The Farrell Cohomology of $\mathrm{SP}(p-1,\mathbb{Z})$

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Abstract. Let p be an odd prime with odd relative class number h^-. In this article we compute the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$, the first p-rank one case. This allows us to determine the p-period of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$, which is $2y$, where $p-1=2^r y$, y odd. The p-primary part of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$ is given by the Farrell cohomology of the normalizers of the subgroups of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$. We use the fact that for odd primes p with h^- odd a relation exists between representations of $\mathbb{Z}/p\mathbb{Z}$ in $\mathrm{Sp}(p-1,\mathbb{Z})$ and some representations of $\mathbb{Z}/p\mathbb{Z}$ in $\mathrm{U}((p-1)/2)$.

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1 Introduction

We define a homomorphism

$$\begin{array}{rccl} \phi: & \mathrm{U}(n) & \longrightarrow & \mathrm{Sp}(2n,\mathbb{R}) \\ & X=A+iB & \longmapsto & \begin{pmatrix} A & B \\ -B & A \end{pmatrix} =: \phi(X) \end{array}$$

where A and B are real matrices. Then ϕ is injective and maps $\mathrm{U}(n)$ on a maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. This homomorphism allows to consider each representation

$$\widetilde{\rho}: \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{U}((p-1)/2)$$

as a representation

$$\phi\circ\widetilde{\rho}: \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Sp}(p-1,\mathbb{R}).$$

In an article of Busch [6] it is determined which properties $\widetilde{\rho}$ has to fulfil for $\phi\circ\widetilde{\rho}$ to be conjugate in $\mathrm{Sp}(p-1,\mathbb{R})$ to a representation

$$\rho: \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Sp}(p-1,\mathbb{Z}).$$

Theorem 2.2. *Let* $X \in \mathrm{U}((p-1)/2)$ *be of odd prime order* p. *We define* $\phi : \mathrm{U}((p-1)/2) \to \mathrm{Sp}(p-1, \mathbb{R})$ *as above. Then* $\phi(X) \in \mathrm{Sp}(p-1,\mathbb{R})$ *is conjugate to* $Y \in \mathrm{Sp}(p-1,\mathbb{Z})$ *if and only if the eigenvalues* $\lambda_1, \ldots, \lambda_{(p-1)/2}$ *of* X *are such that*

$$\{\lambda_1, \ldots, \lambda_{(p-1)/2}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{(p-1)/2}\}$$

is a complete set of primitive p*-th roots of unity.*

The proof of Theorem 2.2 involves the theory of cyclotomic fields. For the p-primary component of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$, the following holds:

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)} \cong \prod_{P \in \mathfrak{P}} \widehat{\mathrm{H}}^*(N(P),\mathbb{Z})_{(p)}$$

where $\mathfrak{P}$ is a set of representatives for the conjugacy classes of subgroups of order p of $\mathrm{Sp}(p-1,\mathbb{Z})$ and $N(P)$ denotes the normalizer of $P \in \mathfrak{P}$. This property also holds if we consider $\mathrm{GL}(p-1,\mathbb{Z})$ instead of the symplectic group. This fact was used by Ash in [1] to compute the Farrell cohomology of $\mathrm{GL}(n,\mathbb{Z})$ with coefficients in $\mathbb{F}_p$ for $p-1 \leqslant n < 2p-2$. Moreover, we have

$$\widehat{\mathrm{H}}^*(N(P),\mathbb{Z})_{(p)} \cong \left(\widehat{\mathrm{H}}^*(C(P),\mathbb{Z})_{(p)}\right)^{N(P)/C(P)}$$

where $C(P)$ is the centralizer of P. We will determine the structure of $C(P)$ and of $N(P)/C(P)$. After that we will compute the number of conjugacy classes of those subgroups for which $N(P)/C(P)$ has a given structure. Here again arithmetical questions are involved. In the articles of Brown [2] and Sjerve and Yang [9] is shown that the number of conjugacy classes of elements of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$ is $2^{(p-1)/2}h^-$ where h^- denotes the relative class number of the cyclotomic field $\mathbb{Q}(\xi)$, ξ a primitive p-th root of unity. If h^- is odd, each conjugacy class of matrices of order p in $\mathrm{Sp}(p-1,\mathbb{R})$ that lifts to $\mathrm{Sp}(p-1,\mathbb{Z})$ splits into h^- conjugacy classes in $\mathrm{Sp}(p-1,\mathbb{Z})$. The main results in this article are

Theorem 3.7. *Let* p *be an odd prime for which* h^- *is odd. Then*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)} \cong \prod_{\substack{k \mid p-1 \\ k \text{ odd}}} \left(\prod_{1}^{\widetilde{\mathcal{K}}_k} \mathbb{Z}/p\mathbb{Z}[x^k, x^{-k}]\right),$$

where $\widetilde{\mathcal{K}}_k$ *denotes the number of conjugacy classes of subgroups of order* p *of* $\mathrm{Sp}(p-1,\mathbb{Z})$ *for which* $|N/C| = k$. *Moreover* $\widetilde{\mathcal{K}}_k \geqslant \mathcal{K}_k$, *where* $\mathcal{K}_k$ *is the number of conjugacy classes of subgroups of* $\mathrm{U}((p-1)/2)$ *with* $|N/C| = k$. *As usual* N *denotes the normalizer and* C *the centralizer of the corresponding subgroup.*

Theorem 3.8. *Let* p *be an odd prime for which* h^- *is odd and let* y *be such that* $p-1 = 2^r y$ *and* y *is odd. Then the period of* $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)}$ *is* $2y$.

Corresponding results have been shown for other groups, for example $\mathrm{GL}(n,\mathbb{Z})$ in the p-rank one case [1], the mapping class group [8] and the outerautomorphism group of the free group in the p-rank one case [7].

This article presents results of my doctoral thesis, which I wrote at the ETH Zürich under the supervision of G. Mislin. I thank G. Mislin for the suggestion of this interesting subject.

2 The symplectic group

2.1 Definition

Let R be a commutative ring with 1. The general linear group $\mathrm{GL}(n,R)$ is defined to be the multiplicative group of invertible $n\times n$-matrices over R.

Definition. The symplectic group $\mathrm{Sp}(2n,R)$ over the ring R is the subgroup of matrices $Y \in \mathrm{GL}(2n,R)$ that satisfy

$$Y^{\mathrm{T}}JY = J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n\times n$-identity matrix.

It is the group of isometries of the skew-symmetric bilinear form

$$\begin{array}{llcl} \langle\ ,\ \rangle : & R^{2n}\times R^{2n} & \longrightarrow & R \\ & (x,y) & \longmapsto & \langle x,y\rangle := x^{\mathrm{T}}Jy. \end{array}$$

It follows from a result of Bürgisser [4] that elements of odd prime order p exist in $\mathrm{Sp}(2n,\mathbb{Z})$ if and only if $2n \geqslant p-1$.

Proposition 2.1. *The eigenvalues of a matrix $Y \in \mathrm{Sp}(p-1,\mathbb{Z})$ of odd prime order p are the primitive p-th roots of unity, hence the zeros of the polynomial*

$$m(x) = x^{p-1} + \cdots + x + 1.$$

Proof. If λ is an eigenvalue of Y, we have $\lambda = 1$ or $\lambda = \xi$, a primitive p-th root of unity, and the characteristic polynomial of Y divides $x^p - 1$ and has integer coefficients. Since $m(x)$ is irreducible over $\mathbb{Q}$, the claim follows. □

2.2 A relation between $\mathrm{U}\left(\frac{p-1}{2}\right)$ and $\mathrm{Sp}(p-1,\mathbb{Z})$

Let $X \in \mathrm{U}(n)$, i.e., $X \in \mathrm{GL}(n,\mathbb{C})$ and $X^*X = I_n$ where $X^* = \overline{X}^{\mathrm{T}}$ and I_n is the $n\times n$-identity matrix. We can write $X = A + iB$ with $A, B \in \mathrm{M}(n,\mathbb{R})$, the ring of real $n\times n$-matrices. We now define the following map

$$\begin{array}{lccl} \phi : & \mathrm{U}(n) & \longrightarrow & \mathrm{Sp}(2n,\mathbb{R}) \\ & X = A + iB & \longmapsto & \begin{pmatrix} A & B \\ -B & A \end{pmatrix} =: \phi(X). \end{array}$$

The map ϕ is an injective homomorphism. Moreover, it is well-known that ϕ maps $\mathrm{U}(n)$ on a maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$.

Theorem 2.2. *Let $X \in \mathrm{U}((p-1)/2)$ be of odd prime order p. We define $\phi : \mathrm{U}((p-1)/2) \to \mathrm{Sp}(p-1,\mathbb{R})$ as above. Then $\phi(X) \in \mathrm{Sp}(p-1,\mathbb{R})$ is conjugate to $Y \in \mathrm{Sp}(p-1,\mathbb{Z})$ if and only if the eigenvalues $\lambda_1,\dots,\lambda_{(p-1)/2}$ of X are such that*

$$\left\{\lambda_1,\dots,\lambda_{(p-1)/2},\overline{\lambda}_1,\dots,\overline{\lambda}_{(p-1)/2}\right\}$$

is a complete set of primitive p-th roots of unity.

Proof. See [5] or [6]. □

In the proof of Theorem 2.2 we used the following facts. For a primitive p-th root of unity ξ, we consider the cyclotomic field $\mathbb{Q}(\xi)$. It is well-known that $\mathbb{Q}(\xi+\xi^{-1})$ is the maximal real subfield of $\mathbb{Q}(\xi)$, and that $\mathbb{Z}[\xi]$ and $\mathbb{Z}[\xi+\xi^{-1}]$ are the rings of integers of $\mathbb{Q}(\xi)$ and $\mathbb{Q}(\xi+\xi^{-1})$ respectively. Let $(\mathfrak{a},a)$ denote a pair where $\mathfrak{a} \subseteq \mathbb{Z}[\xi]$ and $a \in \mathbb{Z}[\xi]$ are chosen such that $\mathfrak{a} \neq 0$ is an ideal in $\mathbb{Z}[\xi]$ and $\mathfrak{a}\overline{\mathfrak{a}} = (a)$, a principal ideal. Here $\overline{\mathfrak{a}}$ denotes the complex conjugate of $\mathfrak{a}$. We define an equivalence relation on the set of those pairs by $(\mathfrak{a},a) \sim (\mathfrak{b},b)$ if and only if $\lambda,\mu \in \mathbb{Z}[\xi]\setminus\{0\}$ exist such that $\lambda\mathfrak{a} = \mu\mathfrak{b}$ and $\lambda\overline{\lambda}a = \mu\overline{\mu}b$. We denote by $[\mathfrak{a},a]$ the equivalence class of the pair $(\mathfrak{a},a)$ and by $\mathcal{P}$ the set of equivalence classes $[\mathfrak{a},a]$.

Let $\mathcal{S}_p$ denote the set of conjugacy classes of elements of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$. Sjerve and Yang have shown in [9] that a bijection exists between $\mathcal{P}$ and $\mathcal{S}_p$. If $Y \in \mathrm{Sp}(p-1,\mathbb{Z})$ is a matrix of order p, then the equivalence class $[\mathfrak{a},a] \in \mathcal{P}$ corresponding to the conjugacy class of Y in $\mathrm{Sp}(p-1,\mathbb{Z})$ can be determined in the following way. Let $\alpha = (\alpha_1,\dots,\alpha_{p-1})^{\mathrm{T}}$ be an eigenvector of Y corresponding to the eigenvalue $\xi = e^{i2\pi/p}$, that is $Y\alpha = \xi\alpha$. Then $\alpha_1,\dots,\alpha_{p-1}$ is a basis of an ideal $\mathfrak{a} \subseteq \mathbb{Z}[\xi]$. Sjerve and Yang [9] proved that this ideal $\mathfrak{a}$ has the property $[\mathfrak{a},a] \in \mathcal{P}$. Let h and h^+ be the class numbers of $\mathbb{Q}(\xi)$ and $\mathbb{Q}(\xi+\xi^{-1})$ respectively. Then $h^- := h/h^+$ denotes the relative class number. Sjerve and Yang [9] showed that the number of conjugacy classes of matrices of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$ is $h^-\,2^{(p-1)/2}$. The number of conjugacy classes in $\mathrm{U}((p-1)/2)$ of unitary matrices that satisfy the condition in Theorem 2.2 is $2^{(p-1)/2}$.

Let $\mathcal{U}_p$ denote the set of conjugacy classes of matrices in $\mathrm{U}((p-1)/2)$ that satisfy the condition on the eigenvalues that is given in Theorem 2.2. A consequence of Theorem 2.2 is that it is possible to define a map

$$\Psi : \mathcal{S}_p \longrightarrow \mathcal{U}_p$$

and that this map is surjective. Therefore the map

$$\psi : \mathcal{P} \longrightarrow \mathcal{U}_p$$

is surjective either.

For a given choice of the ideal $\mathfrak{a}$ (for example $\mathfrak{a} = \mathbb{Z}[\xi]$), we denote by $\mathcal{P}_{\mathfrak{a}}$ the set of those classes $[\mathfrak{a},a] \in \mathcal{P}$, where $\mathfrak{a}$ corresponds to our choice. If the restriction

$$\psi|_{\mathcal{P}_{\mathfrak{a}}} : \mathcal{P}_{\mathfrak{a}} \longrightarrow \mathcal{U}_p$$

is surjective each conjugacy class in $\mathcal{U}_p$ of matrices that satisfy Theorem 2.2 yields h^- conjugacy classes in $\mathrm{Sp}(p-1,\mathbb{Z})$. In general $\psi|_{\mathcal{P}_\mathfrak{a}}$ is not surjective. It is a result of Busch, [5], [6], that $\psi|_{\mathcal{P}_\mathfrak{a}}$ is surjective if h^- is odd. If h^- is even and h^+ is odd, we have no surjectivity of $\psi|_{\mathcal{P}_\mathfrak{a}}$. This happens for example for the primes 29 and 113.

2.3 Subgroups of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$

It follows from Theorem 2.2 that a mapping exists that sends the conjugacy classes of matrices $Y \in \mathrm{Sp}(p-1,\mathbb{Z})$ of odd prime order p onto the conjugacy classes of matrices X in $\mathrm{U}((p-1)/2)$ that satisfy the condition on the eigenvalues described in Theorem 2.2. This mapping is surjective.
It is clear that $\det X = e^{l2\pi i/p}$ for some $1 \leqslant l \leqslant p$. If $X \in \mathrm{U}((p-1)/2)$ satisfies the condition on the eigenvalues, then so does X^k, $k = 1,\ldots,p-1$. If $\det X = e^{l2\pi i/p}$ for some $1 \leqslant l \leqslant p-1$, then

$$\left\{\det X,\ldots,\det X^{p-1}\right\} = \left\{e^{i2\pi/p},\ldots,e^{i(p-1)2\pi/p}\right\}$$

and the X^k are in different conjugacy classes. If $\det X = 1$, it is possible that some k exists such that X and X^k are in the same conjugacy class. In this section we will analyse when and how many times this happens. The number of conjugacy classes of matrices $X \in \mathrm{U}((p-1)/2)$ that satisfy the condition required in Theorem 2.2 is $2^{(p-1)/2}$. Herewith we will be able to compute the number of conjugacy classes of subgroups of matrices of order p in $\mathrm{U}((p-1)/2)$. We remember that the number of conjugacy classes of matrices of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$ is $2^{(p-1)/2}h^-$. If $h^- = 1$, a bijection exists between the conjugacy classes of matrices of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$ and the conjugacy classes of matrices of order p in $\mathrm{U}((p-1)/2)$ that satisfy the condition required in Theorem 2.2.
Let $X \in \mathrm{U}((p-1)/2)$ with $X^p = 1$, $X \neq 1$. Then X generates a subgroup S of order p in $\mathrm{U}((p-1)/2)$. If $\det X = 1$, it is possible that X is conjugate to $X' \in S$ with $X \neq X'$. Two matrices in $\mathrm{U}((p-1)/2)$ are conjugate to each other if and only if they have the same eigenvalues. The set of eigenvalues of X is

$$\left\{e^{ig_1 2\pi/p},\ldots,e^{ig_{(p-1)/2}2\pi/p}\right\}$$

where $1 \leqslant g_l \leqslant p-1$ for $l = 1,\ldots,\frac{p-1}{2}$ and for all $l \neq j$, $l,j = 1,\ldots,(p-1)/2$, $g_l \neq p-g_j$ and $g_l \neq g_j$. From now on we consider the g_j as elements of $(\mathbb{Z}/p\mathbb{Z})^*$. The matrix X is conjugate to X^κ for some κ if the eigenvalues of X and X^κ are the same. This is equivalent to

$$\{g_1,\ldots,g_{(p-1)/2}\} = \{\kappa g_1,\ldots,\kappa g_{(p-1)/2}\} \subset (\mathbb{Z}/p\mathbb{Z})^*$$

where g_j and κg_j, $j = 1,\ldots,(p-1)/2$, denote the corresponding congruence classes.

We introduce some notation that will be used in the whole section. Let

$$G := \{g_1, \dots, g_{(p-1)/2}\} \subset (\mathbb{Z}/p\mathbb{Z})^*,$$
$$\kappa G := \{\kappa g_1, \dots, \kappa g_{(p-1)/2}\} \subset (\mathbb{Z}/p\mathbb{Z})^*$$

for some $\kappa \in (\mathbb{Z}/p\mathbb{Z})^*$. Let x be a generator of the multiplicative cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$ and let K be a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ with $|K| = k$. Then K is cyclic and k divides $p-1$. Let $m := (p-1)/k$, then x^m generates K.
First we prove the following proposition.

Proposition 2.3. *Let $G \subset (\mathbb{Z}/p\mathbb{Z})^*$ be a subset with $|G| = (p-1)/2$. The following are equivalent.*

i) For all $g_j, g_l \in G$, $g_j \neq -g_l$ and $\kappa \in (\mathbb{Z}/p\mathbb{Z})^*$ exists with $\kappa G = G$, $\kappa \neq 1$.

ii) An integer $h \in \mathbb{N}$, $1 \leqslant h \leqslant (p-1)/2$, and $n_j \in (\mathbb{Z}/p\mathbb{Z})^*$, $j = 1, \dots, h$, exist with

$$G = \bigcup_{j=1}^{h} n_j K$$

where

- *$K \subset (\mathbb{Z}/p\mathbb{Z})^*$ is the subgroup generated by κ,*
- *the order of K is odd,*
- *for $\kappa' \in K$ and all $j, l = 1, \dots, h$, $n_j \neq -n_l\kappa'$,*
- *and for all $j = 2, \dots, h$, $n_j \notin K$.*

Then we will analyse the uniqueness of this decomposition of G. This will enable us to determine the number of $G \subset (\mathbb{Z}/p\mathbb{Z})^*$ with $|G| = (p-1)/2$ and $G = \kappa G$ for some $1 \neq \kappa \in (\mathbb{Z}/p\mathbb{Z})^*$. Herewith we will determine the number of conjugacy classes of subgroups of order p in $\mathrm{U}((p-1)/2)$ whose group elements satisfy the condition of Theorem 2.2.

Definition. Let $\kappa \in (\mathbb{Z}/p\mathbb{Z})^*$ and let K be the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by κ. Let $G \subset (\mathbb{Z}/p\mathbb{Z})^*$ be a subset with $|G| = (p-1)/2$. We say that K decomposes G if G, κ and K fulfil the conditions of Proposition 2.3.

So K decomposes G if the order of the group K is odd and G is a disjoint union of cosets $n_1K, \dots, n_hK$ of K in $(\mathbb{Z}/p\mathbb{Z})^*$ for which for all n_j, n_l, $j, l = 1, \dots, h$, holds $n_jK \neq -n_lK$.

Lemma 2.4. *Let $G \subset (\mathbb{Z}/p\mathbb{Z})^*$ with $|G| = (p-1)/2$. Then $1 \neq \kappa \in (\mathbb{Z}/p\mathbb{Z})^*$ exists with $\kappa G = G$ if and only if $1 \leqslant h \leqslant (p-1)/2$ and $n_j \in (\mathbb{Z}/p\mathbb{Z})^*$, $j = 1, \dots, h$, exist with*

$$G = \bigcup_{j=1}^{h} n_j K$$

where $n_j \notin K$ for $j = 2, \dots, h$, and K is the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ that is generated by κ.

Proof. $\underline{\Leftarrow}$: Let $\kappa^l \in K$. Then

$$\kappa^l G = \kappa^l \bigcup_{j=1}^{h} n_j K = \bigcup_{j=1}^{h} n_j \kappa^l K = \bigcup_{j=1}^{h} n_j K = G.$$

$\underline{\Rightarrow}$: Without loss of generality we assume that $1 \in G$. If $1 \notin G$, $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$ exists with $1 \in \lambda G$ because $(\mathbb{Z}/p\mathbb{Z})^*$ is a multiplicative group. Of course $\kappa\lambda G = \lambda G$. Moreover, it is easy to see that if λG is a union of cosets of K, this is also true for G. The equation $\kappa G = G$ implies that $KG = G$. If $1 \in G$, then $K \subseteq G$ since $KG = G$. If $K = G$, we have finished the proof. If $K \neq G$, we consider $G_1' = G \setminus K$. For all $\kappa^l \in K$ we have $\kappa^l K = K$ and

$$\kappa^l G_1' = \kappa^l (G \setminus K) = G \setminus K = G_1'.$$

Now $\lambda_1 \in (\mathbb{Z}/p\mathbb{Z})^*$ exists with $1 \in \lambda_1 G_1' =: G_1$. Then $G = K \cup \lambda_1^{-1} G_1$ and we can repeat the construction on G_1 instead of G. This procedure finishes after $h := (p-1)/2k$ steps. Let $n_1 := 1$ and for $j = 2, \ldots, h$ let $n_j := n_{j-1}\lambda_{j-1}^{-1}$. Then $G = \bigcup_{j=1}^{h} n_j K$. □

Let $G = \{g_1, \ldots, g_{(p-1)/2}\} \subset (\mathbb{Z}/p\mathbb{Z})^*$ with $|G| = (p-1)/2$ and $\kappa G = G$ for some $\kappa \in (\mathbb{Z}/p\mathbb{Z})^*$ with $\kappa \neq 1$, $\kappa^k = 1$. The following lemma will give an answer to the question when G satisfies the conditions $g_l \neq g_j$, $g_l \neq -g_j$ for all $j \neq l$ with $j, l = 1, \ldots, \frac{p-1}{2}$.

LEMMA 2.5. *Let $G = \bigcup_{j=1}^{h} n_j K \subset (\mathbb{Z}/p\mathbb{Z})^*$ be defined like in Lemma 2.4. Then for all $g_j, g_l \in G$ holds $g_j \neq -g_l$ if and only if $-1 \notin K$ and for all $\kappa \in K$ and all $j, l = 1, \ldots, h$ holds $n_j \neq -n_l\kappa$.*

Proof. $\underline{\Rightarrow}$: Suppose $-1 \in K$. Then $-1 = \kappa^l$ for some l and $n_1 = -n_1\kappa^l$. But then we have found $g_1 := n_1 \in G$ and $g_2 := n_1\kappa^l \in G$ with $g_1 = -g_2$.
$\underline{\Leftarrow}$: Suppose $g_j, g_l \in G$ exist with $g_j = -g_l$. Let $g_j = n_j\kappa^j$, $g_l = n_l\kappa^l$. Then $n_j\kappa^j = -n_l\kappa^l$, and we have found $\kappa^{j-l} \in K$ with $n_l = -n_j\kappa^{j-l}$. □

Which subgroups $K \subseteq (\mathbb{Z}/p\mathbb{Z})^*$ satisfy the condition $-1 \notin K$?

LEMMA 2.6. *Let $K \subseteq (\mathbb{Z}/p\mathbb{Z})^*$ be a subgroup of order k. Then $-1 \notin K$ if and only if k is odd.*

Proof. The group $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p-1$ and K is a cyclic group. Let x be a generator of K, then $x^k = 1$. If k is even, $k/2 \in \mathbb{Z}$ and $x^{k/2} \in K$. But then $(x^{k/2})^2 = x^k = 1$ and therefore $x^{k/2} = -1 \in K$ since -1 is the element of order 2 in $(\mathbb{Z}/p\mathbb{Z})^*$. On the other hand if $-1 \in K$, then K contains an element of order 2. But then k is even, since the order of any element of K divides the order of K. □

Proof of Proposition 2.3. A subgroup K decomposes a set G as required in Lemma 2.5 if and only if the order of K is odd. Moreover, the order of K divides $p-1$. Now Proposition 2.3 follows from Lemma 2.4 and Lemma 2.5. □

We did not yet analyse the uniqueness of the decomposition of a set G. It is evident that the n_j can be permuted and multiplied with any $\kappa^l \in K$, but we will see that K and h are not uniquely determined. The next lemma states that if K decomposes G then so does any nontrivial subgroup of K.

Lemma 2.7. *Let $G = \bigcup_{j=1}^h n_j K \subset (\mathbb{Z}/p\mathbb{Z})^*$, $|G| = (p-1)/2$, be such that K decomposes G (Proposition 2.3). Let $|K| = k$ be not a prime and let $K' \neq K$ be a nontrivial subgroup of K. Then K' decomposes G.*

Proof. Since K' is a subgroup of K, K can be written as a union of cosets of K' in K. Moreover, G is a union of cosets of K in $(\mathbb{Z}/p\mathbb{Z})^*$. Therefore

$$G = \bigcup_{j=1}^{h} n_j K = \bigcup_{i=1}^{h'} n'_i K'.$$

Since K decomposes G, we have $n_l K \neq -n_j K$ for all $l, j = 1, \ldots, h$. This implies that $n'_l K' \neq -n'_i K'$ for all $i, l = 1, \ldots, h'$. So K' decomposes G. □

Our next aim is to determine the number of sets G. Therefore we consider for a given G the group K with $|K|$ maximal and K decomposes G.

Lemma 2.8. *Let $K \subset (\mathbb{Z}/p\mathbb{Z})^*$ be a nontrivial subgroup of odd order k. Then $2^{(p-1)/2k}$ different sets G exist such that K decomposes G and $|G| = (p-1)/2$.*

Proof. The order of $K \subset (\mathbb{Z}/p\mathbb{Z})^*$ is odd. Then it follows from Lemma 2.6 that $-1 \notin K$. Consider the cosets $n_j K$ of K in $(\mathbb{Z}/p\mathbb{Z})^*$. Since $-1 \notin K$, we have $n_j K \neq -n_j K$. So n_j, $j = 1, \ldots, (p-1)/2k$, exist such that

$$(\mathbb{Z}/p\mathbb{Z})^* = \bigcup_{j=1}^{(p-1)/2k} (n_j K \cup -n_j K).$$

The group K decomposes G if and only if G is a union of cosets of K and $m_j K \subseteq G$ implies that $-m_j K \nsubseteq G$ for $m_j = \pm n_j$, $j = 1, \ldots, (p-1)/2k$. Therefore $2^{(p-1)/2k}$ sets G exist such that K decomposes G. □

Definition. Let $K \subset (\mathbb{Z}/p\mathbb{Z})^*$ be a group of odd order k. We define $\mathcal{N}_k$ to be the number of $G \subset (\mathbb{Z}/p\mathbb{Z})^*$ such that K decomposes G but any K' with $K \subset K' \subset (\mathbb{Z}/p\mathbb{Z})^*$, $K \neq K'$, does not decompose G.

To determine $\mathcal{N}_k$ we have to subtract the number $\mathcal{N}_{k'}$ from $2^{(p-1)/2k}$ for each odd $k' \neq k$ with $k|k'$, $k'|p-1$. The integer k' is the order of the group K' with $K \subset K'$. Therefore we get a recursive formula

$$\mathcal{N}_k = 2^{(p-1)/2k} - \sum_{\substack{k' \text{ odd},\ k' > k \\ k|k',\ k'|p-1}} \mathcal{N}_{k'}.$$

Now it remains to determine $\mathcal{N}_y$. Let $y \in \mathbb{Z}$ be such that $p-1 = 2^r y$ and y is odd. Then

$$\mathcal{N}_y = 2^{(p-1)/2y} = 2^{2^{r-1}}.$$

Let $p-1 = 2^r p_1^{r_1} \dots p_l^{r_l}$ be a factorisation of $p-1$ into primes where $p_1, \dots, p_l$ are odd and $p_i \neq p_j$ for all $i \neq j$ with $i, j = 1, \dots, l$. Since $p-1$ is even, $r \geqslant 1$. Let K be of order $k = p_1^{s_1} \dots p_l^{s_l}$ where $0 \leqslant s_j \leqslant r_j$ for $j = 1, \dots, l$. Let x be a generator of $(\mathbb{Z}/p\mathbb{Z})^*$. Then K is generated by x^m, $m = 2^r p_1^{r_1 - s_1} \dots p_l^{r_l - s_l}$. If $k' = p_1^{t_1} \dots p_l^{t_l}$ where $s_j \leqslant t_j \leqslant r_j$ for $j = 1, \dots, l$, then K is a proper subgroup of K' of order k' if $s_j < t_j$ for some $1 \leqslant j \leqslant l$. Herewith $-1 + \prod_{j=1}^{l}(r_j - s_j + 1)$ groups K' exist such that K is a proper subgroup of K'. So the number of sets G that are decomposed by K and for which no $K' \supsetneq K$ exists such that K' decomposes G is

$$\mathcal{N}_k = 2^{(p-1)/2k} - \sum_{y \in T_k} \mathcal{N}_y$$

where

$$T_k := \left\{ y \in \mathbb{N} \;\middle|\; y \text{ odd},\ k|y,\ y \neq k \text{ and } y|p-1 \right\}.$$

Now we have to determine the number of sets G that satisfy the conditions of Proposition 2.3. Let this be the number $\mathcal{N}_G$. One easily sees that

$$\mathcal{N}_G = \sum_{\substack{K \subset (\mathbb{Z}/p\mathbb{Z})^* \\ |K| \neq 1 \\ |K| \text{ odd}}} \mathcal{N}_{|K|} = \sum_{\substack{k|p-1 \\ k \neq 1 \\ k \text{ odd}}} \mathcal{N}_k.$$

Now let $G \subset (\mathbb{Z}/p\mathbb{Z})^*$ with $|G| = (p-1)/2$, such that for all $g_i, g_j \in G$, $g_i \neq -g_j$. Let $\mathcal{N}_1$ be the number of sets G for which no $\kappa \in (\mathbb{Z}/p\mathbb{Z})^*$, $\kappa \neq 1$, exists such that $\kappa G = G$. Then

$$\mathcal{N}_1 = 2^{(p-1)/2} - \mathcal{N}_G = 2^{(p-1)/2} - \sum_{\substack{1 \neq k |\, p-1 \\ k \text{ odd}}} \mathcal{N}_k.$$

We have seen that each set G corresponds to the set of eigenvalues of a matrix in $\mathrm{U}((p-1)/2)$ that satisfies Theorem 2.2.

Definition. We define a matrix $X_G \in \mathrm{U}\left(\frac{p-1}{2}\right)$ with the eigenvalues

$$\left\{ e^{ig_1 2\pi/p}, \dots, e^{ig_{(p-1)/2} 2\pi/p} \right\}$$

where $G = \{g_1, \dots, g_{(p-1)/2}\} \subset (\mathbb{Z}/p\mathbb{Z})^*$. We used the same notation for the elements of $(\mathbb{Z}/p\mathbb{Z})^*$ and their representatives in $\mathbb{Z}$.

Let the maximal order of K that decomposes G be k. Then G yields k elements of the group generated by X_G. As a result we have:

Proposition 2.9. *The number of conjugacy classes of subgroups of order p in* $\mathrm{U}((p-1)/2)$ *whose group elements satisfy the necessary and sufficient condition is*

$$\mathcal{K}(p) = \frac{1}{p-1} \sum_{\substack{k \ odd \\ k|p-1}} k\mathcal{N}_k.$$

3 The Farrell cohomology

3.1 An introduction to Farrell cohomology

An introduction to the Farrell cohomology can be found in the book of Brown [3]. The Farrell cohomology is a complete cohomology for groups with finite virtual cohomological dimension (vcd). It is a generalisation of the Tate cohomology for finite groups. If G is finite, the Farrell cohomology and the Tate cohomology of G coincide. It is well-known that the groups $\mathrm{Sp}(2n, \mathbb{Z})$ have finite vcd.

Definition. An elementary abelian p-group of rank $r \geqslant 0$ is a group that is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$.

It is well-known that $\widehat{\mathrm{H}}^i(G, \mathbb{Z})$ is a torsion group for every $i \in \mathbb{Z}$. We write $\widehat{\mathrm{H}}^i(G, \mathbb{Z})_{(p)}$ for the p-primary part of this torsion group, i.e., the subgroup of elements of order some power of p. We will use the following theorem.

Theorem 3.1. *Let G be a group such that* $\operatorname{vcd} G < \infty$ *and let p be a prime. Suppose that every elementary abelian p-subgroup of G has rank* $\leqslant 1$. *Then*

$$\widehat{\mathrm{H}}^*(G, \mathbb{Z})_{(p)} \cong \prod_{P \in \mathfrak{P}} \widehat{\mathrm{H}}^*(N(P), \mathbb{Z})_{(p)}$$

where $\mathfrak{P}$ is a set of representatives for the conjugacy classes of subgroups of G of order p and $N(P)$ denotes the normalizer of P.

Proof. See Brown's book [3]. □

We also have

$$\widehat{\mathrm{H}}^*(G, \mathbb{Z}) \cong \prod_p \widehat{\mathrm{H}}^*(G, \mathbb{Z})_{(p)}$$

where p ranges over the primes such that G has p-torsion.
A group G of finite virtual cohomological dimension is said to have periodic cohomology if for some $d \neq 0$ there is an element $u \in \widehat{\mathrm{H}}^d(G, \mathbb{Z})$ that is invertible in the ring $\widehat{\mathrm{H}}^*(G, \mathbb{Z})$. Cup product with u then gives a periodicity isomorphism $\widehat{\mathrm{H}}^i(G, M) \cong \widehat{\mathrm{H}}^{i+d}(G, M)$ for any G-module M and any $i \in \mathbb{Z}$. Similarly we say that G has p-periodic cohomology if the p-primary component $\widehat{\mathrm{H}}^*(G, \mathbb{Z})_{(p)}$, which is itself a ring, contains an invertible element of non-zero degree d. Then we have

$$\widehat{\mathrm{H}}^i(G, M)_{(p)} \cong \widehat{\mathrm{H}}^{i+d}(G, M)_{(p)},$$

and the smallest positive d that satisfies this condition is called the p-period of G.

PROPOSITION 3.2. *The following are equivalent:*

i) G has p-periodic cohomology.

ii) Every elementary abelian p-subgroup of G has rank $\leqslant 1$.

Proof. See Brown's book [3]. □

3.2 NORMALIZERS OF SUBGROUPS OF ORDER p IN Sp($p-1,\mathbb{Z}$)

In order to use Theorem 3.1, we have to analyse the structure of the normalizers of subgroups of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$. We already analysed the conjugacy classes of subgroups of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$. Let N be the normalizer and let C be the centralizer of such a subgroup. Then we have a short exact sequence

$$1 \longrightarrow C \longrightarrow N \longrightarrow N/C \longrightarrow 1.$$

Moreover, it follows from the discussion in the paper of Brown [2] that for p an odd prime

$$C \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2p\mathbb{Z},$$

and therefore N is a finite group. We will use the following proposition.

PROPOSITION 3.3. *Let*

$$1 \longrightarrow U \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence with Q a finite group of order prime to p. Then

$$\widehat{\mathrm{H}}^*(G,\mathbb{Z})_{(p)} \cong \left(\widehat{\mathrm{H}}^*(U,\mathbb{Z})_{(p)}\right)^Q.$$

Proof. See Brown [3], the Hochschild-Serre spectral sequence. □

Applying this to our case, we get

$$\widehat{\mathrm{H}}^*(N,\mathbb{Z})_{(p)} \cong \left(\widehat{\mathrm{H}}^*(C,\mathbb{Z})_{(p)}\right)^{N/C}.$$

Therefore we have to determine N/C and its action on $C \cong \mathbb{Z}/2p\mathbb{Z}$. From now on, if we consider subgroups or elements of order p in $\mathrm{U}((p-1)/2)$, we mean those that satisfy the condition of Theorem 2.2. In what follows we assume that p is an odd prime for which $h^- = 1$, because in this case we have a bijection between the conjugacy classes of subgroups of order p in $\mathrm{U}((p-1)/2)$ and those in $\mathrm{Sp}(p-1,\mathbb{Z})$. Therefore, in order to determine the structure of the conjugacy classes of subgroups of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$, we can consider the corresponding conjugacy classes in $\mathrm{U}((p-1)/2)$. We have already seen that in a subgroup of $\mathrm{U}((p-1)/2)$ of order p different elements can be in the same conjugacy class. Let $\mathcal{N}_k$ be the number of conjugacy classes of elements of order p in $\mathrm{U}((p-1)/2)$ where k powers of one element are in the same conjugacy class. Let $\mathcal{K}_k$ be the number of conjugacy classes of subgroups of $\mathrm{U}((p-1)/2)$ with $|N/C| = k$, where N denotes the normalizer and C the centralizer of this subgroup. Then the number $\mathcal{K}(p)$ of conjugacy classes of subgroups of order p in $\mathrm{U}((p-1)/2)$ is

$$\mathcal{K}(p) = \sum_{\substack{k|p-1,\\ k \text{ odd}}} \mathcal{K}_k.$$

If $|N/C| = k$, then

$$N/C \cong \mathbb{Z}/k\mathbb{Z} \subseteq \mathbb{Z}/(p-1)\mathbb{Z} \cong \mathrm{Aut}(\mathbb{Z}/2p\mathbb{Z})$$

where $k|p-1$ and k is odd. This means that N/C is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$. So we get the short exact sequence

$$1 \longrightarrow \mathbb{Z}/2p\mathbb{Z} \longrightarrow N \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow 1.$$

Moreover, we have an injection $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/2p\mathbb{Z} \hookrightarrow N$. Applying the proposition to this case yields

$$\widehat{\mathrm{H}}^*(N, \mathbb{Z})_{(p)} \cong \left(\widehat{\mathrm{H}}^*(\mathbb{Z}/2p\mathbb{Z}, \mathbb{Z})_{(p)}\right)^{\mathbb{Z}/k\mathbb{Z}}.$$

The action of $\mathbb{Z}/k\mathbb{Z}$ on $\mathbb{Z}/2p\mathbb{Z}$ is given by the action of $\mathbb{Z}/k\mathbb{Z}$ as a subgroup of the group of automorphisms of $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/2p\mathbb{Z}$.

Lemma 3.4. *The Farrell cohomology of* $\mathbb{Z}/l\mathbb{Z}$ *is*

$$\widehat{\mathrm{H}}^*(\mathbb{Z}/l\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}\,[x, x^{-1}]$$

where $\deg x = 2$, $x \in \widehat{\mathrm{H}}^2(\mathbb{Z}/l\mathbb{Z}, \mathbb{Z})$, *and* $\langle x\rangle \cong \mathbb{Z}/l\mathbb{Z}$.

Proof. See Brown's book [3]. For finite groups the Farrell cohomology and the Tate cohomology coincide. □

Proposition 3.5. *Let* p *be an odd prime and let* $k \in \mathbb{Z}$ *divide* $p-1$. *Then*

$$\left(\widehat{\mathrm{H}}^*(\mathbb{Z}/2p\mathbb{Z}, \mathbb{Z})_{(p)}\right)^{\mathbb{Z}/k\mathbb{Z}} \cong \mathbb{Z}/p\mathbb{Z}[x^k, x^{-k}]$$

where $x \in \widehat{\mathrm{H}}^2(\mathbb{Z}/2p\mathbb{Z}, \mathbb{Z})$.

Proof. For an odd prime p

$$\widehat{\mathrm{H}}^*(\mathbb{Z}/2p\mathbb{Z}, \mathbb{Z})_{(p)} = \left(\mathbb{Z}/2p\mathbb{Z}[x, x^{-1}]\right)_{(p)} = \mathbb{Z}/p\mathbb{Z}[x, x^{-1}].$$

We have to consider the action of $\mathbb{Z}/k\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}[x, x^{-1}]$. We have $px = 0$ and $x \in \widehat{\mathrm{H}}^2(\mathbb{Z}/2p\mathbb{Z}, \mathbb{Z})$. The action is given by $x \mapsto qx$ with q such that $(q, p) = 1$,

$q^k \equiv 1 \pmod p$ and k is the smallest number such that this is fulfilled. The action of $\mathbb{Z}/k\mathbb{Z}$ on

$$\widehat{\mathrm{H}}^{2m}(\mathbb{Z}/2p\mathbb{Z},\mathbb{Z})_{(p)} \cong (\langle x^m\rangle) \cong \mathbb{Z}/p\mathbb{Z}$$

is given by

$$x^m \mapsto q^m x^m.$$

The $\mathbb{Z}/k\mathbb{Z}$-invariants of $\widehat{\mathrm{H}}^*(\mathbb{Z}/2p\mathbb{Z},\mathbb{Z})_{(p)}$ are the $x^m \in \widehat{\mathrm{H}}^{2m}(\mathbb{Z}/2p\mathbb{Z},\mathbb{Z})_{(p)}$ with $x^m \mapsto x^m$, or equivalently $q^m \equiv 1 \pmod p$. Herewith we get

$$\begin{aligned}\widehat{\mathrm{H}}^*(N,\mathbb{Z})_{(p)} &\cong \left(\widehat{\mathrm{H}}^*(\mathbb{Z}/2p\mathbb{Z},\mathbb{Z})_{(p)}\right)^{\mathbb{Z}/k\mathbb{Z}} \cong \left(\mathbb{Z}/p\mathbb{Z}[x,x^{-1}]\right)^{\mathbb{Z}/k\mathbb{Z}} \\ &\cong \mathbb{Z}/p\mathbb{Z}[x^k,x^{-k}].\end{aligned}$$

□

Proposition 3.6. *Let p be an odd prime for which $h^- = 1$. Then*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)} \cong \prod_{\substack{k\mid p-1\\ k\ odd}} \left(\prod_1^{\mathcal{K}_k} \mathbb{Z}/p\mathbb{Z}[x^k,x^{-k}]\right),$$

where $\mathcal{K}_k$ is the number of conjugacy classes of subgroups of $\mathrm{U}((p-1)/2)$ *with $|N/C| = k$. As usual N denotes the normalizer and C the centralizer of this subgroup.*

Proof. Let p be a prime with $h^- = 1$. Then a bijection exists between the conjugacy classes of matrices of order p in $\mathrm{U}((p-1)/2)$ that satisfy the conditions of Theorem 2.2 and the conjugacy classes of matrices of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$. Now this proposition follows from Theorem 3.1. □

Now it remains to determine $\mathcal{K}_k$, the number of conjugacy classes of subgroups of $\mathrm{U}((p-1)/2)$ of order p with $N/C \cong \mathbb{Z}/k\mathbb{Z}$. Therefore we need $\mathcal{N}_k$, the number of conjugacy classes of elements $X \in \mathrm{U}((p-1)/2)$ of order p for which $1 = j_1 < \cdots < j_k < p$ exist such that the X^{j_l}, $l = 1,\ldots,k$, are in the same conjugacy class than X and k is maximal. One such class yields k elements in a group for which $|N/C| = k$ and therefore

$$\mathcal{K}_k = k\mathcal{N}_k\,\frac{1}{p-1}.$$

We recall the formula for $\mathcal{N}_k$:

$$\mathcal{N}_k = 2^{\frac{p-1}{2k}} - \sum_{\substack{k' \text{ odd},\ k'>k\\ k\mid k',\ k'\mid p-1}} \mathcal{N}_{k'}.$$

Now we have everything we need to compute the p-primary part of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$ for some examples of primes with $h^- = 1$.

3.3 Examples with $3 \leqslant p \leqslant 19$

$p = 3$: It is $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$. One conjugacy class exists with $N = C$. Therefore

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(2,\mathbb{Z}),\mathbb{Z})_{(3)} \cong \mathbb{Z}/3\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(2,\mathbb{Z})$ has 3-period 2.

$p = 5$: One conjugacy class exists with $N = C$. Therefore

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})_{(5)} \cong \mathbb{Z}/5\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(4,\mathbb{Z})$ has 5-period 2.

$p = 7$: One conjugacy class exists with $N/C \cong \mathbb{Z}/3\mathbb{Z}$, and one class exists with $N = C$. Therefore

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(6,\mathbb{Z}),\mathbb{Z})_{(7)} \cong \mathbb{Z}/7\mathbb{Z}[x^3,x^{-3}] \times \mathbb{Z}/7\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(6,\mathbb{Z})$ has 7-period 6.

$p = 11$: One conjugacy class exists with $N/C \cong \mathbb{Z}/5\mathbb{Z}$ and 3 classes exist with $N = C$. Therefore

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(10,\mathbb{Z}),\mathbb{Z})_{(11)} \cong \mathbb{Z}/11\mathbb{Z}[x^5,x^{-5}] \times \prod_1^3 \mathbb{Z}/11\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(10,\mathbb{Z})$ has 11-period 10.

$p = 13$: One conjugacy class exists with $N/C \cong \mathbb{Z}/3\mathbb{Z}$ and 5 classes exist with $N = C$. Therefore

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(12,\mathbb{Z}),\mathbb{Z})_{(13)} \cong \mathbb{Z}/13\mathbb{Z}[x^3,x^{-3}] \times \prod_1^5 \mathbb{Z}/13\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(12,\mathbb{Z})$ has 13-period 6.

$p = 17$: 16 conjugacy classes exist with $N = C$. Therefore

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(16,\mathbb{Z}),\mathbb{Z})_{(17)} \cong \prod_1^{16} \mathbb{Z}/17\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(16,\mathbb{Z})$ has 17-period 2.

$p = 19$: One conjugacy class exists with $N/C \cong \mathbb{Z}/9\mathbb{Z}$, one class exists with $N/C \cong \mathbb{Z}/3\mathbb{Z}$, and 28 classes exist with $N = C$.

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(18,\mathbb{Z}),\mathbb{Z})_{(19)} \cong \mathbb{Z}/19\mathbb{Z}[x^9,x^{-9}] \times \mathbb{Z}/19\mathbb{Z}[x^3,x^{-3}] \times \prod_1^{28} \mathbb{Z}/19\mathbb{Z}[x,x^{-1}],$$

and $\mathrm{Sp}(18,\mathbb{Z})$ has 19-period 18.

3.4 The p-primary part of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$

Let p be an odd prime and let ξ be a primitive p-th root of unity. Let h^- be the relative class number of the cyclotomic field $\mathbb{Q}(\xi)$. In this section we compute $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)}$ and its period for any odd prime p for which h^- is odd.

Theorem 3.7. *Let p be an odd prime for which h^- is odd. Then*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)} \cong \prod_{\substack{k|p-1\\ k \text{ odd}}} \left(\prod_{1}^{\widetilde{\mathcal{K}}_k} \mathbb{Z}/p\mathbb{Z}[x^k,x^{-k}]\right),$$

where $\widetilde{\mathcal{K}}_k$ denotes the number of conjugacy classes of subgroups of order p of $\mathrm{Sp}(p-1,\mathbb{Z})$ *for which $|N/C|=k$. Moreover $\widetilde{\mathcal{K}}_k \geqslant \mathcal{K}_k$, where $\mathcal{K}_k$ is the number of conjugacy classes of subgroups of* $\mathrm{U}((p-1)/2)$ *with $|N/C|=k$. As usual N denotes the normalizer and C the centralizer of the corresponding subgroup.*

Proof. We have seen in Section 2.2 that if h^- is odd, a bijection exists between the conjugacy classes of matrices of order p in $\mathrm{U}((p-1)/2)$ that satisfy the conditions of Theorem 2.2 and the conjugacy classes of matrices of order p in $\mathrm{Sp}(p-1,\mathbb{Z})$ that correspond to the equivalence classes $[\mathbb{Z}[\xi],u] \in \mathcal{P}$. Each conjugacy class of subgroups of order p in $\mathrm{U}((p-1)/2)$ whose group elements satisfy the condition required in Theorem 2.2 yields at least one conjugacy class in $\mathrm{Sp}(p-1,\mathbb{Z})$. This implies that the p-primary part of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$ is a product

$$\prod_{\substack{k|p-1,\\ k \text{ odd}}} \left(\prod_{1}^{\widetilde{\mathcal{K}}_k} \mathbb{Z}/p\mathbb{Z}[x^k,x^{-k}]\right)$$

where $\widetilde{\mathcal{K}}_k$ denotes the number of conjugacy classes of subgroups of order p of $\mathrm{Sp}(p-1,\mathbb{Z})$ that satisfy $|N/C|=k$. Let $\mathcal{K}_k$ be the number of such subgroups in $\mathrm{U}((p-1)/2)$. Because h^- is odd, each such subgroup gives at least one such subgroup of $\mathrm{Sp}(p-1,\mathbb{Z})$. Therefore, if h^- is odd, $\widetilde{\mathcal{K}}_k \geqslant \mathcal{K}_k$. If h^- is even, it may be possible that no subgroup of $\mathrm{Sp}(p-1,\mathbb{Z})$ of order p exists for which $|N/C|=k$. □

Theorem 3.8. *Let p be an odd prime for which h^- is odd and let y be such that $p-1=2^r y$ and y is odd. Then the period of $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)}$ is $2y$.*

Proof. By Theorem 3.7 we know that the p-primary part of the Farrell cohomology of $\mathrm{Sp}(p-1,\mathbb{Z})$ is

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})_{(p)} \cong \prod_{\substack{k|p-1\\ k \text{ odd}}} \left(\prod_{1}^{\widetilde{\mathcal{K}}_k} \mathbb{Z}/p\mathbb{Z}[x^k,x^{-k}]\right).$$

Moreover, $\widetilde{\mathcal{K}}_k \geqslant 1$ and the period of $\mathbb{Z}/p\mathbb{Z}[x^k, x^{-k}]$ is $2k$. Herewith the period of the p-primary part of the Farrell cohomology is $2y$. □

If p is a prime for which h^- is even, the p-period of $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})$ is $2z$ where z is odd and divides $p-1$. The period is not necessarily $2y$ because there may be no subgroup of order p in which y elements are conjugate in $\mathrm{Sp}(p-1,\mathbb{Z})$ even if we know that they are conjugate in $\mathrm{Sp}(p-1,\mathbb{R})$.

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