 427–442, (1986).

[Ang88] S. Angenent. The zero set of a solution of a parabolic equation. *J. reine angew. Math.*, 390: 79–96, (1988).

[BF88] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics Reported*, 1: 57–89, (1988).

[BF89] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: The complete solution. *J. Diff. Eq.*, 81: 106–135, (1989).

[BV89] A.V. Babin and M.I. Vishik. *Attractors in Evolutionary Equations.* Nauka, Moscow 1989.

[Cha74] N. Chafee. A stability analysis for a semilinear parabolic partial equation. *J. Differential Equations*, 15: 522–540, (1974).

[CI74] N. Chafee and E. Infante. A bifurcation problem for a nonlinear parabolic equation. *J. Applicable Analysis*, 4: 17–37, (1974).

[Fie94] B. Fiedler. Global attractors of one-dimensional parabolic equations: sixteen examples. *Tatra Mountains Math. Publ.*, 4: 67–92, (1994).

[FMP89] B. Fiedler and J. Mallet-Paret. Connections between Morse sets for delay-differential equations. *J. reine angew. Math.*, 397: 23–41, (1989).

[FO88] G. Fusco and W.M. Oliva. Jacobi matrices and transversality. *Proc. Royal Soc. Edinburgh* A, 109: 231–243, (1988).

[FP90] B. Fiedler and P. Poláčik. Complicated dynamics of scalar reaction diffusion equations with a nonlocal term. *Proc. Royal Soc. Edinburgh* A, 115: 167–192, (1990).

[FR91] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *J. Diff. Eq.*, 91: 75–94, (1991).

[FiRo94] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations. Preprint, to appear in J. Diff. Eqs., (1994).

[FiRo96] B. Fiedler and C. Rocha. Orbit equivalence of global attractors of semilinear parabolic differential equations. Preprint, (1996).

[Hal88] J.K. Hale. *Asymptotic Behavior of Dissipative Systems.* Math. Surv. 25. AMS Publications, Providence, 1988.

[Hen81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lect. Notes Math. 840. Springer-Verlag, New York, 1981.

[Hen85] D. Henry. Some infinite dimensional Morse-Smale systems defined by parabolic differential equations. *J. Diff. Eq.*, 59: 165–205, (1985).

[HR87] J. Hale and C. Rocha. Interaction of diffusion and boundary conditions. *Nonlinear Analysis*: 11: 633–649, (1987).

[Joh82] F. John. *Partial Differential Equations.* Springer-Verlag, New York, 1982.

[Lad91] O.A. Ladyzhenskaya. *Attractors for Semi-groups and Evolution Equations.* Cambridge University Press, 1991.

[Mat79] H. Matano. Asymptotic behavior and stability solutions of semilinear diffusion equations. *Publ. Res. Inst. Math. Sc. Kyoto Univ.*, 15: 401–454, (1979).

[Mat82] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semi-linear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sec. IA*, 29: 401–441, (1982).

[Mat84] H. Matano. Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems. *J. Fac. Sci. Univ. Tokyo Sec. IA-Math.*, 3: 645–673, (1984).

[Mat88] H. Matano. Asymptotic behavior of solutions of semilinear heat equations on S^1. In *Nonlinear Diffusion Equations and their Equilibrium States II.* Springer Verlag, New York, 1988.

[MM83] H. Matano and M. Mimura. Pattern formation in competition-diffusion systems in nonconvex domains. *Publ. Res. Inst. Math. Sci.*, 19: 1049–1079, (1983).

[Nad90] N.S. Nadirashvili. On the dynamics of nonlinear parabolic equations. *Soviet Math. Dokl.*, 40: 636–639, (1990).

[Oli92] W.M. Oliva. Stability of Morse-Smale maps. Preprint, (1992).

[P69] J. Palis Jr. On Morse-Smale dynamical systems. *Topology*, 8: 385–404, (1969).

[PdM82] J. Palis Jr. and W. de Melo. *Geometric Theory of Dynamical Systems.* Springer Verlag, New York, 1982.

[Pol95] P. Poláčik. High-dimensional ω-limit sets and chaos in scalar parabolic equations. *J. Diff. Eq.*, 119: 24–53, (1995).

[PS70] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis.* Proc. Symp. in Pure Math. vol. XIV. AMS, Providence, 1970.

[Roc91] C. Rocha. Properties of the attractor of a scalar parabolic PDE. *J. Dyn. Diff. Eq.*, 3: 575–591, (1991).

[Stu36] C. Sturm. Sur une classe d'équations à différences partielles. *J. Math. Pure Appl.*, 1: 373–444, (1836).

[Tem88] R. Teman. *Infinite Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1988.

Bernold Fiedler
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin
Germany
fiedler@math.fu-berlin.de