Stabilization of Periods of Eisenstein Series and Bessel Distributions on $GL(3)$ Relative to $U(3)$

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Received: February 7, 2000
Revised: June 1, 2000

Communicated by Don Blasius

Abstract. We study the regularized periods of Eisenstein series on $GL(3)$ relative to $U(3)$. A stabilization procedure is used to express the periods in terms of L-functions. This is combined with the relative trace formula Jacquet and Ye to obtain new identities on Bessel distributions.

1991 Mathematics Subject Classification: 11F30, 11F67, 11F70, 11F72
Keywords and Phrases: Eisenstein series, Bessel distributions, periods, stabilization

1 Introduction

Let G be a reductive group over a number field F and H the fixed point set of an involution θ of G. The *period* of a cusp form ϕ on G is defined as the integral

$$\Pi^H(\phi) = \int_{H(F)\backslash(H(\mathbb{A})\cap G(\mathbb{A})^1)} \phi(h)\, dh,$$

known to converge by [AGR]. Recall that a cuspidal representation $\pi = \otimes\pi_v$ of G is said to be *distinguished* by H if there exists an element ϕ in the space V_π of π such that $\Pi^H(\phi) \neq 0$. It is known in some cases that the period factors as a product of local H_v-invariant functionals, even when there is no local uniqueness for such functionals (cf. [J2]), and that it is related to special

[1] Supported by NSF Grant DMS 97-29992
[2] Partially supported by NSF Grant 9700950

values of L-functions. Periods of more general automorphic forms, such as an Eisenstein series, are also of interest. Although the above integral need not converge, it should be possible to regularize it, as has been carried out in [JLR] and [LR] when (G, H) is a Galois pair, i.e., when H is the fixed point set of a Galois involution on G. The regularized period of cuspidal Eisenstein series was computed for the pair $(GL(n)_E, GL(n)_F)$ in [JLR]. It is either identically zero or can be expressed as a ratio of Asai L-functions (up to finitely many local factors at the ramified places). In general, however, the result will be more complicated.

In this paper we study the pair $(GL(3)_E, U(3))$ in detail in order to illustrate some phenomena which are likely to appear in the general case. Our goal is two-fold. First, we introduce a stabilization procedure to express the period of an Eisenstein series induced from a Borel subgroup as a sum of terms, each of which is factorizable with local factors given almost everywhere by ratios of L-factors. This is reminiscent of the procedure carried out in [LL] for the usual trace formula. Second, we define a stable version of the relative Bessel distributions occurring in the relative trace formula developed by Jacquet-Ye. We then use the comparison of trace formulae carried out in [JY] to prove some identities between our stable relative Bessel distributions on $GL(3)_E$ and Bessel distributions on $GL(3)_F$.

The main motivation for this work comes form the relative trace formula (RTF), introduced by Jacquet to study distinguished representations. One expects in general that the distinguished representations are precisely those in the image of a functorial lifting from a group G' determined by the pair (G, H). For example, G' is $GL(n)_F$ for the pair $(GL(n)_E, U(n))$. To that end one compares the RTF for G with the Kuznetzov trace formula (KTF) for G'. This was first carried out in [Y] for the group $GL(2)$. The cuspidal contribution to the RTF appears directly as sum of relative Bessel distributions (defined below) attached to distinguished representations of G. It should match term by term with the corresponding sum in the KTF of Bessel distributions attached to cuspidal representations of G'. Examples ([J1], [JY], [GJR]) suggest that the contribution of the continuous spectrum of G can also be written as integrals of relative Bessel distributions built out of regularized periods of Eisenstein series. However, these terms cannot be matched up with the continuous part of the KTF directly. Rather, as we show in our special case, the matching can be carried out using the stable relative Bessel distributions.

We now describe our results in greater detail. Assume from now on that (G, H) is a Galois pair, that is $G = \mathrm{Res}_{E/F} H$ where E/F is a quadratic extension and θ is the involution induced by the Galois conjugation of E/F. We also assume that H is quasi-split. By abuse of notation, we treat G as a group over E, identifying it with H_E. The regularized period of an automorphic form ϕ, also denoted $\Pi^H(\phi)$, can be defined using a certain truncation operator $\Lambda_m^T \phi$ depending on a parameter T in the positive Weyl chamber. For T sufficiently regular, the integral

$$\int_{H(F)Z\backslash H(\mathbb{A})} \Lambda_m^T \phi(h)\; dh$$

is a polynomial exponential function of T, i.e., it has the form $\sum p_j(T)e^{\langle \lambda_j, T\rangle}$ for certain polynomials p_j and exponents λ_j. Under some restrictions on the exponents of ϕ, the polynomial $p_0(T)$ is constant and $\Pi^H(\phi)$ is defined to be its value.

When $\phi = E(\varphi, \lambda)$ is a cuspidal Eisenstein series, $\Pi^H(\phi)$ can be expressed in terms of certain linear functionals $J(\eta, \varphi, \lambda)$ called *intertwining periods* ([JLR], [LR]). To describe this, consider for simplicity the case of an Eisenstein series induced from the Borel subgroup $B = TN$. We assume that B, T and N are θ-stable. Given a character χ of $T(E)\backslash T(\mathbb{A}_E)$, trivial on $Z(\mathbb{A}_E)$, and λ in the complex vector space $\mathfrak{a}^*_{0,\mathbb{C}}$ spanned by the roots of G, the Eisenstein series

$$E(g, \varphi, \lambda) \;=\; \sum_{\gamma \in B(E)\backslash G(E)} \varphi(\gamma g)\; e^{\langle \lambda, H(\gamma g)\rangle}$$

converges for $\operatorname{Re}\lambda$ sufficiently positive. Here, as usual, $\varphi : G(\mathbb{A}) \to \mathbb{C}$ is a smooth function such that $\varphi(bg) = \delta_B(b)^{\frac{1}{2}}\chi(b)\varphi(g)$. According to a result of T. Springer [S], each double coset in $B(E)\backslash G(E)/H(F)$ has a representative η such that $\eta\theta(\eta)^{-1}$ lies in the normalizer $N_G(T)$ of T. Denoting the class of $\eta\theta(\eta)^{-1}$ in the Weyl group W by $\left[\eta\theta(\eta)^{-1}\right]$, we obtain a natural map

$$\iota : B(E)\backslash G(E)/H(F) \;\; \to \;\; W$$

sending $B(E)\eta H(F)$ to $\left[\eta\; \theta(\eta)^{-1}\right]$. For such η, set

$$H_\eta \;\;=\;\; H \,\cap\, \eta^{-1}B\eta.$$

The intertwining period attached to η is the integral

$$J(\eta, \varphi, \lambda) \quad = \quad \int_{H_\eta(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} e^{\langle \lambda, H(\eta h)\rangle}\;\; \varphi(\eta h)\; dh$$

where dh is a semi-invariant measure on the quotient $H_\eta(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$. The result of [LR] alluded to above is that for suitable χ and λ the integral defining $J(\eta, \lambda, \varphi)$ converges and that

$$\Pi^H(E(\varphi, \lambda)) \;\; = \delta_\theta \;\cdot\; c \;\cdot\; \sum_{\iota(\eta) = w} J(\eta, \varphi, \lambda) \tag{1}$$

where w is the *longest element* in W and $c = \mathrm{vol}(H_\eta(F)Z(\mathbb{A}_F)\backslash H_\eta(\mathbb{A}_F))$. Here, δ_θ is 1 if θ acts on $\mathfrak{a}_0^*$ as $-w$ and it is 0 otherwise.

If $G = GL(2)_E$ and $H = GL(2)_F$, $\iota^{-1}(w)$ consists of a single coset $B(E)\eta H(F)$ and $\Pi^H(E(\varphi, \lambda))$ is either zero or proportional to $J(\eta, \varphi, \lambda)$. More generally, if

$G = GL(n)_E$ and $H = GL(n)_F$, the regularized period of a cuspidal Eisenstein series is either zero or is proportional to a single intertwining period. This intertwining period factors as a product of local integrals which are equal almost everywhere to a certain ratio of Asai L-functions ([JLR]). For general groups, however, the sum in (1) is infinite and $\Pi^H(E(\varphi,\lambda))$ cannot be expressed directly in terms of L-functions. This occurs already for $G = SL(2)_E$ and $H = SL(2)_F$. This is related to the fact that base change is not necessarily one-to-one for induced representations. See [J2] for a discussion of the relation between non-uniqueness of local H-invariant functionals and the non-injectivity of base change for the pair $(GL(3)_E, U(3))$.

For the rest of this paper, let $G = GL(3)_E$, $G' = GL(3)_F$, and let $H = U(3)$ be the quasi-split unitary group in three variables relative to a quadratic extension E/F and the Hermitian form

$$\Phi = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Let T and T' be the diagonal subgroups of G and G', respectively, and $\mathrm{Nm} : T \to T'$ the norm mapping. We shall fix a unitary character χ of $T(\mathbb{A}_E)$ which is a base change lifting with respect to Nm of a unitary character of $T'(F)Z'(\mathbb{A}_F)\backslash T'(\mathbb{A}_F)$. We write $\mathcal{B}(\chi) = \{\nu\}$ for the set of four characters of $T'(F)Z'(\mathbb{A}_F)\backslash T'(\mathbb{A}_F)$ such that $\chi = \nu \circ \mathrm{Nm}$. The stable intertwining periods $J^{st}(\nu,\varphi,\lambda)$ are defined in §8. Each functional $J^{st}(\nu,\varphi,\lambda)$ is factorizable and invariant under $H(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the ring of finite adeles. Our first main result is that with a suitable normalization of measures, we have the following

Theorem 1.

$$\Pi^H(E(\varphi,\lambda)) \quad = \sum_{\nu\in\mathcal{B}(\chi)} J^{st}(\nu,\varphi,\lambda)$$

In particular, this expresses the left-hand side as a sum of factorizable distributions.

In Proposition 3, §7, we show that the local factors at the unramified places are given by ratios of L-functions. This is done by a lengthy calculation which can fortunately be handled using Mathematica. Hence we obtain a description of $\Pi^H(E(\varphi,\lambda))$ in terms of L-functions.

We now describe our results on Bessel distributions. In general, if (π, V) is a unitary admissible representation of $G(\mathbb{A})$ and $L_1, L_2 \in V^*$ are linear functionals, we may define a distribution on the space of compactly-supported, **K**-finite functions by the formula

$$O(f) = \sum_{\{\phi\}} L_1(\pi(f)\phi)\, \overline{L_2(\phi)}$$

where $\{\phi\}$ is an orthonormal basis of V consisting of $\mathbf{K}$-finite vectors. The sum is then finite and O is independent of the choice of basis. The distributions occurring in the KTF and RTF are all of this type. They are referred to as Bessel distributions and relative Bessel distributions in the two cases, respectively.
In this paper, the representations of G and G' that we consider are all assumed to have trivial central character. Correspondingly we will consider factorizable functions $f = \prod f_v$ (resp. $f' = \prod f'_v$) on $G(\mathbb{A}_E)$ (resp. $G'(\mathbb{A})$) of the usual type such that f_v (resp. f'_v) is invariant under the center Z_v (resp. Z'_v) of G_v (resp. G'_v) for all v. In particular, we define $\pi(f) = \int_{G(E)Z(E)\backslash G(\mathbb{A}_E)} f(g)\pi(g)dg$ and similarly for $\pi'(f')$. The *relative Bessel distribution* attached to a cuspidal representation (π, V_π) of G is defined by

$$\widetilde{B}(f,\pi) = \sum_{\{\phi\}} \Pi^H(\pi(f)\phi)\ \overline{\mathbb{W}(\phi)} \tag{2}$$

and the *Bessel distribution* attached to a cuspidal representation (π', V') of G' is defined by

$$B(f',\pi') = \sum_{\{\phi'\}} \mathbb{W}'(\pi'(f')\phi')\overline{\mathbb{W}'(\phi')}.$$

Here $\{\phi\}$ and $\{\phi'\}$ are orthonormal bases of V and V', respectively, and $\mathbb{W}(\phi)$ and $\mathbb{W}'(\phi')$ are the Fourier coefficients defined in §2. They depend on the choice of an additive character ψ which will remain fixed.
Jacquet and Ye have studied the comparison between the relative trace formula on G and the Kuznetzov trace formula on G' under the assumption that E splits at all real places of F ([J1], [JY]). They define a local notion of matching functions $f_v \leftrightarrow f'_v$ for all v and prove the identity

$$RTF(f) = KTF(f')$$

for all global function $f = \prod_v f_v$ and $f' = \prod f'_v$ such that $f \leftrightarrow f'$. By definition, $f \leftrightarrow f'$ if $f_v \leftrightarrow f'_v$ for all v. It follows from the work of Jacquet-Ye that if $f \leftrightarrow f'$, then

$$\widetilde{B}(f,\pi) = B(f',\pi') \tag{3}$$

for any cuspidal representation π' of $G'(\mathbb{A}_F)$ with base change lifting π on $G(\mathbb{A}_E)$. Our goal is to formulate and prove an analogous result for Eisensteinian automorphic representations.
Assume that χ is *unitary* and let

$$I(\chi,\lambda) = \mathrm{Ind}_{B(\mathbb{A}_E)}^{G(\mathbb{A}_E)}\ \chi\cdot e^{\langle\lambda,H(\cdot)\rangle}$$

be an induced representation of $G(\mathbb{A}_E)$. In this case, we define a relative Bessel distribution in terms of the regularized period as follows:

$$\widetilde{B}(f,\chi,\lambda) = \sum_{\{\varphi\}} \Pi^H(E(I(f,\chi,\lambda)\varphi,\lambda))\ \overline{\mathcal{W}}(\varphi,\lambda)$$

where $\{\varphi\}$ runs through an orthonormal basis of $I(\chi, \lambda)$ and $\mathcal{W}(\varphi, \lambda) = \mathbb{W}(E(\varphi, \lambda))$. Throughout the paper we will use the notation $\overline{\mathcal{W}}(\cdot, \cdot)$ for the complex conjugate of $\mathcal{W}(\cdot, \cdot)$. For $\nu \in \mathcal{B}(\chi)$, the Bessel distribution is defined by

$$B'(f', \nu, \lambda) = \sum_{\{\varphi'\}} \mathcal{W}'(I(f', \nu, \lambda)\varphi, \lambda) \overline{\mathcal{W}'}(\varphi', \lambda),$$

where $\mathcal{W}'(\varphi', \lambda)$ is defined similarly. However, the equality (3) no longer holds. In fact, it is not well-defined since there is more than one automorphic representation π' whose base change lifting is $I(\chi, \lambda)$. However, for $\nu \in \mathcal{B}(\chi)$ we may define

$$\tilde{B}^{st}(f, \nu, \lambda) = \sum_{\{\varphi\}} J^{st}(\nu, \varphi, \lambda) \overline{\mathcal{W}}(\varphi, \lambda). \tag{4}$$

With this definition, Theorem 1 allows us to write

$$\widetilde{B}(f, \chi, \lambda) = \sum_{\nu \in \mathcal{B}(\chi)} \tilde{B}^{st}(f, \nu, \lambda).$$

Our next main result is the following

THEOREM 2. *Assume that the global quadratic extension E/F is split at the real archimedean places. Fix a* unitary *character χ and $\nu \in \mathcal{B}(\chi)$. Then*

$$\tilde{B}^{st}(f, \nu, \lambda) = B'(f', \nu, \lambda)$$

for all matching functions $f \leftrightarrow f'$.

There is a local analogue of this Theorem. The distributions $\tilde{B}^{st}(f, \nu, \lambda)$ are factorizable. Their local counterparts are defined in terms of Whittaker functionals and local intertwining periods. Let E/F be a quadratic extension of p-adic fields. For any character μ of F^*, set

$$\gamma(\mu, s, \psi) = \frac{L(\mu, s)}{\epsilon(\mu, s, \psi) L(\mu^{-1}, 1-s)}.$$

Denote by ω the character of F^* attached to E/F by class field theory. For any character $\nu = (\nu_1, \nu_2, \nu_3)$ of $T'(F)$ and $\lambda \in i\mathfrak{a}_0^*$, set

$$\gamma(\nu, \lambda, \psi) = \gamma(\nu_1 \nu_2^{-1} \omega, s_1, \psi) \gamma(\nu_2 \nu_3^{-1} \omega, s_2, \psi) \gamma(\nu_1 \nu_3^{-1} \omega, s_3, \psi). \tag{5}$$

with notation as in §2.

THEOREM 3. *Let E/F be a quadratic extension of p-adic fields. There exists a constant $d_{E/F}$ depending only on the extension E/F with the property: for all* unitary *characters ν of $T'(F)$,*

$$\tilde{B}^{st}(f, \nu, \lambda) = d_{E/F} \gamma(\nu, \lambda, \psi) B'(f', \nu, \lambda).$$

whenever $f \leftrightarrow f'$. Moreover, if E/F is unramified and $p \neq 2$ then $d_{E/F} = 1$.

We determine the constant $d_{E/F}$ for E/F unramified and $p \neq 2$ by taking f to be the identity in the Hecke algebra and directly comparing both sides of the equality. As remarked, this involves an elaborate Mathematica calculation. Determining $d_{E/F}$ in general would require more elaborate calculations which we have not carried out. We remark, however, that in a global situation we do have $\prod d_{E_v/F_v} = 1$.

1.1 Acknowledgement

The first author would like to thank the Institute for Advanced Study in Princeton for its kind hospitality he enjoyed during the academic year 1998–99.

2 Notation

Throughout, E/F will denote a quadratic extension of global or local fields of characteristic zero. In the local case we also consider $E = F \oplus F$. In the global case we make the following assumption on the extension E/F:

$$E \text{ splits at every real place of } F. \tag{6}$$

The character of F attached to E by class field theory will be denoted ω. In the local case, if $E = F \oplus F$, then ω is trivial, and $\mathrm{Nm} : E \to F$ is the map $(x, y) \to xy$.

As in the introduction, $H = U(3)$ denotes the quasi-split unitary group with respect to E/F and the Hermitian form Φ, and we set $G = GL(3)_E$ and $G' = GL(3, F)$. We shall fix some notation and conventions for the group G. Similar notation and conventions will be used for G' with a prime added.

We write B for the Borel subgroup of G of upper triangular matrices and $B = TN$ for its Levi decomposition where T is the diagonal subgroup. Let W be the Weyl group of G. The standard maximal compact subgroup of $G(\mathbb{A})$ will be denoted $\mathbf{K}$. In the local case we write K. We have the Iwasawa decompositions $G(\mathbb{A}) = T(\mathbb{A})N(\mathbb{A})\mathbf{K} = N(\mathbb{A})T(\mathbb{A})\mathbf{K}$. We fix the following Haar measures. Let dn and dt be the Tamagawa measures on $N(\mathbb{A})$ and $T(\mathbb{A})$, respectively. Then $\mathrm{vol}(N(F)\backslash N(\mathbb{A})) = 1$. We fix dk on $\mathbf{K}$ by the requiring $\mathrm{vol}(\mathbf{K}) = 1$. Let dg the Haar measure $dt\, dn\, dk$. We define Haar measures for G' and H similarly.

Let α_1, α_2 be the standard simple roots and set $\alpha_3 = \alpha_1 + \alpha_2$. Denote the associated co-roots $\alpha_1^\vee$, $\alpha_2^\vee$, $\alpha_3^\vee$. Let $\mathfrak{a}_0^*$ be the real vector space spanned by the roots, and let $\mathfrak{a}_0$ be the dual space. For $\lambda \in \mathfrak{a}_0^*$ set $s_i = \langle \lambda, \alpha_i^\vee \rangle$ for $i = 1, 2, 3$. If M is a Levi subgroup containing T, let $\mathfrak{a}_M \subset \mathfrak{a}_0$ be the subspace spanned by the co-roots of the split component of the center of M. The map $H : G(\mathbb{A}) \to \mathfrak{a}_0$ is characterized, as usual, by the condition $e^{\langle \alpha_i, H(ntk) \rangle} = |\alpha_i(t)|$. We write $d(a, b, c)$ for the diagonal element

$$d(a, b, c) = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}.$$

If M is a Levi subgroup of G, π is an admissible representation of M (locally or globally) and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ we write $I(\pi,\lambda)$ for the representation of G unitary induced from the representation $m \to \pi(m)e^{\langle\lambda,H(m)\rangle}$. In the global case, we let $E(g,\varphi,\lambda)$ be the Eisenstein series on $G(\mathbb{A})$ induced by φ.
If χ is a unitary character of $T(\mathbb{A})$, we identify the induced space $I(\chi) = I(\chi,\lambda)$, with the pre-Hilbert space of smooth functions $\varphi : G(\mathbb{A}) \to \mathbb{C}$ such that $\varphi(ntg) = \delta_B^{1/2}(t)\chi(t)\varphi(g)$ for $n \in N(\mathbb{A})$ and $t \in T(\mathbb{A})$. We use the notation δ_Q to denote the modulus function of a group Q. The scalar product is given by

$$(\varphi_1,\varphi_2) = \int_{B(\mathbb{A})\backslash G(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\; dg = \int_{\mathbf{K}} \varphi_1(k)\overline{\varphi_2(k)}\; dk.$$

The representation $I(\chi,\lambda)$ is defined by

$$I(g,\chi,\lambda)\varphi(g') = e^{-\left\langle\lambda,H(g')\right\rangle}e^{\left\langle\lambda,H(g'g)\right\rangle}\varphi(g'g).$$

It is unitary if $\lambda \in i\mathfrak{a}_0^*$. Similar notation will be used in the local case. Let $w \in W$ and let $w\chi(t) = \chi(wtw^{-1})$. The (unnormalized) intertwining operator

$$M(w,\lambda) : I(\chi,\lambda) \to I(w\chi,w\lambda)$$

is defined by

$$[M(w,\lambda)\varphi](g) = \int_{(N\cap w^{-1}Nw)\backslash N} \varphi(wng)\; dn.$$

It is absolutely convergent in a suitable cone and admits a meromorphic continuation in λ.
Let $\mathcal{B}(\chi) = \{\nu\}$ be the set of four Hecke characters of the diagonal subgroup $T'(\mathbb{A}_F)$ of $G'(\mathbb{A}_F)$ such that ν is trivial on the center $Z'(\mathbb{A}_F)$ and $\chi = \nu\circ\mathrm{Nm}$. In the non-archimedean case, let $\mathcal{H}_G$ be the Hecke algebra of compactly-supported, bi -K-invariant functions on G. Let $\hat{f}(\chi,\lambda)$ be the Satake transform, i.e., $\hat{f}(\chi,\lambda)$ is the trace of f acting on $I(\chi,\lambda)$. Define $\mathcal{H}_{G'}$ and $\hat{f}'(\nu,\lambda)$ similarly. We define the base change homomorphism

$$\mathrm{bc} : \mathcal{H}_G \to \mathcal{H}_{G'}$$

in the usual way. By definition, if $f' = \mathrm{bc}(f)$ then $\hat{f}(\nu\circ Nm,\lambda) = \hat{f}'(\nu,\lambda)$ for any unramified character ν of $T'(F)$.
We fix a non-trivial additive character ψ of $F\backslash\mathbb{A}_F$. The ψ-Fourier coefficient of an automorphic form on G is defined by

$$\mathbb{W}(\phi) = \int_{N(E)\backslash N(\mathbb{A}_E)} \phi(n)\;\overline{\psi_N(n)}\; dn$$

where

$$\psi_N(\begin{pmatrix}1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1\end{pmatrix}) = \psi(\mathrm{tr}\, x + \mathrm{tr}\, y).$$

The ψ-Fourier coefficient $\mathbb{W}'(\phi)$ of an automorphic form on G' is defined in a similar way with respect to the character

$$\psi_{N'}(\begin{pmatrix}1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1\end{pmatrix}) = \psi(x+y).$$

If $\varphi \in I(\pi, \lambda)$, we set

$$\mathcal{W}(\varphi, \lambda) = \mathbb{W}(E(\varphi, \lambda)).$$

In this paper, we use a different Hermitian form from the one used in [JY]. This forces us to modify the definition of matching functions, namely we have to take a left translate of f by τ where ${}^t\overline{\tau}\tau = \Phi$.

Part I
The regularized Period

3 Double Cosets

Let $\theta : G \to G$ be the involution

$$\theta(g) \quad = \quad \Phi^{-1}\ {}^t\overline{g}^{-1}\Phi.$$

Then $H = U(3)$ is the fixed-point set of θ. Note that θ preserves B, T, N and $\mathbf{K}$. We shall consider the space

$$\mathcal{S}_0 = \{s \in G : \ \theta(s) = s^{-1}\}$$

and its quotient modulo scalars

$$\mathcal{S} = \{s \in G : \ \theta(s) = s^{-1}\}/F^*.$$

Remark 1. The space $\mathcal{S}_0$ is a translate by Φ^{-1} of the space of non-degenerate Hermitian forms. Indeed, $s \in \mathcal{S}_0$ if and only if $\Phi^{-1}\ {}^t\overline{s} = s\Phi^{-1}$, i.e. $s\Phi^{-1}$ is Hermitian.

The group G acts on $\mathcal{S}$ via $s \to \ gs\theta(g)^{-1}$. This is compatible with the action on Hermitian forms. The stabilizer of s is the subgroup

$$H_s = \{g \in G : \quad gs\Phi^{-1}\ {}^t\overline{g} = \lambda s\phi^{-1} \quad \text{for some} \quad \lambda \in F^*\}.$$

Thus H_s is the unitary similitude group of the Hermitian form $s\Phi^{-1}$. There are only finitely many equivalence classes of Hermitian forms modulo scalars and hence G has finitely many orbits in $\mathcal{S}$. We obtain a bijection

$$\coprod_{\{s\}} G/H_s \to \mathcal{S}$$
$$g \to gs\theta(g)^{-1}$$

where $\{s\}$ is a set of orbit representatives. In fact, our assumption (6) implies that there is only one orbit.
Consider the B-orbits in $\mathcal{S}$. According to a result of Springer ([S]), every B-orbit in $\mathcal{S}$ intersects the normalizer $N_G(T)$ of the diagonal subgroup T. Moreover, the map $\mathcal{C} \mapsto \mathcal{C} \cap N_G(T)$ is a bijection between B-orbits of $\mathcal{S}$ and T-orbits of $\mathcal{S} \cap N_G(T)$. Thus we may define a map

$$\iota : \quad B \text{ orbits of } \mathcal{S} \quad \rightarrow \quad W$$

sending $\mathcal{C}$ to the T-coset of $\mathcal{C} \cap N_G(T)$.
Set $w = \Phi$ and regard w as an element of W. We shall be interested in $\iota^{-1}(w)$. Suppose that $\eta \in G(E)$ satisfies

$$\eta\theta(\eta)^{-1} = tw$$

where $t = d(t_1, t_2, t_3) \in T(E)$. In this case, $\theta(tw)tw = 1$, or $\theta(w)tw = \theta(t)^{-1}$, and hence $t_1, t_2, t_3 \in F^*$. If $\alpha \in T(E)$, then

$$\alpha(tw)\theta(\alpha)^{-1} = \alpha(tw)w\overline{\alpha}w^{-1} = (\mathrm{Nm}\,\alpha)tw \tag{7}$$

since $w = w^{-1}$. This yields the following

LEMMA 1. *There is a bijection (depending on the choice of w)*

$$\iota^{-1}(w) \longleftrightarrow T'(F)/Z'(F)\,\mathrm{Nm}(T(E))$$

defined by sending $\mathcal{C}$ to $\{t : tw \in \mathcal{C} \cap N_G(T)\}$ modulo $Z'(F)\,\mathrm{Nm}(T(E))$.

In the local case, $\iota^{-1}(w)$ consists of the open orbits.
Set

$$B_\eta = \eta H \eta^{-1} \cap B$$

and

$$H_\eta = H \cap \eta^{-1} B \eta.$$

Then $B_\eta = \{b \in B : \theta(b) = twbw^{-1}t^{-1}\}$ and hence

$$B_\eta = \{d(a, b, c) : \quad a, b, c \in E^1\}$$

where E^1 is the group of norm one elements in E^*. The subgroup B_η is thus independent of η.

4 FOURIER INVERSION AND STABILIZATION

For E/F a quadratic extension of local fields or number fields, or for $E = F \oplus F$, we set

$$A(F) = T'(F)/Z'(F)\,\mathrm{Nm}(T(E)).$$

By Lemma 1, $A(F)$ parameterizes the B-orbits in $\iota^{-1}(w)$. Note that $A(F) \simeq (F^*/NE^*)^2$. If $E = F \oplus F$, then $A(F)$ is trivial. In the global case, we define $A(\mathbb{A}_F)$ as the direct sum of the corresponding local groups

$$A(\mathbb{A}_F) = \bigoplus_v A(F_v)$$

where v ranges over all places of F. View $A(F)$ as a subgroup of $A(\mathbb{A}_F)$ embedded diagonally. Note that $[A(\mathbb{A}_F) : A(F)] = 4$.
For an absolutely summable function g on $A(\mathbb{A}_F)$, we may define the Fourier transform

$$\widehat{g}(\kappa) = \sum_{x \in A(\mathbb{A}_F)} \kappa(x)\ g(x)$$

for any character κ of $A(\mathbb{A}_F)$. Let X be the set of four characters of $A(\mathbb{A}_F)$ trivial on $A(F)$. Then the following Fourier inversion formula holds

$$\sum_{x \in A(F)} g(x) \quad = \quad \frac{1}{4} \sum_{\kappa \in X} \widehat{g}(\kappa).$$

Suppose in addition that g is of the form

$$g(x) = \prod_v g_v(x_v)$$

where g_v is a function on $A(F_v)$ for all v and the infinite product converges absolutely. Define the local Fourier transform

$$\widehat{g_v}(\kappa) = \sum_{x_v \in A(F_v)} \kappa(x_v)\ g_v(x_v)$$

for any character κ of $A(F_v)$. We shall write κ_v for the restriction of a character $\kappa \in A(\mathbb{A}_F)$ to $A(F_v)$. Then we have the following

LEMMA 2. $\widehat{g}(\kappa) = \prod_v \widehat{g}_v(\kappa_v)$

Proof. Let $\{S_n\}_{n=1}^{\infty}$ be an ascending sequence of finite sets of places of F whose union is the set of all places of F. For any finite set of places S let $A_S(\mathbb{A}_F)$ be the subgroup of elements $x = (x_v) \in A(\mathbb{A}_F)$ such that $x_v = 1$ for $v \notin S$. By definition,

$$g(x) = \lim_{n \to \infty} \prod_{v \in S_n} g_v(x_v)$$

and

$$\begin{aligned}\widehat{g}(\kappa) &= \lim_{n\to\infty} \sum_{x\in A_{S_n}(\mathbb{A}_F)} \kappa(x)\ g(x) \\ &= \lim_{n\to\infty} \prod_{v\in S_n} \left(\sum_{x_v\in A(F_v)} \kappa_v(x_v)\ g_v(x_v) \right) \prod_{v\notin S_n} g_v(1) \\ &= \lim_{n\to\infty} \left(\prod_{v\in S_n} \widehat{g}_v(\kappa_v) \right) \prod_{v\notin S_n} g_v(1) = \prod_v \widehat{g}_v(\kappa_v)\end{aligned}$$

since $\prod_{v\notin S_n} g_v(1)$ converges to 1 as $n \to \infty$. □

5 Stable Local Period

Let us consider the local case. Fix $\eta \in G(E)$ such that $\eta\theta(\eta)^{-1} = tw$ for some $t \in T(F)$. To define the stable local period, we assume that the inducing character $\chi = (\chi_1, \chi_2, \chi_3)$ satisfies $\chi_j\big|_{E^1} \equiv 1$. It is shown in [LR] that the integral

$$J(\eta, \varphi, \lambda) = \int_{H_\eta(F)\backslash H(F)} e^{\langle \lambda, H(\eta h)\rangle}\ \ \varphi(\eta h)\ \ dh$$

where $\varphi \in I(\chi, \lambda)$ converges for $\operatorname{Re}\lambda$ positive enough. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a character of T' such that $\chi_j = \nu_j \circ \mathrm{Nm}$ for $j = 1, 2, 3$. Set

$$\Delta_{\nu,\lambda}(\eta) = \nu(t)\omega(t_1 t_3)\ \ e^{\frac{1}{2}\langle \lambda+\rho, H(t)\rangle}.$$

By (7), we have

$$\Delta_{\nu,\lambda}(\alpha\eta) = \ \chi(\alpha)\ e^{\langle \lambda+\rho, H(\alpha)\rangle}\ \Delta_{\nu,\lambda}(\eta),$$

for all $\alpha \in T(E)$ and the expression

$$\Delta_{\nu,\lambda}(\eta)^{-1} \int_{H_\eta(F)\backslash H(F)} e^{\langle \lambda, H(\eta h)\rangle}\ \ \varphi(\eta h)\ \ dh \tag{8}$$

depends only on the double coset $B\eta H$ and the measure on $H_\eta(F)\backslash H(F)$. The stable local period is defined to be the distribution

$$J^{st}(\nu, \varphi, \lambda) = \sum_{\iota(\eta)=w} \Delta_{\nu,\lambda}(\eta)^{-1} \int_{H_\eta(F)\backslash H(F)} e^{\langle \lambda, H(\eta h)\rangle}\ \ \varphi(\eta h)\ \ dh.$$

The split case

If $E = F \oplus F$, then

$$G(E) = GL_3(F) \times GL_3(F) = G'(F) \times G'(F).$$

In this case, the local period can be expressed in terms of an intertwining operator. We have $\theta(h_1,h_2) = (\vartheta(h_2),\vartheta(h_1))$ where $\vartheta(h) = \Phi^{-1}\,{}^t h^{-1}\Phi$, and

$$H = \{(h,\vartheta(h)) : h \in G'\}.$$

Furthermore, $B(E)\backslash G(E)/H(F) \simeq W$, so the stabilization is trivial. We can take $\eta = (1,w)$. Then $H_\eta = \{(t,\vartheta(t)) : t \in T'\}$.
Let ν be a unitary character of T'. Its base change to T is $\chi = (\nu,\nu)$. Let $\varphi = \varphi_1 \otimes \varphi_2 \in I(\nu \otimes \nu,(\lambda,\lambda)) = I'(\nu,\lambda) \otimes I'(\nu,\lambda)$. Recall that K' is the standard maximal compact subgroup of $G'(F)$.

Proposition 1. *We have*

$$J^{st}(\nu,\varphi_1\otimes\varphi_2,\lambda) = \int_{K'} \varphi_1(k)(M(w,\lambda)\varphi_2)(\vartheta(k))\ dk.$$

Proof. By definition

$$\begin{aligned}
J(\eta,\varphi,\lambda) &= \int_{H_\eta\backslash H} e^{\langle(\lambda,\lambda),H(\eta h)\rangle}\varphi(\eta h)\ dh \\
&= \int_{T'\backslash G'} e^{\langle\lambda,H(h)+H(w\vartheta(h))\rangle}\varphi_1(h)\varphi_2(w\vartheta(h))\ dh \\
&= \int_{K'} \varphi_1(k)\left(\int_N e^{\langle\lambda,H(w\vartheta(n))\rangle}\varphi_2(w\vartheta(n)\vartheta(k))\ dn\right)\ dk \\
&= \int_{K'} \varphi_1(k)(M(w,\lambda)\varphi_2)(\vartheta(k))\ dk.
\end{aligned}$$

as required. □

6 Meromorphic Continuation of Local Periods in the p-adic case

Let E/F be quadratic extension of p-adic fields and let $q = q_F$ be the cardinality of the residue field of F.

Proposition 2. *$J^{st}(\nu,\varphi,\lambda)$ is a rational function in q^λ.*

It suffices to show that each integral $J(\eta,\varphi,\lambda)$ is a rational function in q^λ. We shall follow the discussion in [GPSR], pp. 126–130, where a similar assertion is established for certain zeta integrals.
The key ingredient is a theorem of J. Bernstein, which we now recall. Let V be a vector space of countable dimension over $\mathbb{C}$, Y an irreducible variety over $\mathbb{C}$ with ring of regular functions $\mathbb{C}[Y]$, and I an arbitrary index set. By a system of linear equations in V^* indexed by $i \in I$ we mean a set of equations for $\ell \in V^*$ of the form $\ell(v_i) = a_i$ where $v_i \in V$ and $a_i \in \mathbb{C}$. Consider an *algebraic family* Ξ of systems parameterized by Y. In other words, for each $i \in I$ we have functions $v_i(y) \in V \otimes \mathbb{C}[Y]$ and $a_i(y) \in \mathbb{C}[Y]$ defining a system Ξ_y: $\ell(v_i(y)) = a_i(y)$, for each $y \in Y$.

Let $K = \mathbb{C}(Y)$ be the fraction field of $\mathbb{C}[Y]$. If $L \in \mathrm{Hom}_{\mathbb{C}}(V, K)$ and $v(y) \in V \otimes \mathbb{C}[Y]$, then $L(v(y))$ may be viewed as an element of K. We will say that $L \in \mathrm{Hom}_{\mathbb{C}}(V, K)$ is a meromorphic solution of the family Ξ if $L(v_i(y)) = a_i(y)$ for all $i \in I$. Then Bernstein's Theorem is the following statement.

THEOREM 4. *In the above notation, suppose that the system Ξ_y has a unique solution $\ell_y \in V^*$ for all y in some non-empty open set $\Omega \subset Y$ (in the complex topology). Then the family Ξ has a unique meromorphic solution $L \in \mathrm{Hom}_{\mathbb{C}}(V, K)$. Furthermore, outside a countable set of hypersurfaces in Y, $\ell_y(v) = (L(v))(y)$.*

To use the Theorem, we need the following input. Recall that there are 4 open B-orbits in G/H. The next lemma shows that generically each one supports at most one H-invariant functional.

LEMMA 3. *(a). If $\mathrm{Re}\,\lambda$ is sufficiently positive, then there exists a unique (up to a constant) H-invariant functional ℓ on $I(\chi, \lambda)$ such that $\ell(\varphi) = 0$ if $\varphi|_{B\eta H} = 0$.*

(b). There exists $\varphi_0 \in I(\chi, \lambda)$ such that $J(\eta, \varphi_0, \lambda) = cq^{n\lambda}$ for some non-zero $c \in \mathbb{C}$ and $n \in \mathbb{Z}$.

Proof. To prove (a), first note that $J(\eta, \varphi, \lambda)$ defines a non-zero H-invariant functional whenever the integral defining it converges. To prove uniqueness, let $\{\eta_i\}$ be a set of representatives for the open orbits in $B\backslash G/H$ and let V be the H-invariant subspace of $\varphi \in I(\chi, \lambda)$ whose support is contained in the union $\coprod_{\eta_i} B\eta_i H$. The argument in [JLR], pp. 212–213 shows that an H-invariant functional vanishing on V is identically zero if $\mathrm{Re}\,\lambda$ is sufficiently positive. On the other hand, V decomposes as a direct sum over the η_i's of $\mathrm{ind}_{H_{\eta_i}}^H(\chi_\lambda^{\eta_i})$ where

$$(\chi_\lambda^{\eta_i})(h) = \chi(\eta_i h \eta_i^{-1}) e^{\langle \lambda, H(\eta_i h \eta_i^{-1}) \rangle}.$$

It remains to show that the space of H-invariant functionals on $\mathrm{ind}_{H_{\eta_i}}^H(\chi_\lambda^{\eta_i})$ is at most one-dimensional. However, the dual of $\mathrm{ind}_{H_\eta}^H(\chi_\lambda^\eta)$ is $\mathrm{Ind}_{H_\eta}^H((\chi_\lambda^\eta)^{-1})$ and the dimension of H-invariant vectors in the latter is at most one by Frobenius reciprocity.

To prove (b), let V be a small open subgroup of G. The map $B \times H \to G$ defined by $(b, h) \mapsto b\eta h$ is proper since the stabilizer B_η is compact. We infer that the set $U = \eta H \cap B\eta V$ is a small neighborhood of $B_\eta \eta$ and hence the weight function $H(x)$ takes the constant value $H(\eta)$ on U. Then we may take for φ_0 any non-negative, non-zero function supported in $B\eta V$. □

To finish the proof of Proposition 2, let $V = I(\chi)$ and $Y = \mathbb{C}^*$. Then $\mathbb{C}[Y]$ may be identified with the ring of polynomials in $q^{\pm\lambda}$. Fix φ_0, c and n as in Lemma 3 and consider the following conditions on a linear functional $\ell \in V^*$:

1. ℓ is H-invariant, i.e. $\ell(I(h, \chi, \lambda)\varphi - \varphi) = 0$ for all φ and $h \in H$.

2. $\ell(\varphi) = 0$ if $\varphi\big|_{B\eta H} = 0$.

3. $\ell(\varphi_0) = cq^{n\lambda}$.

These conditions form an algebraic family of systems of linear equations as above. The functional $J(\eta, \varphi, \lambda)$ converges for $\operatorname{Re}\lambda > \lambda_0$ and it is the unique solution for the system by Lemma 3. The proposition now follows from Bernstein's Theorem.

7 Unramified Computation

Suppose now that E/F is an unramified extension of p-adic fields with $p \neq 2$ and that χ is unramified. Let φ_0 be the K-invariant section of $I(\chi, \lambda)$ such that $\varphi_0(e) = 1$. Recall that $s_i = \langle \lambda, \alpha_i^\vee \rangle$ for $i = 1, 2, 3$.
Recall our convention that the Haar measure dh on $H(F)$ is defined via the Iwasawa decomposition and assigns measure one to K_H. In the following Proposition, we assume that the measure on $H_\eta(F)\backslash H(F)$ is the quotient of the dh by the measure on $H_\eta(F)$ such that $\mathrm{vol}(H_\eta(F)) = 1$.

Proposition 3. *The stable local period $J^{st}(\nu, \varphi_0, \lambda)$ is equal to*

$$\frac{L(\nu_1\nu_2^{-1}\omega, s_1)L(\nu_2\nu_3^{-1}\omega, s_2)L(\nu_1\nu_3^{-1}\omega, s_3)}{L(\nu_1\nu_2^{-1}, s_1+1)L(\nu_2\nu_3^{-1}, s_2+1)L(\nu_1\nu_3^{-1}, s_3+1)}$$

Sketch of proof. Without loss of generality we may assume that $\chi = 1$. Let $\eta_1, \eta_2, \eta_3, \eta_4$ be the matrices

$$\begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{\epsilon}{2} & 0 & 1 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} \frac{\epsilon}{2} & 0 & 1 \\ -\frac{1}{2} & 0 & \epsilon^{-1} \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ \frac{\epsilon}{2} & 0 & 1 \\ -\frac{1}{2} & 0 & \epsilon^{-1} \end{pmatrix}$$

respectively, where $\epsilon \in F^* - NE^*$, e.g., ϵ has odd valuation. They form a set of representatives for the double cosets in $B\backslash G/H$ over w. The matrices $\eta_i\theta(\eta_i)^{-1}$ are

$$\begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & \epsilon \\ & 1 & \\ -\epsilon^{-1} & & \end{pmatrix}, \begin{pmatrix} & & \epsilon \\ & -\epsilon^{-1} & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & \epsilon & \\ -\epsilon^{-1} & & \end{pmatrix}$$

respectively.
By definition,

$$J^{st}(\nu, \varphi_0, \lambda) = \sum_j I_j$$

where

$$I_j = \Delta_{\nu,\lambda}^{-1}(\eta_j) \int_{H_{\eta_j}(F)\backslash H(F)} e^{\langle \lambda+\rho, H(\eta_j h)\rangle}\, dh. \tag{9}$$

Since $H_{\eta_j}(F)$ has measure one, we may use Iwasawa decomposition to write I_j as $\Delta_{\nu,\lambda}(\eta_j)^{-1}$ times

$$\sum_n q_E^{2n}$$
$$\int_E \int_{\bar\beta=-\beta} \exp(\left\langle \lambda+\rho, H(\eta_j \begin{pmatrix} 1 & x & \frac{x\bar x}{2}+\beta \\ 0 & 1 & \bar x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^n & & \\ & 1 & \\ & & \overline{\varpi}^{-n} \end{pmatrix}) \right\rangle)\, d\beta\, dx$$

where ϖ is a uniformizer of E and q_E is the cardinality of the residue field of E.

The I_j can be evaluated explicitly. For $j = 2,3,4$, the integrands depend only on $v(x)$, $v(\beta)$ and n, but for $j = 1$ it depends also on $v(\frac{x\bar x}{4} \pm 1)$. In any case, the integrations and the summation over β, x and n can be computed as geometric series. This is tedious to carry out by hand, especially in the first case, but we did it using Mathematica with the following results. The term I_1 is independent of ν and is equal to

$$\Big[1 - q^{-2(s_1+s_2)} - 2q^{-2(1+s_1+s_2)} + q^{-1-2s_1} + q^{-2-2s_2-4s_1} + q^{-1-2s_2} - 2q^{-1-2(s_1+s_2)} - q^{-3-2(s_1+s_2)} + q^{-2-2s_1-4s_2} + q^{-4s_1-4s_2-3}\Big] / ((1-q^{-2s_1})(1-q^{-2s_2})(1-q^{-2(s_1+s_2)})), \tag{10}$$

while

$$I_2 = \nu_1(\epsilon)^{-1}\nu_3(\epsilon) \cdot \frac{q^{-2-s_1-s_2}(1+q)}{1-q^{-2(s_1+s_2)}},$$
$$I_3 = -\nu_1(\epsilon)^{-1}\nu_2(\epsilon) \cdot \frac{q^{-1-s_1}(1+q)(1-q^{-2(1+s_1+s_2)})}{(1-q^{-2s_1})(1-q^{-2(s_1+s_2)})},$$
$$I_4 = -\nu_2(\epsilon)^{-1}\nu_3(\epsilon) \cdot \frac{q^{-1-s_2}(1+q)(1-q^{-2(1+s_1+s_2)})}{(1-q^{-2s_2})(1-q^{-2(s_1+s_2)})}$$

respectively. Summing up the contributions (also done with Mathematica) we get

$$\frac{(1-q^{-(s_1+1)})(1-q^{-(s_2+1)})(1-q^{-(s_1+s_2+1)})}{(1+q^{-s_1})(1+q^{-s_2})(1+q^{-(s_1+s_2)})} \quad \text{for } \nu_1=\nu_2=\nu_3=1$$
$$\frac{(1-q^{-(s_1+1)})(1+q^{-(s_2+1)})(1+q^{-(s_1+s_2+1)})}{(1+q^{-s_1})(1-q^{-s_2})(1-q^{-(s_1+s_2)})} \quad \text{for } \nu_1=\nu_2=\omega,\ \nu_3=1$$
$$\frac{(1+q^{-(s_1+1)})(1+q^{-(s_2+1)})(1-q^{-(s_1+s_2+1)})}{(1-q^{-s_1})(1-q^{-s_2})(1+q^{-(s_1+s_2)})} \quad \text{for } \nu_1=\nu_3=\omega,\ \nu_2=1$$
$$\frac{(1+q^{-(s_1+1)})(1-q^{-(s_2+1)})(1+q^{-(s_1+s_2+1)})}{(1-q^{-s_1})(1+q^{-s_2})(1-q^{-(s_1+s_2)})} \quad \text{for } \nu_2=\nu_3=\omega,\ \nu_1=1$$

as required. □

Remark 2. We have also computed $J^{st}(\nu, \varphi_0, \lambda)$ when E/F is ramified but ν is unramified and $-1 \in \operatorname{Nm} E^*$. In this case the matrices η_i still provide representatives for $B\backslash G/H$. We find that $J^{st}(\nu, \varphi_0, \lambda)$ is equal to

$$0 \quad \text{for } \nu_1 = \nu_2 = \nu_3 = 1$$
$$\frac{q^{-(s_2+1/2)}(1-q^{-(s_1+1)})}{(1-q^{-s_2})(1-q^{-(s_1+s_2)})} \quad \text{for } \nu_1 = \nu_2 = \omega,\ \nu_3 = 1$$
$$\frac{q^{-1/2}(1-q^{-(s_1+s_2+1)})}{(1-q^{-s_1})(1-q^{-s_2})} \quad \text{for } \nu_1 = \nu_3 = \omega,\ \nu_2 = 1$$
$$\frac{q^{-(s_1+1/2)}(1-q^{-(s_2+1)})}{(1-q^{-s_1})(1-q^{-(s_1+s_2)})} \quad \text{for } \nu_2 = \nu_3 = \omega,\ \nu_1 = 1$$

However, this is not sufficient to evaluate $d_{E/F}$. We would also need to determine the function f' on G' that matches φ_0. Finally if $-1 \notin \operatorname{Nm} E^*$ then the representatives are more difficult to write down and the computation is more elaborate. We have not attempted to do this.

8 Stabilization of Periods

We return to the global situation. Let

$$c = \operatorname{vol}(H_\eta(F)Z(\mathbb{A}_F)\backslash H_\eta(\mathbb{A})) = \operatorname{vol}(E^1\backslash E^1(\mathbb{A}))^2.$$

The following identity

$$\Pi^H(E(\varphi,\lambda)) = c \sum_{\iota(\eta)=w} J(\eta,\varphi,\lambda)$$

is proved in [LR]. It is valid whenever $\operatorname{Re}\lambda$ is positive enough. From now on, we assume that the Haar measure on $H_\eta(\mathbb{A})$ is the Tamagawa measure. This measure has the property that $\operatorname{vol}(H_\eta(F_v)) = 1$ for almost all v and furthermore,

$$c = 4 \tag{11}$$

by Ono's formula for the Tamagawa number of a torus [O].
Fix a character $\nu_0 \in \mathcal{B}(\chi)$ to serve as a base-point. Recall from §4 that the set of double cosets over w is parameterized by the group $A(F)$ both locally and globally. For $a \in A(F)$, let η_a be a representative for the double coset corresponding to a such that $\eta_a\theta(\eta_a)^{-1} = tw$ with $t \in T'(F)$. Fix λ and $\varphi = \otimes\varphi_v \in I(\chi,\lambda) = \otimes I(\chi_v,\lambda)$, and let g_v be the function on $A(F_v)$ defined as follows:

$$g_v(a) = \Delta_{\nu_0,\lambda}(\eta_a)^{-1} \int_{H_{\eta_a}(F_v)\backslash H(F_v)} e^{\langle\lambda, H_v(\eta_a h)\rangle}\ \varphi_v(\eta_a h)\ dh. \tag{12}$$

The product function $g = \prod_v g_v$ on $A(\mathbb{A})$ is integrable over $A(\mathbb{A}_F)$ for $\mathrm{Re}\,\lambda$ positive enough. Indeed, Proposition 3 applied to $|\nu_0|$ shows that almost all factors of the integral are local factors of a quotient of products of L-functions. For the same reason, we may define the global stable intertwining period for $\mathrm{Re}\,\lambda$ positive enough as the absolutely convergent product

$$J^{st}(\nu_0, \varphi, \lambda) = \prod_v J_v^{st}(\nu_{0v}, \varphi_v, \lambda).$$

For any character κ of $A(\mathbb{A})$ trivial on $A(F)$ we have $\widehat{g}(\kappa) = J^{st}(\kappa\nu_0, \varphi, \lambda)$. Observe that the characters $\nu = \kappa\nu_0$ comprise $\mathcal{B}(\chi)$. The Fourier inversion formula of Section 4 together with (11) gives

$$\Pi^H(E(\varphi, \lambda)) = \sum_{\nu \in \mathcal{B}(\chi)} J^{st}(\nu, \varphi, \lambda). \tag{13}$$

By Propositions 1, 2 and 3, each $J^{st}(\nu, \varphi, \lambda)$ admits a meromorphic continuation, and hence, the identity (13) is valid for all λ. These assertions make up Theorem 1, which is now proved.

Remark 3. It seems unlikely that the individual terms $J(\eta, \varphi, \lambda)$ have a meromorphic continuation to the entire complex plane. This is motivated by the following old result of Estermann ([E]). Let $P(x)$ be a polynomial with integer coefficients with $P(0) = 1$. Then either $P(x)$ is a product of cyclotomic polynomials or else the Euler product $\prod_p P(p^{-s})$ has the imaginary axis as its natural boundary. The function $J(\eta, \varphi, \lambda)$ has Euler product where the factors in the inert places are almost always (10).

Part II
Relative Bessel Distributions

We now turn to the relative Bessel distributions, starting with the local case. Thus we assume that F is a local field. The Whittaker functional is defined by the integral

$$\mathcal{W}(\varphi, \lambda) = \int_N e^{\langle \lambda, H(wn) \rangle} \varphi(wn) \overline{\psi_N(n)}\, dn$$

for $\varphi \in I(\chi, \lambda)$. It converges absolutely for $\mathrm{Re}\,\lambda$ sufficiently large, and defines a rational function in q^λ.

Definition 1. Let ν be a unitary character of T' which base changes to χ. The stable relative Bessel distribution is defined by

$$\tilde{B}^{st}(f, \nu, \lambda) = \sum_{\varphi} J^{st}(\nu, I(f, \chi, \lambda)\varphi, \lambda) \overline{\mathcal{W}}(\varphi, \lambda)$$

for $\lambda \in i\mathfrak{a}_0^*$ where $\{\varphi\}$ is an orthonormal basis for $I(\chi, \lambda)$.

Similarly, the local Bessel distributions on G' are defined in terms of the Whittaker functionals on G' as follows:

$$B'(f',\nu,\lambda) = \sum_{\varphi'} \mathcal{W}'(I(f',\nu,\lambda)\varphi',\lambda)\overline{\mathcal{W}'}(\varphi',\lambda).$$

Remark 4. Let us check that Theorem 3 is compatible with a change of additive character. In doing this, we include the additive character in the notation. Let ψ' be the character $\psi'(x) = \psi(ax)$. Then $\psi'_N(\cdot) = \psi_N(t_0^{-1} \cdot t_0)$ where $t_0 = d(a^{-1}, 1, a)$, and

$$\mathcal{W}^{\psi'}(\varphi,\lambda) = \chi(wt_0w^{-1})e^{\langle w\lambda+\rho, H(t_0)\rangle}\mathcal{W}^{\psi}(I(t_0^{-1},\chi,\lambda)\varphi,\lambda).$$

Hence

$$\tilde{B}^{st}_{\psi'}(f,\nu,\lambda) = \overline{\chi(wt_0w^{-1})}e^{\left\langle w\overline{\lambda}+\rho, H(t_0)\right\rangle}\tilde{B}^{st}_{\psi}(f_{t_0},\nu,\lambda)$$

where $f_{t_0}(x) = f(xt_0)$. Similarly,

$$B'_{\psi'}(f',\nu,\lambda) = |\nu(wt_0w^{-1})e^{\left\langle w\lambda+\rho, H'(t_0)\right\rangle}|^2 B'_{\psi}(f'_{t_0},\nu,\lambda)$$

where $f'_{t_0}(x) = f'(t_0^{-1}xt_0)$. It follows from the definition in [JY] that if $f \leftrightarrow f'$ with respect to ψ then $f_{t_0} \leftrightarrow f'_{t_0}$ with respect to ψ'. On the other hand,

$$\gamma(\nu,\lambda,\psi') = (\nu_1\nu_2^{-1})(a)|a|^{s_1}(\nu_2\nu_3^{-1})(a)|a|^{s_2}(\nu_1\nu_3^{-1})(a)|a|^{s_3}\gamma(\nu,\lambda,\psi).$$

It remains to note that

$$\overline{\chi(wt_0w^{-1})}e^{\left\langle w\overline{\lambda}+\rho, H(t_0)\right\rangle} = (\nu_1\nu_2^{-1})(a)|a|^{s_1}(\nu_2\nu_3^{-1})(a)|a|^{s_2}(\nu_1\nu_3^{-1})(a)|a|^{s_3} \\ |\nu(wt_0w^{-1})e^{\left\langle w\lambda+\rho, H'(t_0)\right\rangle}|^2.$$

Our goal is to prove Theorems 2 and 3. Let us start with the split case $E = F \oplus F$. By a special case of a result of Shahidi ([Sh]) we have the following local functional equations. For any $\varphi \in I'(\nu,\lambda)$

$$\mathcal{W}(M(w,\lambda)\varphi, w\lambda) = \gamma(\nu,\lambda,\psi)\mathcal{W}(\varphi,\lambda) \tag{14}$$

where $\gamma(\nu,\lambda,\psi)$ is defined in (5). Recall that $\omega \equiv 1$ in this case.

PROPOSITION 4. *We have*

$$\tilde{B}^{st}(f_1 \otimes f_2,\nu,\lambda) = \gamma_v(\nu,\lambda)B'(f,\nu,\lambda)$$

where

$$f(g) = \int_{H(F)} f_1(hg)f_2(\vartheta(h))\, dh.$$

Proof. The involution ϑ on G' preserves B', T', N'. It induces the principal involution on the space spanned by the roots. We also let $(\vartheta\varphi)(g) = \varphi(\vartheta(g))$. This is a self-adjoint involution on $I'(\nu)$. By Proposition 1

$$J(\eta, \varphi_1 \otimes \varphi_2, \lambda) = (\vartheta(M(w, \lambda)\varphi_2), \overline{\varphi_1}),$$

and hence

$$\tilde{B}^{st}(f_1 \otimes f_2, \nu, \lambda) \tag{15}$$

is equal to

$$\sum_{i,j} (\vartheta(M(w, \lambda)I'(\nu, f_2, \lambda)\varphi_j), \overline{I'(\nu, f_1, \lambda)\varphi_i}) \times \overline{\mathcal{W}}(\varphi_i, \lambda) \cdot \overline{\mathcal{W}}(\varphi_j, \lambda).$$

The following identity holds for any operator A, functional l and orthonormal basis $\{e_i\}$ on a Hilbert space:

$$\sum_i (Ae_i, v)\overline{l(e_i)} = \overline{l(A^* v)}.$$

Hence (15) is equal to

$$\sum_i \overline{\mathcal{W}}((\vartheta \circ M(w, \lambda)I'(\nu, f_2, \lambda))^* \overline{I'(\nu, f_1, \lambda)\varphi_i}, \lambda)\overline{\mathcal{W}}(\varphi_i, \lambda) \tag{16}$$

We have the following simple relations

$$\begin{gathered}
\mathcal{W}(\vartheta(\varphi), \vartheta(\lambda)) = \overline{\mathcal{W}}(\overline{\varphi}, \overline{\lambda}) \text{ because } \psi_{N'}(\vartheta(n)) = \overline{\psi_{N'}(n)} \\
\vartheta \circ M(w, \lambda) = M(w, \vartheta(\lambda)) \circ \vartheta \\
\vartheta \circ I'(\nu, f, \lambda) = I'(\nu, \vartheta(f), \vartheta(\lambda)) \circ \vartheta \\
I'(\nu, f, \lambda)^* = I'(\nu, f^\vee, -\overline{\lambda}) \text{ where } f^\vee(g) = \overline{f(g^{-1})} \\
M(w, \lambda)^* = M(w, -w\overline{\lambda})
\end{gathered}$$

We get for any φ,

$$\begin{aligned}
(\vartheta \circ M(w, \lambda)I'(\nu, f_2, \lambda))^* \overline{\varphi} &= (M(w, \vartheta(\lambda))I'(\nu, \vartheta(f_2), \vartheta(\lambda)) \circ \vartheta)^* \overline{\varphi} \\
&= (I'(\nu, \vartheta(f_2), w\vartheta(\lambda))M(w, \vartheta(\lambda))\vartheta)^* \overline{\varphi} \\
&= (\vartheta \circ M(w, -w\overline{\vartheta(\lambda)})I'(\nu, \vartheta(f_2)^\vee, -w\overline{\vartheta(\lambda)}))\overline{\varphi} \\
&= \overline{(\vartheta \circ M(w, \lambda)I'(\nu, \vartheta(f_2^\circ), \lambda))\varphi}
\end{aligned}$$

with $f^\circ(g) = f(g^{-1})$. Hence,

$$\begin{aligned}
&\overline{\mathcal{W}}((\vartheta \circ M(w, \lambda)I'(\nu, f_2, \lambda))^* \overline{I'(\nu, f_1, \lambda)\varphi_i}, \lambda) \\
=&\overline{\mathcal{W}}(\overline{\vartheta \circ M(w, \lambda)I'(\nu, \vartheta(f_2^\circ), \lambda)I'(\nu, f_1, \lambda)\varphi_i}, \lambda) \\
=&\mathcal{W}(M(w, \lambda)I'(\nu, \vartheta(f_2^\circ), \lambda)I'(\nu, f_1, \lambda)\varphi_i, \overline{\vartheta(\lambda)}) \\
=&\gamma(\nu, \lambda, \psi)\mathcal{W}(I'(\nu, \vartheta(f_2^\circ), \lambda)I'(\nu, f_1, \lambda)\varphi_i, w\overline{\vartheta(\lambda)}) \\
=&\gamma(\nu, \lambda, \psi)\mathcal{W}(I'(\nu, \vartheta(f_2^\circ) * f_1, \lambda)\varphi_i, -\overline{\lambda})
\end{aligned}$$

and since $\lambda \in i\mathfrak{a}_0^*$, (16) becomes

$$\gamma(\nu,\lambda,\psi)\sum_i \mathcal{W}(I'(\nu,\vartheta(f_2^\circ)*f_1,\lambda)\varphi_i,\lambda)\overline{\mathcal{W}}(\varphi_i,\lambda)=\gamma(\nu,\lambda,\psi)B'(f,\nu,\lambda)$$

as required. □

We next consider the unramified non-archimedean case. Recall that

$$\mathrm{bc}:\mathcal{H}_G\to\mathcal{H}_{G'}$$

is the base change homomorphism. Proposition 3 gives the following

Proposition 5. *Assume that $f\in\mathcal{H}_G$ and that ν is a unitary, unramified character. Then*

$$\tilde{B}^{st}(f,\nu,\lambda)=\gamma(\nu,\lambda,\psi)B'(\mathrm{bc}(f),\nu,\lambda). \tag{17}$$

Proof. The left hand side is

$$J^{st}(\nu,I(f,\chi,\lambda)\varphi_0,\lambda)\overline{\mathcal{W}}(\varphi_0,\lambda)=\hat{f}(\chi,\lambda)J^{st}(\nu,\varphi_0,\lambda)\overline{\mathcal{W}}(\varphi_0,\lambda).$$

Recall that we may assume that the additive character ψ is unramified. By the formula for the unramified Whittaker functional and Proposition 3

$$J^{st}(\nu,\varphi_0,\lambda)\overline{\mathcal{W}}(\varphi_0,\lambda)=\frac{L(\nu_1\nu_2^{-1}\omega,s_1)L(\nu_2\nu_3^{-1}\omega,s_2)L(\nu_1\nu_3^{-1}\omega,s_3)}{L(\nu_1\nu_2^{-1},s_1+1)L(\nu_2\nu_3^{-1},s_2+1)L(\nu_1\nu_3^{-1},s_3+1)}$$

times

$$\overline{(L(\chi_1\chi_2^{-1},s_1+1)L(\chi_2\chi_3^{-1},s_2+1)L(\chi_1\chi_3^{-1},s_3+1))}^{-1}$$

If μ_0 is a character of F^*, $\mu=\mu_0\circ Nm$, and $s\in i\mathbb{R}$ we have

$$\begin{aligned}\frac{L(\mu_0\varpi,s)}{L(\mu_0,s+1)}(\overline{L(\mu,s+1)})^{-1}&=\frac{L(\mu_0\varpi,s)}{L(\mu_0,s+1)}(\overline{L(\mu_0,s+1)L(\mu_0\varpi,s+1)})^{-1}\\&=|L(\mu_0,s+1)|^{-2}\frac{L(\mu_0\varpi,s)}{L((\mu_0\varpi)^{-1},1-s)}.\end{aligned}$$

On the other hand

$$\begin{aligned}B'(\mathrm{bc}(f),\nu,\lambda)&=\mathcal{W}'(I'(\nu,bc(f),\lambda)\varphi_0')\overline{\mathcal{W}'}(\varphi_0')\\&=\widehat{\mathrm{bc}(f)}(\nu,\lambda)|L(\nu_1\nu_2^{-1},s_1+1)L(\nu_2\nu_3^{-1},s_2+1)L(\nu_1\nu_3^{-1},s_3+1)|^{-2}\\&=\hat{f}(\chi,\lambda)|L(\nu_1\nu_2^{-1},s_1+1)L(\nu_2\nu_3^{-1},s_2+1)L(\nu_1\nu_3^{-1},s_3+1)|^{-2},\end{aligned}$$

and the statement follows. □

In the global setup we define

$$\tilde{B}^{st}(f,\nu,\lambda)=\prod_v \tilde{B}^{st}_v(f_v,\nu_v,\lambda)$$

for $f=\otimes f_v$. Note that this is compatible with (4). Similarly

$$B'(f',\nu,\lambda)=\prod_v B'_v(f'_v,\nu_v,\lambda).$$

9 The Relative Trace Formula

The relative trace formula identity, established by Jacquet reads

$$RTF(f)=KTF(f') \tag{18}$$

where

$$RTF(f)=\int_{H\backslash H(\mathbb{A})}\int_{N(E)\backslash N(\mathbb{A}_E)} K_f(h,n)\psi_N(n)\ dh\ dn$$
$$KTF(f')=\int_{N'(F)\backslash N'(\mathbb{A})}\int_{N'(F)\backslash N'(\mathbb{A})} K'_{f'}(n_1,n_2)\psi_{N'}(n_1n_2)\ dn_1\ dn_2$$

for f, f' matching. In ([J1]), Jacquet established the following spectral expansion for $RTF(f)$, at least for **K**-finite functions f. It is a sum over terms indexed by certain pairs $Q=(M,\pi)$ consisting of a standard Levi subgroup M and a cuspidal representation π of $M(\mathbb{A})$.
If $M=G$, then Q contributes if π is H-distinguished. In this case, the contribution is

$$\sum_\varphi \Pi^H(\pi(f)\varphi)\overline{\mathbb{W}(\varphi)}$$

where $\{\varphi\}$ is an orthonormal basis of the space V_π of π. We will say that a Hecke character of $GL(1)_E$ is distinguished if it is trivial on $E^1(\mathbb{A})$, that is, it is the base change of a Hecke character of $GL(1)_F$. If M is the Levi factor of a maximal parabolic subgroup, then $\pi=\sigma\otimes\kappa$ where σ is a cuspidal representation of of $GL(2)_E$ and κ is a Hecke character of $GL(1)_E$. The pair Q contributes if σ is distinguished relative to some unitary group in two variables relative to E/F in $GL(2)_E$ and κ is distinguished. In this case, the contribution is

$$\int_{i\mathfrak{a}^*_P}\sum_\varphi \Pi^H(E(I(f,\pi,\lambda)\varphi,\lambda))\overline{\mathbb{W}(E(\varphi,\lambda))}\ d\lambda.$$

Finally, if M is the diagonal subgroup, then π is a triple of characters which we denote $\chi=(\chi_1,\chi_2,\chi_3)$. There are two kinds of contributions. The first one, which we call *fully continuous* is in the case where each χ_j is distinguished. In the notation of relative Bessel distributions, the contribution is

$$\frac{1}{6}\int_{i\mathfrak{a}_0^*}\tilde{B}(f,\chi,\lambda)\ d\lambda.$$

On the other hand, the *residual* contribution comes from triplets χ such that χ_2 is distinguished, $\chi_3(x)=\chi_1(\overline{x})$ but χ_1, χ_3 are not distinguished. Up to a volume factor it is

$$\int_{i(\alpha_3^\vee)^\perp)}\sum_{\varphi}\int_{_H\mathbf{K}} I(f,\chi,\lambda)\varphi(k)\ dk\cdot\overline{\mathcal{W}}(\varphi,\lambda)\ d\lambda.$$

Moreover the sum over χ and the integrals are absolutely convergent.
The spectral decomposition of the $KT(f')$ is

$$\sum_{M',\pi'}\int_{i\mathfrak{a}_{M'}^*}\sum_{\varphi'}\mathcal{W}'(I(f',\pi',\lambda')\varphi',\lambda)\overline{\mathcal{W}'}(\varphi',\lambda)\ d\lambda'.$$

The fully continuous part is

$$\frac{1}{6}\cdot\sum_{\nu}\int_{i\mathfrak{a}_0^*}B'(f',\nu,\lambda)\ d\lambda.$$

There is no contribution from the residual spectrum because the representations occurring in it are not generic, i.e., the ψ-Fourier coefficients of the residual automorphic forms all vanish. This follows from the description of the residual spectrum by Moeglin and Waldspurger ([MW]).

Remark 5. For all $w\in W$, we have $\tilde{B}(f,\chi,\lambda)=\tilde{B}(f,w\chi,w\lambda)$. Indeed, by the functional equation for the Eisenstein series we have

$$\begin{aligned}&\Pi^H(E(I(f,\chi,\lambda)\varphi,\lambda))\overline{\mathcal{W}}(\varphi,\lambda)\\ &=\Pi^H(E(M(w,\lambda)I(f,\chi,\lambda)\varphi,w\lambda))\overline{\mathcal{W}}(M(w,\lambda)\varphi,w\lambda)\\ &=\Pi^H(E(I(f,w\chi,w\lambda)M(w,\lambda)\varphi,w\lambda))\overline{\mathcal{W}}(M(w,\lambda)\varphi,w\lambda).\end{aligned}$$

We may change the orthonormal basis $\{\varphi\}$ to $\{M(w,\lambda)\varphi\}$ because $M(w,\lambda)$ is unitary for $\lambda\in i\mathfrak{a}_0^*$.
A similar remark applies to the other contributions. In particular, all the expressions above depend on Q only up to conjugacy.
Recall that

$$\tilde{B}(f,\chi,\lambda)=\sum_{\nu\in\mathcal{B}(\chi)}\tilde{B}^{st}(f,\nu,\lambda)$$

and therefore

$$\sum_{\varphi}\Pi^H(E(I(f,\chi,\lambda)\varphi,\lambda))\overline{\mathcal{W}}(\varphi,\lambda)=\sum_{\nu\in\mathcal{B}(\chi)}\tilde{B}^{st}(f,\nu,\lambda).$$

We shall now prove Theorems 2 and 3 by isolating the term corresponding to ν,λ in the relative trace formula identity. We proceed in several steps.

10 Separation of Continuous Spectrum

PROPOSITION 6. *For any χ and λ and matching functions $f \leftrightarrow f'$,*

$$\sum_{\nu \in \mathcal{B}(\chi)} \tilde{B}^{st}(f, \nu, \lambda) = \sum_{\nu \in \mathcal{B}(\chi)} B(f', \nu, \lambda). \tag{19}$$

To prove this, we modify the usual linear independence of characters argument ([L]). In the following lemma, let G be a reductive group over a global field F and S a set of places containing all the archimedean places and the "bad" places of G. Let $\mathfrak{X}$ be a countable set of pairs (M, π) consisting of a Levi subgroup M and a cuspidal representation π of $M(\mathbb{A})$ which is unramified outside S. For each $(M, \pi) \in \mathfrak{X}$ let a subspace $A_\pi \subset i\mathfrak{a}_M^*$ and a continuous function $g_\pi(\cdot)$ on A_π be given. We make the following hypotheses:

1. If (M_1, π_1), $(M_2, \pi_2) \in \mathfrak{X}$ with $\pi_1 \neq \pi_2$ and $\lambda_i \in A_{\pi_i}$, $i = 1, 2$, then $I(\pi_1, \lambda_1)^S$ and $I(\pi_2, \lambda_2)^S$ have no sub-quotient in common.

2. If $(M, \pi) \in \mathfrak{X}$, $\lambda, \lambda' \in A_\pi$ and $I(\pi, \lambda)^S \simeq I(\pi, \lambda')^S$ then $g_\pi(\lambda) = g_\pi(\lambda')$.

Let $\mathcal{H}^S$ denote the Hecke algebra of G^S relative to hyperspecial maximal compact subgroups of G_v for $v \notin S$. For $f \in \mathcal{H}^S$ and σ^S an unramified representation of G^S, we set $\hat{f}^S(\sigma^S) = \mathrm{tr}(\sigma^S(f^S))$.

LEMMA 4 (Generalized linear independence of characters). *Suppose that*

$$\sum_\pi \int_{A_\pi} |g_\pi(\lambda)| \; d\lambda < \infty \tag{20}$$

and that for any $f \in \mathcal{H}^S$

$$\sum_\pi \int_{A_\pi} \hat{f}^S(I(\pi, \lambda)^S) g_\pi(\lambda) \; d\lambda = 0 \tag{21}$$

Then $g_\pi(\lambda) = 0$ for all $\pi \in \mathfrak{X}$.

Proof. Let U be any set of places containing at least two places v_1, v_2 with distinct residual characteristics p_1 and p_2 such that $U \cap S = \emptyset$. Each π defines a map

$$T_{\pi,U} : A_\pi \to G_U^{un}$$

sending λ to $I(\pi, \lambda)_U$. Applying (21) with $f^U = 1$ gives

$$\sum_\pi \int_{A_\pi} \hat{f}_U(T_{\pi,U}(\lambda)) g_\pi(\lambda) \; d\lambda = 0. \tag{22}$$

Let μ_π be the push-forward under $T_{\pi,U}$ of the measure $g_\pi(\lambda)d\lambda$. The Stone-Weierstrass Theorem implies that the image of $\mathcal{H}_U$ under the Satake transform gives a dense set of continuous functions on $\widehat{G}_U^{un}$ relative to the sup norm. Therefore (21) vanishes for all continuous functions on $\hat{G}_U^{un}$. The Riesz Representation Theorem implies that

$$\sum_\pi \mu_\pi = 0.$$

Let $Z_{\pi,U}$ be the image of $T_{\pi,U}$. Then $\mu_{\pi_1}(Z_{\pi,U}) = 0$ unless $Z_{\pi,U} \supset Z_{\pi_1,U}$. We now claim that for any subset $Z \subset \widehat{G}_U^{un}$ we have

$$\sum_{\pi : Z_{\pi,U}=Z} \mu_\pi = 0. \tag{23}$$

We argue by induction on $\dim Z$ (i.e., $\dim A_\pi$ where $Z = Z_{\pi,U}$). For Z zero dimensional this follows from the fact that the atomic part of μ is $\sum_{\pi:A_\pi=\{0\}} \mu_\pi$. For the induction step, we can assume that there are no π with $\dim A_\pi < \dim Z$. The restriction of μ to Z is then given by the left-hand side of (23), and our claim is proved.

For each place v, let X_v be the group of unramified characters of the maximal split torus T_v in $M_0(F_v)$ where M_0 is a Levi factor of a fixed minimal parabolic subgroup P_0 of $G(F_v)$ (contained in a globally defined minimal parabolic). We identify X_v with the vector space $\mathfrak{a}_v^* = X^*(T_v) \otimes \mathbb{C}$ modulo the lattice $L_v = \frac{2\pi i}{\ln q_v} X^*(T_v)$. Attached to each unramified representation σ of M_v is an orbit of characters in X_v under the Weyl group of M_v. Let λ_σ be a representative of this orbit. Let W_v be the Weyl group of G_v and let W_U be the Weyl group $\prod_{v\in U} W_v$ of G_U. If σ is an unramified representation of G_U, let λ_σ be the element (λ_{σ_v}) in the product $X_U = \prod X_v$. We observe that the natural map $\mathfrak{a}_P^* \to X_U$ is injective since $\ln p_1$ and $\ln p_2$ are linearly independent over $\mathbb{Q}$. We identify $\mathfrak{a}_P^*$ with its image in X_U.

We now claim that if $Z_{\pi,U} = Z_{\pi_0,U}$, then there exists an element $w \in W_U$ such that

1. $w\lambda_{\pi_0,U} - \lambda_{\pi,U} \in A_\pi$.
2. $A_\pi = wA_{\pi_0}$.
3. If $x \in A_{\pi_0}$ and $\lambda = w(x + \lambda_{\pi_0,U}) - \lambda_{\pi,U}$, then $T_{\pi,U}(\lambda) = T_{\pi_0,U}(x)$.

To prove this, observe that for each $\lambda \in A_{\pi_0}$, there exist $w_\lambda \in W_U$ and $\lambda' \in A_\pi$ such that

$$w_\lambda(\lambda + \lambda_{\pi_0,U}) = \lambda' + \lambda_{\pi,U}$$

in X_U. There are only finitely many possibilities for w_λ, and hence there exists $w \in W_U$ such that $w_\lambda = w$ for a set of λ whose closure has a non-empty interior.

Since the condition $w_\lambda = w$ is a closed condition, it holds on an open set and hence everywhere on A_{π_0}. Taking $\lambda = 0$ gives (1). Parts (2) and (3) follow. Set $Z = Z_{\pi_0,U}$ for a fixed π_0. We rewrite (23) as follows:

$$\sum_{\pi:Z_{\pi,U}=Z} \int_{A_\pi} \hat{f}_U(T_{\pi,U}(\lambda))g_\pi(\lambda)\ d\lambda = \sum_{\pi:Z_{\pi,U}=Z} \int_{A_{\pi_0}} \hat{f}_U(T_{\pi_0,U}(x))g_\pi(\lambda)\ dx$$
$$= \int_{A_{\pi_0}} \hat{f}_U(T_{\pi_0,U}(x)) \left(\sum_{\pi:Z_{\pi,U}=Z} g_\pi(\lambda) \right)\ dx = 0$$

where $\lambda + \lambda_{\pi,U} = w(x + \lambda_{\pi_0,U})$. The sum taken inside the integral is absolutely convergent for almost all λ by Fubini's theorem. It follows that the push-forward to $\widehat{G}_U^{un}$ with respect to $T_{\pi_0,U}$ of the measure

$$\left(\sum_{\pi:Z_{\pi,U}=Z} g_\pi(\lambda) \right)\ dx$$

is zero. We conclude that for almost all $\lambda_0 \in A_{\pi_0}$, we have

$$\sum_{(\pi,\lambda):T_{\pi,U}(\lambda)=T_{\pi_0,U}(\lambda_0)} g_\pi(\lambda) = 0. \tag{24}$$

It remains to prove that g_{π_0} is identically zero. Fix U as above and fix a set $Z = Z_{\pi_0,U}$. Suppose that $g_{\pi_0}(\lambda_0) \neq 0$. Let Y be the set of pairs (π, λ) such that $T_{\pi,U}(\lambda) = T_{\pi_0,U}(\lambda_0)$. We may choose a finite subset $Y' \subset Y$ such such that

$$\sum_{Y-Y'} |g_\pi(\lambda)| < |g_{\pi_0}(\lambda_0)|/2.$$

By choosing $U' \supset U$ sufficiently large, we can ensure that $T_{\pi,U'}(\lambda) \neq T_{\pi_0,U'}(\lambda_0)$ for all $(\pi, \lambda) \in Y'$ such that $I(\pi, \lambda)^S$ and $I(\pi_0, \lambda_0)^S$ have distinct unramified constituents. By Assumption (1), this holds if $\pi \neq \pi_0$. Equality (24) for U' yields

$$\sum_{\lambda:T_{\pi_0,U'}(\lambda)=T_{\pi_0,U'}(\lambda_0)} g_{\pi_0}(\lambda) + \sum_{(\pi,\lambda):\pi\neq\pi_0,T_{\pi,U'}(\lambda)=T_{\pi_0,U'}(\lambda_0)} g_\pi(\lambda) = 0. \tag{25}$$

By Assumption (2), the first term is a positive integer multiple of multiple of $g_{\pi_0}(\lambda_0)$. The second term is bounded by $|g_{\pi_0}(\lambda_0)|/2$. This is a contradiction. □

Proof of Proposition 6. Fix a character χ and let S be a finite set of places containing the archimedean places such that χ and E/F are unramified outside

S. We shall consider functions $f = f_S \otimes f^S$ where f_S is fixed and f^S varies in the Hecke algebra $\mathcal{H}^S$. Write (18) as an equality

$$RT(f) - KT(f') = 0.$$

Using the fundamental lemma we can write this in the form (21). To apply Lemma 4, we must check that the Assumptions hold. Assumption (1) is a consequence of the classification theorem of Jacquet-Shalika [JS] applied to $GL(3)$. To check Assumption (2), observe that if $I(\pi,\lambda)^S \simeq I(\pi,\lambda')^S$, then $(\pi,\lambda) = (w\pi, w\lambda')$ by [JS]. Remark 5 then implies that $g_\pi(\lambda) = g_\pi(\lambda')$. □

11 Decomposable distributions

We still have to derive identities for the individual ν's in (19). We note that (19) is an equality between sums of four decomposable distributions. We have the following elementary

Lemma 5. *Let V_1, V_2, V_3 be vector spaces. Consider vectors $x_i, x_i' \in V_1$, $y_i, y_i' \in V_2$, $z_i, z_i' \in V_3$ for $i = 1, \dots, n$ such that*

$$\sum_{i=1}^n x_i \otimes y_i \otimes z_i = \sum_{i=1}^n x_i' \otimes y_i' \otimes z_i'.$$

If each of the sets $\{x_i\}, \{y_i\}$ and $\{z_i\}$ is linearly independent, then there exists a permutation σ of $\{1,\dots,n\}$ such that $x_i' \otimes y_i' \otimes z_i' = x_{\sigma(i)} \otimes y_{\sigma(i)} \otimes z_{\sigma(i)}$ for all i.

Proof. The hypothesis implies that the span of $\{x_i'\}$ is equal to the span of $\{x_i\}$, and similarly for the y's and z's. In particular, the sets $\{x_i'\}$, $\{y_i'\}$, $\{z_i'\}$ are linearly independent. Write $x_i' = \sum_j \alpha_{ij} x_j$. Since the sum $\sum x_j \otimes V_2 \otimes V_3$ is direct, we must have

$$x_j \otimes y_j \otimes z_j = \sum_i \alpha_{ij} x_j \otimes y_i' \otimes z_i',$$

for all j and hence $y_j \otimes z_j = \sum_i \alpha_{ij} y_i' \otimes z_i'$. Writing y_j and z_j in terms of the linearly independent sets $\{y_i'\}$ and $\{z_i'\}$, we see that α_{ij} is non-zero for exactly one i. □

We also have the following

Lemma 6. *For v inert the distributions $\tilde{B}_v^{st}(f,\nu,\lambda)$, $\nu \in \mathcal{B}(\chi)$ are linearly independent for λ generic.*

Proof. Suppose that L' and $L_1, ..., L_n$ are linear functionals on V_π where (π, V) is an irreducible unitary representation of a reductive group G over a local field. Assume that L' is non-zero and set

$$O_i(f) = \sum_{\{\varphi\} \subset V_\pi} L_i(\pi(f)\varphi)\overline{L'(\varphi)}$$

for $i = 1, \ldots, n$. Then the O_i's are linearly independent distributions if and only if the L_i are linearly independent. Indeed, any relation among the O_i's would imply that

$$\sum_{\varphi} L(\pi(f)\varphi)\overline{L'(\varphi)} = 0 \tag{26}$$

for some linear combination L of the L_i's. Fix a compact open K small enough so that $L'|_{V^K} \neq 0$. Then (26) implies that

$$\sum_{\{\varphi\} \subset V^K} L(\pi(f)\varphi)\overline{L'(\varphi)} = 0$$

for any $f \in \mathcal{H}(G, K)$. We can rewrite this as

$$L(\pi(f)\varphi_0) = 0$$

for some $0 \neq \varphi_0 \in V^K$ and all $f \in \mathcal{H}(G, K)$. This implies that $L|_{V^K} = 0$.
Thus, in order to prove the Lemma, it suffices to show that the functionals $\{J_v^{st}(\nu_v, \varphi_v, \lambda)\}_{\nu \in \mathcal{B}(\chi)}$ are linearly independent. However, it is clear that $\{J(\eta, \varphi_v, \lambda)\}_\eta$ is a linearly independent set in the range of convergence, since they are given by integrals over disjoint open orbits. Confining ourselves to $\varphi \in V^K$ where K is small enough, the condition of linear independence can be expressed in terms of a non-vanishing of some determinant which is a meromorphic function in λ (in fact rational in q^λ). Thus, it holds for generic λ. Finally, in order to prove the same thing for the J_v^{st}, it suffices to check that the matrix of coefficients $(\Delta_{\nu,\lambda}(\eta)^{-1})_{\nu,\eta}$ is non-singular. However the determinant of this matrix is easily seen to be a non-zero constant multiple of a power of q^λ times the determinant of the character table of $A(F_v)$. □

COROLLARY 1. *There exists a permutation τ_χ of $\mathcal{B}(\chi)$ such that*

$$\tilde{B}^{st}(f, \nu, \lambda) = B'(f', \tau_\chi(\nu), \lambda) \tag{27}$$

for all λ.

Proof. First, as was noted by Jacquet in ([J2], §4) one may use the localization principle to infer that $B'(f', \nu, \lambda)$ depends only on the orbital integrals used in the definition of matching functions, and hence only on f. Choose any two finite places u, u' of F which are inert in E and view each term in the equality

$$\sum_{\nu \in \mathcal{B}(\chi)} \tilde{B}^{st}(f, \nu, \lambda) = \sum_{\nu \in \mathcal{B}(\chi)} B'(f', \nu, \lambda)$$

of Proposition 6 as a decomposable distribution in f with three factors, where the first two factors are the components at u and u' and the third factor is the product of all components away from u and u'. Lemma 6 allows us to apply Lemma 5 to conclude that (27) holds with the permutation a-priori depending on $\lambda \in i\mathfrak{a}_0^*$. Each side of (27) is the restriction to $i\mathfrak{a}_0^*$ of a meromorphic function on $\mathfrak{a}_{0,\mathbb{C}}^*$. Thus, the permutation does not depend on λ, because there are only finitely many of them. □

12 Uniform distribution of Hecke characters

We need to prove that $\tau_\chi(\nu) = \nu$. We first prove a Lemma which is interesting in its own right. Let F be a number field and S a finite set of places including the archimedean ones. We write $S = S_\infty \cup S_f$. Embed $\mathbb{R}_+$ in $F \otimes_\mathbb{Q} \mathbb{R}$ by $x \mapsto 1 \otimes x$. For any ideal I of $\mathcal{O}_F$ let $\phi(I) = |(\mathcal{O}_F/I)^*|$.

Lemma 7. *Let $\{I_k\}$ be a family of ideals, disjoint from S whose norms $N(I_k)$ tend to ∞. For each k let X_k be the set of Hecke characters of F which are trivial on $\mathbb{R}_+$ and whose conductor divides $I_k J$ for some ideal J whose prime factors lie in S. Then for any $f_S \in C_c^\infty(F_S^*)$ we have*

$$1/\phi(I_k) \sum_{\varrho \in X_k} \hat{f}_S(\varrho_S) \to \lambda_F \int_{\mathbb{R}_+} f_S(t)\ d^*t$$

where $\lambda_F = \mathrm{vol}(F^*\mathbb{R}_+ \backslash \mathbb{I}_F)$.

Proof. This is a simple application of the trace formula for $L^2(\mathbb{R}_+ F^* \backslash \mathbb{I}_F)$. Let $f = f_S \otimes f_k \in C_c^\infty(\mathbb{I}_F)$ where $f_S \in C_c^\infty(F_S^*)$ is fixed and f_k is the characteristic function of $\{x \in \prod_{v \notin S} \mathcal{O}_v^* : x \equiv 1 \pmod{I_k}\}$. The Poisson summation formula gives

$$\lambda_F \sum_\gamma g(\gamma) = \sum_\varrho \hat{f}(\varrho) \tag{28}$$

where $g(x) = \int_{\mathbb{R}_+} f(tx)\ d^*t$. By our choice of f, the sum in the right hand side extends over X_k, and in the left hand side only $\gamma = 1$ contributes provided that k is sufficiently large. □

By standard methods, this Lemma implies that as $k \to \infty$, the set of restrictions ϱ_S of the Hecke characters ϱ in X_k is uniformly distributed in the dual of F_S^*. The Lemma carries over immediately to the torus T', which is a product of copies of the multiplicative group. We shall use this variant to prove the following Corollary. If Q is a finite set of finite places, we denote by $\mathcal{U}_Q$ the space of unramified unitary characters of $T'(F_Q)$ with the usual topology.

Corollary 2. *Given a place $w \notin S$, a unitary character $\eta = (\eta_v)_{v \in S_f}$ of $T'(F_{S_f})$ and an open set $U \subset \mathcal{U}_{S_f}$ there exists a Hecke character ϱ of T' which is unramified outside $S \cup \{w\}$ such that $\varrho_{S_f}^{-1}\eta \in U$.*

13 Proof of Theorems 2 and 3

We now finish the proofs of Theorem 2 and Theorem 3.
We first prove Theorem 3 by choosing a favorable global situation. Suppose that we are given local data which consists of:

- a quadratic extension E^0/F^0, and

- a unitary character μ of $T'(F^0)$.

In principle there is also an additive character of F^0, but we are free to choose it at will by Remark 4. We can find a quadratic extension of number fields E/F and a place v_1 of F such that $E_{v_1}/F_{v_1} \simeq E^0/F^0$. By passing to EK/FK for an appropriate K we can assume in addition that

1. Every real and even place of F splits at E.
2. Let $S_1 = \{v_i\}_{i=1}^l$ be the set (possible empty) of places of F which ramify over E. Then $E_{v_i}/F_{v_i} \simeq E^0/F^0$. We fix such isomorphisms.

Choose a non-trivial additive character ψ of $F\backslash\mathbb{A}_F$. Let w_1 be an auxiliary place of F which is inert in E with residual characteristic p. Assume that $p \not| q_{F^0}$. Let $S_2 = \{w_j\}_{j=1}^m$ be the places of F of residual characteristic p. We may also assume that ψ_v is unramified for $v \in S_2$. Set

$$L_p(\eta, s) = \prod_j L_{w_j}(\eta_{w_j}, s)$$

for any Hecke character η.

Lemma 8. *There exists an open set U_2 of $\mathcal{U}_{S_2}$ such that whenever ν is a Hecke character of T' such that $\nu_{S_2} \in U_2$ we necessarily have $\tau_\chi(\nu) = \nu$ in the notations of Corollary 1.*

To deduce Theorem from Lemma 8, apply Corollary 2 with the following data:

1. $S = S_\infty \cup S_1 \cup S_2$.
2. $\eta_v = \mu$ for $v \in S_1$.
3. $\eta_v = 1$ for $v \in S_2$.
4. $U = U_1 \times U_2$ where U_1 is an open set of $\mathcal{U}_{S_1}$.
5. $w \notin S$ is any place of F which splits at E.

Corollary 2 implies that there exists a ν such that $\nu_{S_2} \in U_2$ and $\nu_{S_2}\mu^{-1} \in U_1$. In particular, $\tau_\chi(\nu) = \nu$ by our claim. The equality (27) yields the proportionality of the local distributions, i.e.

$$\tilde{B}^{st}(f_v, \nu, \lambda) = c_v(\nu_v, \lambda) B'(f'_v, \nu, \lambda).$$

Let

$$d_v(\nu_v, \lambda) = c_v(\nu_v, \lambda)/\gamma_v(\nu_v, \lambda).$$

A-priori, d_v is a rational function in q^λ depending on ν and λ. In the split case Proposition 4 shows that $d_v(\nu_v, \lambda) = 1$. In the unramified case, the same holds by Proposition 5 and Remark 4. Thus, by our conditions we have $d_v(\nu_v, \lambda) = 1$ except possibly for $v = v_i$. Since $\prod_v c_v(\nu_v, \lambda) = 1$ we have

$$\prod_i d_{v_i}(\nu_{v_i}, \lambda) = 1. \tag{29}$$

Write $\nu_{v_i}\mu^{-1} = |\cdot|^{\lambda_i}$ with $\lambda_i \in i\mathfrak{a}_0^*$. The relation (29) implies that

$$\prod_i d(\mu, \lambda + \lambda_i) = 1.$$

If $d(\mu, \cdot)$ were not constant this would impose a non-trivial closed condition on the λ_i's and this would contradict Corollary 2 for an appropriate choice of U_1. Hence $d(\mu, \cdot)$ is a constant. In fact, $d(\mu, \cdot)$ is an l-th root of unity where l is the cardinality of S_1 above. To show that d is independent of μ as well, let μ_1 be given and apply the same corollary with η as before except that $\eta_{v_1} = \mu_1$. Then (29) implies that $d(\mu)^{l-1}d(\mu_1) = 1$ so that $d(\mu_1) = d(\mu)$ as required.

We now prove Lemma 8. Let $\nu' = \tau_\chi(\nu)$ and let $\tilde{S}$ be a finite set of places of F including the archimedean ones, outside of which E/F, ψ and ν are unramified. We are free to chose $\tilde{S}$ so that $S_2 \cap \tilde{S} = \emptyset$. By applying Proposition 3.2 and Lemma 3.1 of [JY], we may fix matching functions $f_{\tilde{S}} \leftrightarrow f'_{\tilde{S}}$ such that $B'_{\tilde{S}}(f'_{\tilde{S}}, I(\nu', \lambda)) \not\equiv 0$ as a function of λ. Let $f = f_{\tilde{S}} \otimes \mathbf{1}_{K^{\tilde{S}}}$ and $f' = f'_{\tilde{S}} \otimes \mathbf{1}_{K'^{\tilde{S}}}$. Then $B'(f', I(\nu', \lambda)) \not\equiv 0$.

Since we assume that all real places of F split at E, the relation (27) gives

$$D^{\nu}_{\tilde{S}}(f, \lambda) L^{\tilde{S}}_{(1)}(\nu, \lambda)\overline{L^{\tilde{S}}_{(2)}(\chi, \lambda)} = \gamma_\infty(\nu, \lambda, \psi) D'^{\nu'}_{\tilde{S}}(f', \lambda)|L^{\tilde{S}}_{(3)}(\nu', \lambda)|^2 \tag{30}$$

where $D^{\nu}_{\tilde{S}}$, $D'^{\nu'}_{\tilde{S}}$ are non-zero rational functions in $\tilde{S}^\lambda = \{q^\lambda : q = q_v, v \in \tilde{S}\}$, $L^{\tilde{S}}_{(1)}(\nu, \lambda)$ is the partial L-function computed in Proposition 3 and $L^{\tilde{S}}_{(2)}(\chi, \lambda)$ (resp. $L^{\tilde{S}}_{(3)}(\nu', \lambda)$) is the L-function giving the Fourier coefficient of the Eisenstein series on G (resp. G'), and finally, γ_∞ is the product of the local γ factors (5) of the archimedean places. As is well known,

$$L^{\tilde{S}}_{(2)}(\chi, \lambda) = \Big(L^{\tilde{S}}(\chi_1\chi_2^{-1}, s_1 + 1)L^{\tilde{S}}(\chi_2\chi_3^{-1}, s_2 + 1)L^{\tilde{S}}(\chi_1\chi_3^{-1}, s_1 + s_2 + 1)\Big)^{-1}$$

and

$$L^{\tilde{S}}_{(3)}(\nu, \lambda) = \Big(L^{\tilde{S}}(\nu_1\nu_2^{-1}, s_1 + 1)L^{\tilde{S}}(\nu_2\nu_3^{-1}, s_2 + 1)L^{\tilde{S}}(\nu_1\nu_3^{-1}, s_1 + s_2 + 1)\Big)^{-1}.$$

Since $\overline{L^{\tilde{S}}_{(2)}(\chi, \lambda)} = L^{\tilde{S}}_{(2)}(\chi^{-1}, \overline{\lambda})$ and $s_i \in i\mathbb{R}$, we obtain a relation

$$\begin{gathered}\frac{L^{\tilde{S}}(\nu_1\nu_2^{-1}\omega, s_1)L^{\tilde{S}}(\nu_2\nu_3^{-1}\omega, s_2)L^{\tilde{S}}(\nu_1\nu_3^{-1}\omega, s_1 + s_2)}{L^{\tilde{S}}(\nu_1\nu_2^{-1}, s_1 + 1)L^{\tilde{S}}(\nu_2\nu_3^{-1}, s_2 + 1)L^{\tilde{S}}(\nu_1\nu_3^{-1}, s_1 + s_2 + 1)} \times \\ (L^{\tilde{S}}(\chi_1^{-1}\chi_2, -s_1 + 1)L^{\tilde{S}}(\chi_2^{-1}\chi_3, -s_2 + 1)L^{\tilde{S}}(\chi_1^{-1}\chi_3, -s_1 - s_2 + 1))^{-1}A_{\tilde{S}}(\lambda) \\ = \gamma_\infty(\nu, \lambda, \psi)|L^{\tilde{S}}(\nu_1'\nu_2'^{-1}, s_1 + 1)L^{\tilde{S}}(\nu_2'\nu_3'^{-1}, s_2 + 1)L^{\tilde{S}}(\nu_1'\nu_3'^{-1}, s_1 + s_2 + 1)|^{-2}\end{gathered} \tag{31}$$

where $A_{\tilde{S}}(\lambda)$ is a rational function in $\tilde{S}^{\lambda}$. Note that for any Hecke character μ_0 of F which base changes to μ we have

$$L^{\tilde{S}}(\mu, s) = L^{\tilde{S}}(\mu_0, s)L^{\tilde{S}}(\mu_0\omega, s).$$

Hence, by the functional equation

$$\frac{L(\mu_0\omega, s)}{L(\mu_0, s+1)\cdot L(\mu^{-1}, -s+1)} = \epsilon(\mu_0\omega, s)^{-1}|L(\mu_0, s+1)|^{-2}$$

for $s \in i\mathbb{R}$. Working with partial L-functions we obtain the relation

$$\left|L^{\tilde{S}}(\nu_1\nu_2^{-1}, s_1+1)\, L^{\tilde{S}}(\nu_2\nu_3^{-1}, s_2+1)\, L^{\tilde{S}}(\nu_1\nu_3^{-1}, s_1+s_2+1)\right|^2 A'_S(\lambda) =$$

$$\left|L^{\tilde{S}}(\nu'_1\nu_2, {}^{-1}, s_1+1)\, L^{\tilde{S}}(\nu'_2\nu'_3{}^{-1}, s_2+1)\, L^{\tilde{S}}(\nu'_1\nu'_3{}^{-1}, s_1+s_2+1)\right|^2$$

where $A'_{\tilde{S}}(\lambda)$ is also a rational function in $\tilde{S}^{\lambda}$. We will assume now that $\nu' \neq \nu$ and obtain a contradiction. Suppose, to be specific, that $\nu'_1 = \nu_1$ but $\nu'_j = \nu_j\omega$ for $j = 2, 3$. Thus we obtain

$$\begin{aligned}|L^{\tilde{S}}(\nu_1\nu_2{}^{-1}, s_1+1)L^{\tilde{S}}(\nu_1\nu_3{}^{-1}, s_1+s_2+1)|^{-2}A'_{\tilde{S}}(\lambda)\\ = |L^{\tilde{S}}(\nu_1\nu_2{}^{-1}\omega, s_1+1)L^{\tilde{S}}(\nu_1\nu_3{}^{-1}\omega, s_1+s_2+1)|^{-2}\end{aligned}$$

where $A'_{\tilde{S}}(\lambda)$ is as before. This implies then that $A'_S(\lambda)$ decomposes into a product of rational functions depending only on s_1 and s_1+s_2 respectively and then the identity is equivalent to two new identities, one of which reads:

$$\left|L^S(\nu_1\nu_2^{-1}, s+1)\right|^2 c(\lambda) = \left|L^S(\nu_1\nu_2^{-1}\omega, s+1)\right|^2$$

where $c(\lambda)$ is a rational function in $q^{\tilde{S}}$. Using the functional equations again and the fact that $s \in i\mathbb{R}$, one can write this in the form

$$L^{\tilde{S}}(\nu_1\nu_2{}^{-1}, s+1)L^{\tilde{S}}(\nu_1\nu_2{}^{-1}, s)c_{\tilde{S}}(s) = L^{\tilde{S}}(\nu_1\nu_2{}^{-1}\omega, s+1)L^{\tilde{S}}(\nu_1\nu_2{}^{-1}\omega, s)$$

where $c_{\tilde{S}}(s)$ is a rational function in q^s, $q|\tilde{S}$. Note that the γ factors at ∞ cancel because E/F splits at all real places. This relation now holds as an equality of meromorphic functions. For $\mathrm{Re}(s)$ large enough, both sides can be expanded as Dirichlet series and we can compare their p-power coefficients to conclude:

$$L_p(\nu_1\nu_2^{-1}, s)L_p(\nu_1\nu_2^{-1}, s+1) = L_p(\nu_1\nu_2^{-1}\omega, s)L_p(\nu_1\nu_2^{-1}\omega, s+1).$$

This imposes a non-trivial constraint on ν_{S_2}. A similar argument gives the other cases. This proves Lemma 8 and hence finishes the proof of Theorem 3. Theorem 2 is an immediate consequence of Theorem 3 and Corollary 1.

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