Galois Representations and Lubin-Tate Groups

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Abstract. Using Lubin-Tate groups, we develop a variant of Fontaine's theory of (φ, Γ)-modules, and we use it to give a description of the Galois stable lattices inside certain crystalline representations.

Introduction

In his Grothendieck Festschrift paper [Fo 1], Fontaine introduced a new way to classify local Galois representation, using the theory of so called (φ, Γ)-modules. To recall this, let k be a perfect field of characteristic p, $K_0 = \mathrm{Fr}\, W(k)$ and K/K_0 a finite, totally ramified extension. Fix an algebraic closure $\bar{K}$ of K. Fontaine's theory starts with an infinite extension K_∞/K which is required to have certain ramification properties. Miraculously, these properties ensure that $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$ can be identified with the absolute Galois group of a local field of *equal* characteristic p, $X(K)$. It is well known that representations of such a Galois group on finite dimensional $\mathbb{F}_p$-vector spaces can be classified rather concretely in terms of finite dimensional vector spaces over $X(K)$ equipped with an étale Frobenius. If K_∞/K is Galois, then $\Gamma = \mathrm{Gal}(K_\infty/K)$ acts naturally on $X(K)$, and one obtains a classification of G_K-representations on finite dimensional $\mathbb{F}_p$-vector spaces by adding a semi-linear action of Γ to the étale φ-modules over $X(K)$.
To obtain a classification of G_K-representations on finite $\mathbb{Z}_p$-modules, one needs to lift the action of φ and Γ on $X(K)$ to commuting operators on a Cohen ring for $X(K)$. This is probably not always possible, but can be done when K_∞ is the p-cyclotomic extension of K. Much of the work on Fontaine's theory by Berger, Colmez, Wach and others has focused on this case. In this paper we focus on the case when K_∞ is generated by the p-power torsion points of a Lubin-Tate group for a finite extension $L/\mathbb{Q}_p$ contained in K. As an application we obtain a description of the G_K-stable lattices in a certain class of crystalline G_K-representations. This is possible using the p-cyclotomic theory only when K is an unramified extension of some $\mathbb{Q}_p(\mu_{p^n})$.

More precisely, let $\mathcal{G}$ be a Lubin-Tate group over $\mathcal{O}_L$, write k_L for the residue field of L, fix a uniformizer π_L of L, and write $R = \varprojlim \mathcal{O}_{\bar{K}}/p$, where the transition maps in the inverse limit are given by Frobenius. The action of G_K on the Tate module $T\mathcal{G}$ of $\mathcal{G}$ gives rise to a character $\chi : \Gamma \to \mathcal{O}_L^\times$. It turns out that, using the periods of $T\mathcal{G}$ one can construct a subring $\mathcal{O}_\mathcal{E} \subset W(\operatorname{Fr} R) \otimes_{W(k_L)} L$ which is naturally a Cohen ring for $X(K)$. The action of $\mathcal{O}_L^\times \subset \mathcal{O}_L$ on $\mathcal{G}$ gives rise to a natural lifting of the action of Γ to $\mathcal{O}_\mathcal{E}$ (via χ), while the action of π_L on $\mathcal{G}$ allows one to lift the q-Frobenius $\varphi_q = \varphi^r$ to $\mathcal{O}_\mathcal{E}$, where $q = |k_L|$.
This allows one to classify G_K-representations on finite $\mathcal{O}_L$-modules in terms of étale (φ_q, Γ)-modules (see Theorem 1.6 below), and is explained in §1 of the paper. At least some part of this construction was certainly known to experts. The construction of the periods involved is in Colmez's paper [Col 1], and some of the ideas go back to Coleman [Co]. This material is also closely related to the subject of Fourquaux's thesis [Fou, §1.4].
In §2,3 we use this classification to give a classification of Galois stable lattices in certain crystalline G_K-representations, assuming that $K \subset K_0 \cdot L_\infty$ where L_∞/L is the field generated by the torsion points of $\mathcal{G}$. To explain the classification, assume for simplicity that $K = K_0 \cdot L$, and let $\mathfrak{S}_L = \mathcal{O}_K[\![u]\!]$. Fix a co-ordinate X on $\mathcal{G}$, and for $a \in \mathcal{O}_L$ denote by $[a] \in \mathcal{O}_L[\![X]\!]$ the power series giving the action of a on $\mathcal{G}$. Then $\gamma \in \Gamma$ acts on $\mathfrak{S}_L$ by $u \mapsto [\chi(\gamma)](u)$, while φ_q acts on $\mathfrak{S}_L$ by $u \mapsto [\pi_L](u)$. Let $Q = [\pi_L](u)/u$. We denote by $\operatorname{Mod}^{\varphi_q,\Gamma}_{/\mathfrak{S}_L}$ the category of finite free $\mathfrak{S}_L$-modules equipped with a continuous semi-linear action of Γ which induces the trivial action of $\mathfrak{M}/u\mathfrak{M}$, and an isomorphism $\varphi_q^*\mathfrak{M}[1/Q] \xrightarrow{\sim} \mathfrak{M}[1/Q]$ such that the map $1 \otimes \varphi_q : \mathfrak{M} \to \mathfrak{M}[1/Q]$ commutes with the action of Γ. Inside this category is a subcategory $\operatorname{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathfrak{S}_L}$ consisting of objects $\mathfrak{M}$ on which the Γ-action is $\mathcal{O}_L$-*analytic*. this means that there is $\mathcal{O}_L$-linear map of Lie algebras $d\Gamma : \operatorname{Lie}\Gamma \to \operatorname{End}_K(\mathfrak{M} \otimes_{\mathfrak{S}_L} K[\![u]\!])$, such that the action of an open subgroup of Γ is obtained by exponentiating $d\Gamma$.
To describe the crystalline representations we allow, consider any crystalline G_K-representation on an L-vector space V. Then

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \oplus_\mathfrak{m} D_{\mathrm{dR}}(V)_\mathfrak{m}$$

where $\mathfrak{m}$ runs over the maximal ideals of $K \otimes_{\mathbb{Q}_p} L$. We say that V is L-crystalline if the filtration on $D_{\mathrm{dR}}(V)_\mathfrak{m}$ is trivial, unless $\mathfrak{m}$ is the kernel of the natural map $K \otimes_{\mathbb{Q}_p} L \hookrightarrow K$ corresponding to the inclusion $L \to K$. One of our main results is then the following

Theorem (0.1). *There is an exact equivalence of $\otimes$-categories between $\operatorname{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathfrak{S}_L}$ and the category of G_K-stable $\mathcal{O}_L$-lattices in L-crystalline G_K-representations.*

The theorem is a generalization of the classification of G_K-stable lattices in crystalline representations in terms of Wach lattices due to Wach [Wa], Colmez [Col 2] and Berger [Be 3], when K_∞ is the p-cyclotomic extension and K is unramified.

It is also analogous to the classification G_{K_∞}-stable lattices, obtained in [Ki] in the case when K_∞ is obtained from K by adjoining the p-power roots of a uniformizer. The advantage of Theorem (0.1) is that it applies without restriction on the ramification of K, and gives a precise description of G_K-stable lattices. Unfortunately, it applies only to a rather special kind of crystalline G_K-representation. It seems likely that in order to obtain a classification valid for any crystalline G_K-representation one needs to consider higher dimensional subrings of $W(\mathrm{Fr}\, R)$, constructed using the periods of all the conjugates of $\mathcal{G}$.

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§1 Étale (φ_q, Γ)-modules

(1.1) Throughout the paper we fix a perfect field k, of characteristic $p > 0$. Let $W = W(k)$, $K_0 = W[1/p]$ and K/K_0 a finite totally ramified extension with ring of integers $\mathcal{O}_K$, and uniformizer π. We also fix an algebraic closure $\bar{K}$ of K with ring of integers $\mathcal{O}_{\bar{K}}$, and set $G_K = \mathrm{Gal}(\bar{K}/K)$.
Let $L/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$ contained in K. Let $\mathcal{O}_L$ denote the ring of integers of L, and $k_L \subset k$ its residue field. Write $\mathcal{O}_{L_0} = W(k_L)$, $L_0 = \mathcal{O}_{L_0}[1/p]$, and $q = p^r = |k_L|$. For an $\mathcal{O}_{L_0}$-algebra A, it will be convenient to write $A_L = A \otimes_{\mathcal{O}_{L_0}} \mathcal{O}_L$.
Let $\mathcal{G}$ be a Lubin-Tate group over L corresponding to a uniformizer $\pi_L \in L$. Fix a local co-ordinate X on $\mathcal{G}$ so that the formal Hopf algebra $\mathcal{O}_{\mathcal{G}}$ may be identified with $\mathcal{O}_L[\![X]\!]$. For $a \in \mathcal{O}_L$ we denote by $[a] \in \mathcal{O}_L[\![X]\!] = \mathcal{O}_{\mathcal{G}}$ the power series giving the endomorphism a of $\mathcal{G}$.
For $n \geq 1$, let $K_n \subset \bar{K}$ denote the subfield generated by the π_L^n-torsion points of $\mathcal{G}$. We set $K_\infty = \cup_n K_n$ and we write $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$. Let $T\mathcal{G}$ denote the p-adic Tate module of $\mathcal{G}$. Then $T\mathcal{G}$ is a free $\mathcal{O}_L$-module of rank 1, and the action of Γ induces a faithful character $\chi : \Gamma \to \mathcal{O}_L^\times$.
We let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ with the transition maps being given by Frobenius. We may also identify R with $\varprojlim \mathcal{O}_{\bar{K}}/\pi_L$ with the transition map being given by the q-Frobenius φ^r. Evaluation of X at π_L-torsion points then induces a map $\iota : T\mathcal{G} \to R$. Namely if $v = (v_n)_{n\geq 0} \in T\mathcal{G}$ with $v_n \in \mathcal{G}[\pi_L^n](\mathcal{O}_{\bar{K}})$ and $\pi_L \cdot v_{n+1} = v_n$, then $\iota(v) = (v_n^*(X))_{n\geq 0}$.

Lemma (1.2). *There is a unique map* $\{\,\} : R \to W(R)_L$ *such that* $\{x\}$ *is a lifting of* x, *and* $\varphi^r(\{x\}) = [\pi_L](x)$. *Moreover* $\{\,\}$ *respects the action of* G_K, *and for* $v \in T\mathcal{G}$ *we have*

(1) *If* $a \in \mathcal{O}_L$ *then* $\{\iota(av)\} = [a](\{\iota(v)\})$.

(2) *The action of G_K on $\{\iota(T\mathcal{G})\}$ factors through Γ and for $\gamma \in \Gamma$*

$$[\chi(\gamma)](\{\iota(v)\}) = \{\iota(\gamma v)\} = \{\gamma \cdot \iota(v)\} = \gamma \cdot \{\iota(v)\}$$

In particular, if $v \in T\mathcal{G}$ is an $\mathcal{O}_L$-generator, there is an embedding $W_L[\![u]\!] \hookrightarrow W(R)_L$ sending u to $\{\iota(v)\}$ which identifies $W_L[\![u]\!]$ with a G_K-stable, φ^r-stable subring of $W(R)_L$ such that $\{\iota(T\mathcal{G})\}$ lies in the image of $W_L[\![u]\!]$.

Proof. The existence and uniqueness of $\{\,\}$ is [Col 1, Lem. 9.3]. The map $\{x\}$ is given by

$$\{x\} = \lim_n [\pi_L^n](\varphi^{-rn}(\tilde{x}))$$

where $\tilde{x} \in W(R)_L$ is any lifting of x. That $\{\,\}$ respects the action of G_K follows by functoriality. In particular, the action of G_K on $\{\iota(T\mathcal{G})\}$ factors through Γ. For (1) note that

$$[\pi_L][a]\{\iota(v)\} = [a][\pi_L]\{\iota(v)\} = [a]\varphi^r\{\iota(v)\} = \varphi^r([a]\{\iota(v)\}).$$

Since $[a]\{\iota(v)\}$ and $\{\iota(av)\}$ both have image $[a](\iota(v))$ in R, this proves (1). (Here R is viewed as a $\mathcal{O}_K$ algebra via $\mathcal{O}_K \to k$.)
Now the first equality in (2) follows from (1), while the other two equalities follows from the compatibility of ι and $\{\,\}$ with the action of G_K.
Finally, since $\iota(v)$ has positive valuation with respect to the canonical valuation on R, $u \mapsto \{\iota(v)\}$ induces a well defined map $W_L[\![u]\!] \to W(R)_L$. Its image is φ-stable by definition of $\{\,\}$ and Γ-stable by (2). If this map had a non-trivial kernel, then so would its reduction modulo π_L. The latter map $k[\![u]\!] \to R$, sending u to $\iota(v)$ is easily seen to be injective, as $\iota(v)$ has positive valuation. □

(1.3) Write $\mathfrak{S}_L = W_L[\![u]\!]$. We fix an $\mathcal{O}_L$-generator $v \in T_p\mathcal{G}$, and we identify $\mathfrak{S}_L$ with a subring of $W(R)_L$ by sending u to $\{\iota(v)\}$.
Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}_L[1/u]$. Then $\mathcal{O}_{\mathcal{E}}$ is a complete discrete valuation ring with uniformizer π_L and residue field $k((u))$. We may view $\mathcal{O}_{\mathcal{E}}$ as a subring of $W(\mathrm{Fr}\,R)_L$. Let $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \subset W(\mathrm{Fr}\,R)_L$ denote the maximal integral, unramified extension of $\mathcal{O}_{\mathcal{E}}$. We denote by $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$, which is again naturally a subring of $W(\mathrm{Fr}\,R)_L$. We write $\mathcal{E}$, $\mathcal{E}^{\mathrm{ur}}$ and $\widehat{\mathcal{E}}^{\mathrm{ur}}$ for the fields of fractions of $\mathcal{O}_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ and $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ respectively. These rings are all stable by φ^r, and by the action of G_K. Moreover the G_K-action on $\mathcal{O}_{\mathcal{E}}$ factors through Γ.

Lemma (1.4). *The residue field of $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ is a separable closure of $k((u))$. There is a natural isomorphism*

$$\mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E}) \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K_\infty).$$

Proof. This is a consequence of the theory of norm fields [Wi]. Since Γ is a p-adic Lie group the theory of *loc. cit* applies [Wi, 1.2.2]. For any finite extension F/K write $X_K(F) = \varprojlim(F \cdot K_n)$ where the maps in the inverse limit are given by the norm. We set $X_K(\bar{K}) = \cup_F X_K(F)$ where the limit runs over finite extensions F/K in $\bar{K}$. Then $X_K(F)$ has the structure of a local field of characteristic p, which is a finite separable extension of $X_K(K)$, $X_K(\bar{K})$ is a separable closure of $X_K(K)$, and the functor X_K induces an isomorphism [Wi, 3.2.2]

$$\mathrm{Gal}(X_K(\bar{K})/X_K(K)) \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K_\infty).$$

On the other hand, there is a natural embedding $X_K(K) \hookrightarrow \mathrm{Fr}\, R$ [Wi ,§4]. To see this explicitly note that one has well defined maps of rings

$$\varprojlim \mathcal{O}_{K_n} \to \varprojlim \mathcal{O}_{K_n}/(v_1) \hookrightarrow \varprojlim \mathcal{O}_{\bar{K}}/\pi_L = R, \tag{1.4.1}$$

where the transition maps in the first two inverse limits are given by the norm, and the final inverse limit by $x \mapsto x^q$.

The image of (1.4.1) is easily seen to be $k[\![u]\!] \subset R$. Hence we may identify $\mathcal{O}_\mathcal{E}/\pi_L\mathcal{O}_\mathcal{E}$ with $X_K(K)$. It follows that $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/\pi_L\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \subset \mathrm{Fr}\, R$ may be identified with $X_K(\bar{K})$. The lemma follows. □

(1.5) Note that the above proof shows that the map ι induces a map

$$T\mathcal{G} \to \varprojlim K_\infty,$$

where the transition maps are given by the norm. This is Coleman's map [Co, Thm. A].

We will write φ_q for the q-Frobenius φ^r (for example on the ring $W(\mathrm{Fr}\, R)$). Now denote by $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}_\mathcal{E}}$ (resp. $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$) the category of finite free (resp. finite torsion) $\mathcal{O}_\mathcal{E}$-modules M, equipped with an isomorphism $(\varphi_q)^*M \xrightarrow{\sim} M$. We denote by $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}_\mathcal{E}}$ (resp. $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$) the category of consisting of a module M in $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}_\mathcal{E}}$ (resp. $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$) equipped with a continuous semi-linear action of Γ which commutes with the action of φ_q.

We denote by $\mathrm{Rep}_{G_{K_\infty}}$ (resp. $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$) the category of finite free (resp. finite torsion) $\mathcal{O}_L$-modules V, equipped with a linear action of G_{K_∞} Similarly, we denote by Rep_{G_K} (resp. $\mathrm{Rep}^{\mathrm{tor}}_{G_K}$) the category of finite free (resp. finite torsion) $\mathcal{O}_K$-modules V, equipped with a linear action of G_K.

For M in $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}_\mathcal{E}}$ (resp. $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$, resp. $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}_\mathcal{E}}$, resp. $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$) we set

$$V(M) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} M)^{\varphi_q=1}.$$

For V in $\mathrm{Rep}_{G_{K_\infty}}$ (resp. $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$ resp. Rep_{G_K} resp. $\mathrm{Rep}^{\mathrm{tor}}_{G_K}$)) we set

$$M_{\mathcal{O}_\mathcal{E}}(V) = (V \otimes_{\mathcal{O}_K} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_{K_\infty}}.$$

Theorem (1.6). *V and $M_{\mathcal{O}_\mathcal{E}}$ are quasi-inverse equivalences between the exact tensor categories* $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}_\mathcal{E}}$ *(resp.* $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$, *resp.* $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}_\mathcal{E}}$, *resp.* $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$*) and* $\mathrm{Rep}_{G_{K_\infty}}$ *(resp.* $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$, *resp.* Rep_{G_K}, *resp.* $\mathrm{Rep}^{\mathrm{tor}}_{G_K}$*)*

Proof. The argument for this is identical to that in [Fo 1, 1.2.6, 3.4.3]. For the convenience of the reader we sketch it: It suffices to prove that V and M induce quasi-inverse, exact tensor equivalences between $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$ and $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$.
We first remark that both functors are exact. It suffices to prove this for objects killed by p. For $M_{\mathcal{O}_\mathcal{E}}$ this follows from the fact that for M in $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$, $1-\varphi_q$ is étale locally (on $\mathrm{Spec}\, k((u))$) surjective. For V this is a consequence of Hilbert's theorem 90, and (1.4).
For M in $\mathrm{Mod}^{\varphi_q,\mathrm{tor}}_{/\mathcal{O}_\mathcal{E}}$, we have a natural map

$$(1.6.1) \qquad (M \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi_q=1} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \to M \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$$

and taking G_{K_∞} invariants of both sides induces a map $M_{\mathcal{O}_\mathcal{E}}(V(M)) \to M$. To show that this map is an isomorphism one reduces to the case of objects killed by p, using the exactness proved above. In this case, (1.6.1) is an isomorphism, because étale locally M is spanned by its φ_q-invariants. Similarly, one obtains an isomorphism $V(M_{\mathcal{O}_\mathcal{E}}(V)) \to M$ for V in $\mathrm{Rep}^{\mathrm{tor}}_{G_{K_\infty}}$, using dévissage, Hilbert theorem 90, and (1.4). □

§2 (φ_q, Γ)-modules and weakly admissible modules

(2.1) We keep the notation of the previous section, so in particular we write $K_{0,L}$ for the field $K_0 \otimes_{L_0} L = \mathrm{Fr}\, W_L \subset K$. In order not to overload notation we will write v_n for $(v_n)^*(X) \in \bar{K}$. We now also assume that $K \subset K_{0,L}(v_n)_{n\geq 0}$. Fix an integer $m \geq 1$ such that $K \subset K_{0,L}(v_m)$.
As in [Ki, 1.1.1], denote by $D[0,1)$ the rigid analytic disk of radius 1, over $K_{0,L}$, and denote by u the co-ordinate on $D[0,1)$. For $I \subset [0,1)$ an interval, denote by $D(I) \subset D[0,1)$ the open subspace whose $\bar{K}$ points consist of $x \in \bar{K}$ with $|x| \in I$. We denote by $\mathcal{O}_I$ the ring of rigid analytic functions on $D(I)$, and we write $\mathcal{O} = \mathcal{O}_{[0,1)}$. We will often use the fact that $D[0,1)$ is a p-adic Stein space, so that a coherent sheaf on $D[0,1)$ can be recovered from its global sections. In particular, we may regard a finite free $\mathcal{O}$-module as a coherent sheaf on $D[0,1)$.
We regard $\mathfrak{S}_L \subset \mathcal{O}$ by $u \mapsto u$. The action of φ_q and Γ on $\mathfrak{S}_L$ have a unique continuous extension to $\mathcal{O}$, regarded with its canonical Frechet topology.[1]
Let $Q = [\pi_L^m](u)/[\pi_L^{m-1}](u)$. Denote by $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}}$ the category of finite free $\mathcal{O}$-modules $\mathcal{M}$ equipped with an isomorphism $\varphi_q^*(\mathcal{M})[1/Q] \xrightarrow{\sim} \mathcal{M}[1/Q]$. We denote by $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$ the category whose objects consist of an object of $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}}$ equipped with a continuous semi-linear action of Γ such that Γ acts trivially

[1] Contrary to our usual conventions, the symbol $\mathcal{O}_L$ will continue to denote the ring of integers of L, rather than $\mathcal{O} \otimes_{L_0} L$.

on $\mathcal{M}/u\mathcal{M}$ and the φ_q-semi-linear map $1\otimes\varphi_q : \mathcal{M} \to \mathcal{M}[1/Q]$ commutes with Γ.
We now explain how to differentiate the action of Γ on an object in $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$ following [Be 1, §IV,V].

Lemma (2.1.1). *The action of* Γ *on* $\mathcal{O}$, *defined above, is continuous. In particular,* $\mathcal{O}$ *with its action of* Γ *and* φ_q *is an object of* $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$.

Proof. For $r \in (0,1)$, denote by $|\cdot|_r$ the sup norm on $\mathcal{O}_{[0,r]}$. If $f(u) = \sum_{i\geq 0} a_i u^i \in \mathcal{O}_{[0,r]}$ then $|f|_r = \sup_i |a_i|r^i$. We have to show that, for any r, as $\gamma \to 1$, $|\gamma(f) - f|_r \to 0$, uniformly in f with $|f|_r \leqslant 1$.
Any $\gamma \in \Gamma$ acts on $\mathcal{O}$ by composition with $[\chi(\gamma)]$. Write

$$[\chi(\gamma)] = \sum_{i=1}^{\infty} b_i X^i = \exp_{\mathcal{G}}(\chi(\gamma)\log_{\mathcal{G}} X)$$

where $b_i \in \mathcal{O}_L$, $\log_{\mathcal{G}}$ denotes the logarithm of $\mathcal{G}$ and $\exp_{\mathcal{G}}$ denotes its inverse.[2]
Then $b_1 = \chi(\gamma)$ and for $i > 1$, b_i is a polynomial in $\chi(\gamma)$, which vanishes at $\chi(\gamma) = 1$. Given $\epsilon > 0$, choose i_0 so that $r^{i_0} < \epsilon$. Then for γ sufficiently close to 1, $|b_i| < \epsilon$ for $1 < i < i_0$, and $|b_1 - 1| < \epsilon$ so $|[\chi(\gamma)](u) - u|_r < \epsilon$. Hence

$$|\gamma(f) - f|_r = |f([\chi(\gamma)](u)) - f(u)|_r \leqslant \sum_{i=1}^{\infty} |a_i([\chi(\gamma)](u)^i - u^i)|_r \leqslant \epsilon |f(u)|_r.$$

The lemma follows. □

Lemma (2.1.2). *Let* $\mathcal{M}$ *be in* $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$. *For each* $r \in (0,1)$ *and* $\gamma \in \Gamma$ *sufficiently close to* 1 *(depending on* r*) the series*

$$\log\gamma = \sum_{i=1}^{\infty} (\gamma - 1)^i (-1)^{i-1}/i$$

induces a well defined operator on $\mathcal{M}|_{D[0,r)}$. *This induces a well defined* $\mathbb{Z}_p$*-linear map of Lie algebras*

$$d\Gamma_{\mathcal{M}} : \mathrm{Lie}\,\Gamma \to \mathrm{End}_{K_0}\mathcal{M}; \quad \beta \mapsto \log(\exp\beta).$$

such that for $\beta \in \mathrm{Lie}\,\Gamma$, $d\Gamma_{\mathcal{O}}(\beta)$ *is a derivation and* $d\Gamma_{\mathcal{M}}(\beta)$ *is a differential operator over* $d\Gamma_{\mathcal{O}}(\beta)$. *That is, for* $m \in \mathcal{M}$, $f \in \mathcal{O}$ *and* $\beta \in \mathrm{Lie}\,\Gamma$,

$$d\Gamma_{\mathcal{M}}(\beta)(fm) = d\Gamma_{\mathcal{O}}(\beta)(f)m + f d\Gamma_{\mathcal{M}}(\beta)(m).$$

Proof. Let $M_0 \subset \mathcal{M}$ be finite free W_L-submodule of rank equal to $d = \mathrm{rk}_{\mathcal{O}}\mathcal{M}$, which spans $\mathcal{M}$. Choosing a basis for M_0, we may identify $\mathcal{M}$ with $\mathcal{O}^d$. As

[2] So if $\mathcal{G} = \mathbb{G}_m$, then $\log_{\mathcal{G}}(X) = \log(1+X)$.

in (2.1.1), choose $r \in (0,1)$ and denote by $|\,|_r$ the norm on $\mathcal{M}_{D[0,r]} = \mathcal{O}^d_{D[0,r]}$ induced by the sup norm on $\mathcal{O}_{D[0,r]}$. For any $\epsilon > 0$, and γ sufficiently small we have $|\gamma(m) - m|_r \leqslant \epsilon|m|_r$ for $m \in M_0$ and $|\gamma(f) - f|_r \leqslant \epsilon|f|_r$ for $f \in \mathcal{O}_{D[0,r]}$ by (2.1.1). Hence

$$|\gamma(mf) - mf|_r \leqslant |\gamma(m) - m|_r|\gamma(f)|_r + |\gamma(f) - f|_r|m|_r \leqslant 2\epsilon|m|_r|f|_r = 2\epsilon|fm|_r.$$

This shows that $\log\gamma$ is well defined.
It follows that the map

$$d\Gamma_{\mathcal{M}} =: \mathrm{Lie}\,\Gamma \to \mathrm{End}_{K_0}\mathcal{M}; \quad \beta \mapsto \log(\exp\beta)$$

is well defined for β sufficiently small, and we extend it to all of $\mathrm{Lie}\,\Gamma$ by $\mathbb{Z}_p$-linearity. That $d\Gamma_{\mathcal{O}}(\beta)$ is a derivation and $d\Gamma_{\mathcal{M}}(\beta)$ is a differential operator over $d\Gamma_{\mathcal{O}}(\beta)$ follows from a simple computation, as does the fact that $d\Gamma_{\mathcal{M}}$ is a map of Lie algebras. Note that the latter statement just means that the differential operators $d\Gamma_{\mathcal{M}}(\beta)$ for $\beta \in \mathrm{Lie}\,\Gamma$ commute. □

(2.1.3) We say that $\mathcal{M}$ in $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$ is *$\mathcal{O}_L$-analytic* if the map $d\Gamma_{\mathcal{M}}$ is $\mathcal{O}_L$-linear, not just $\mathbb{Z}_p$-linear. We denote by $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathcal{O}}$ the full subcategory of $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$ consisting of $\mathcal{O}_L$-analytic objects. One checks easily that this is a $\otimes$-subcategory, which is stable under taking subobjects and quotients.

Lemma (2.1.4). *Let $\mathcal{M}$ be in $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathcal{O}}$.*

(1) *For each $r \in (0,1)$ the operator $N_\nabla =: \log\gamma/\log\chi(\gamma)$ is well defined for $\gamma \neq 1$ sufficiently close to 1, and is independent of γ.*
(2) *The operators in (1) induce a $K_{0,L}$-linear map $N_\nabla : \mathcal{M} \to \mathcal{M}$, which is a differential operator over the derivation $N_\nabla : \mathcal{O} \to \mathcal{O}$, and which commutes with φ_q on $\mathcal{M}$.*
(3) *There is a singular connection ∇ on $\mathcal{M}$ with simple poles at the zeroes of $[\pi_L^n]/u$ for $n \geq 1$ (that is at the non-trivial π_L-power torsion points of $\mathcal{G}$) such that $N_\nabla = \langle\nabla, \frac{\partial F}{\partial Y}(u,0)\log_{\mathcal{G}}u \cdot d/du\rangle$, where $F(X,Y)$ denotes the formal group law of $\mathcal{G}$ with respect to X, and $\frac{\partial F}{\partial Y}(u,0) \in \mathcal{O}_L[\![u]\!]^\times$.*

Proof. For γ sufficiently close to 1, we may write $\gamma = \exp\beta$ with $\beta \in \mathrm{Lie}\,\Gamma$. Since $\beta \mapsto \log(\exp\beta)$ is $\mathcal{O}_L$-linear by assumption. and $\beta \mapsto \log(\chi(\exp(\beta)))$ is obviously $\mathcal{O}_L$-linear, $\log\gamma/\log\chi(\gamma)$ is independent of γ. This proves (1) and (2) follows by viewing $\mathcal{M}$ as a coherent module on $D[0,1)$. The fact that N_∇ commutes with φ_q follows from the fact that φ_q commutes with the action of Γ.
To see (3), we first compute the derivation N_∇ on $\mathcal{O}$. For $\gamma \in \Gamma$ write $a_\gamma = \chi(\gamma) - 1$. Then

$$N_\nabla(u) = \lim_{\gamma \mapsto 1} \frac{[\chi(\gamma)](u) - u}{\log\chi(\gamma)} = \lim_{\mathfrak{a}_\gamma \to 0} \frac{\exp_{\mathcal{G}}((1 + a_\gamma)\log_{\mathcal{G}}u) - u}{\log\chi(\gamma)}$$
$$= \lim_{\mathfrak{a}_\gamma \to 0} \frac{F(u, \exp_{\mathcal{G}}(a_\gamma\log_{\mathcal{G}}u)) - u}{\log\chi(\gamma)} = \frac{\partial F}{\partial Y}(u,0)\log_{\mathcal{G}}(u).$$

Hence N_∇ is given on $\mathcal{O}$ by $N_\nabla(f) = \frac{\partial F}{\partial Y}(u,0)(\log_{\mathcal{G}} u)\frac{df}{du}$. As $\frac{\partial F}{\partial Y}(u,0)$ has constant term 1, and coefficients in $\mathcal{O}_L[\![u]\!]$, it is a unit $\mathcal{O}$.
Now for any $\mathcal{M}$, define $\nabla(m)$ for $m \in \mathcal{M}$ by $\nabla(m) = (\frac{\partial F}{\partial Y}(u,0)\log_{\mathcal{G}} u)^{-1}N_\nabla(m)$. Since N_∇ on $\mathcal{M}$ is a differential operator over the derivation $\frac{\partial F}{\partial Y}(u,0)(\log_{\mathcal{G}} u)\frac{df}{du}$ on $\mathcal{O}$, ∇ is a (singular) connection. A priori $\nabla(m)$ has a simple pole at each $[\pi_L]$-torsion point of $\mathcal{G}$, however since the action of Γ on $\mathcal{M}$ is trivial mod u, the operator N_∇ vanishes mod u, and $\nabla(m)$ has no pole at $u = 0$. This proves (3). $\square$

(2.2) Denote by $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ the category of finite dimensional $K_{0,L}$-vector spaces D equipped with an isomorphism $\varphi_q^* D \xrightarrow{\sim} D$ and a decreasing, separated filtration on $D_K = D \otimes_{K_{0,L}} K$, indexed by $\mathbb{Z}$, by K-subspaces.
Our next task to to show that there is an exact $\otimes$-equivalence between $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ and $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathcal{O}}$. The construction is analogous to that in [Be 2] and [Ki, 1.2]. Since many of the proofs from [Ki] go over verbatim, we often only sketch the argument.[3]
For $n \geq 0$, denote by $\widehat{\mathfrak{S}}_n$ the complete local ring at the point x_n of $D[0,1)$, corresponding to $u = v_{m+n}$. That is, $\widehat{\mathfrak{S}}_n$ is the completion of the localization of $\mathcal{O}$ at the maximal ideal generated by $[\pi_L^{m+n}](u)/[\pi_L^{m+n-1}](u)$. Then $\widehat{\mathfrak{S}}_n$ is a discrete valuation ring with residue field $K_{m+n} = K(v_{m+n}) \supset K$, which is canonically a subfield of $\widehat{\mathfrak{S}}_n$. In particular, $u - v_{m+n}$ is a uniformizer for $\widehat{\mathfrak{S}}_n$.
Let
$$\lambda = \prod_{n\geq 0} \varphi_q^n(Q(u)/Q(0))) = \prod_{n\geq 0}[\pi_L^{m+n}](u)/[\pi_L^{m+n-1}](u)\pi_L,$$
and write $\varphi_{q,W_L} : \mathcal{O} \to \mathcal{O}$ for the $\mathcal{O}_L$-linear automorphism given by applying φ^r to the coefficients of a series in $\mathcal{O}$.
Given D in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ and $n \geq 0$, we denote by ι_n the composite

$$\iota_n : D \otimes_{K_{0,L}} \mathcal{O}[1/\lambda] \overset{\varphi_q^{-n}\otimes\varphi_{q,W_L}^{-n}}{\hookrightarrow} D \otimes_{K_{0,L}} \widehat{\mathfrak{S}}_n[1/\lambda] \xrightarrow{\sim} D_K \otimes_K \widehat{\mathfrak{S}}_n[1/u - v_{m+n}].$$

We set

$$\mathcal{M}(D) = \{d \in D \otimes_{K_{0,L}} \mathcal{O}[1/\lambda] : \forall n \geq 0,\ \iota_n(d) \in \mathrm{Fil}^0(D_K \otimes_K \widehat{\mathfrak{S}}_n[1/u - v_{m+n}])\}.$$

Lemma (2.2.1). *For D in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$, $\mathcal{M}(D)$ is naturally an object of $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathcal{O}}$.*

Proof. That $\mathcal{M}(D)$ is a finite free $\mathcal{O}$-module, and the fact that φ_q on $D \otimes_{K_{0,L}} \mathcal{O}[1/\lambda]$ induces an isomorphism $\varphi_q^*(\mathcal{M}(D))[1/Q] \xrightarrow{\sim} \mathcal{M}(D)[1/Q]$ is proved exactly as in [Ki, 1.2.2].

[3] In fact they often simplify since one only has to consider the case when $N = 0$ in [Ki].

Note that for $\gamma \in \Gamma$,

$$\gamma([\pi_L](u)) = [\pi_L] \circ [\chi(\gamma)](u) = [\pi_L \chi(\gamma)](u).$$

Hence $\gamma(\lambda) = \lambda \circ [\chi(\gamma)]$ has a simple zero at each $[\pi_L^m]$-torsion point which is not a $[\pi_L^{m-1}]$-torsion point. It follows that $\lambda/\gamma(\lambda) \in \mathfrak{S}_L[1/p]^\times$. In particular, if $\gamma \in \Gamma$ acts on $D \otimes_{K_{0,L}} \mathcal{O}$ by $1 \otimes \gamma$, then this induces an action of Γ on $D \otimes_{K_{0,L}} \mathcal{O}[1/\lambda]$. The same argument shows that γ induces an automorphism of $\widehat{\mathfrak{S}}_n$ for $n \geq 0$. As $\varphi_{q,W_L}[\chi(\gamma)] = [\chi(\gamma)]$, one sees that $\mathcal{M}(D)$ is stable by the action of Γ. Finally, this action is $\mathcal{O}_L$-analytic, as the action of Γ on $\mathcal{O}$ is $\mathcal{O}_L$-analytic. $\square$

Lemma (2.2.2). *Let $\mathcal{M}$ be in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$. There exists a unique $K_{0,L}$-linear section $\xi : \mathcal{M}/u\mathcal{M} \to \mathcal{M}[1/\lambda]$ such that the elements of $\xi(\mathcal{M}/u\mathcal{M})$ are Γ-invariant. Moreover, we have*

(1) *ξ is φ_q-equivariant.*
(2) *ξ induces an isomorphism*

$$\mathcal{M}/u\mathcal{M} \otimes_{K_{0,L}} \mathcal{O}[1/\lambda] \xrightarrow{\sim} \mathcal{M}[1/\lambda].$$

(3) *The image of $\xi \otimes 1 : \mathcal{M}/u\mathcal{M} \otimes_{K_{0,L}} \mathcal{O} \to \mathcal{M}[1/\lambda]$ coincides with $(1 \otimes \varphi_q)(\varphi_q^* \mathcal{M})$ over an admissible open neighborhood of $u = v_m$.*

Proof. Consider the connection ∇ on $\mathcal{M}$ defined in (2.1.4)(3). For $r \in (0,1)$ sufficiently small there exists a unique ∇-parallel section $\xi_r : \mathcal{M}/u\mathcal{M} \to \mathcal{M}|_{D[0,r)}$. Since N_∇ commutes with φ_q, the section $\varphi_q \circ \xi_r \circ \varphi_q^{-1}$ is also ∇-parallel, and hence equal to ξ_r. Hence ξ_r is φ_q-invariant. Similarly $\gamma \circ \xi_r \circ \gamma^{-1}$ is ∇-parallel for $\gamma \in \Gamma$, so ξ_r is Γ-invariant.
Now ξ may be constructed from ξ_r exactly as in [Ki, 1.2.6], by repeatedly pulling ξ_r back by φ_q^* and using the isomorphism $\varphi_q^*\mathcal{M}[1/Q] \xrightarrow{\sim} \mathcal{M}[1/Q]$. The claims (2) and (3) also follow exactly as in *loc. cit.* $\square$

(2.2.3) Suppose that $\mathcal{M}$ is in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$. For i an integer denote by $\mathrm{Fil}^i \varphi_q^* \mathcal{M}$ the preimage of $Q^i\mathcal{M}$ under $\varphi_q^*\mathcal{M}[1/Q] \xrightarrow{\sim} \mathcal{M}[1/Q]$. Note that this filtration is Γ-stable. Let $D(\mathcal{M}) = \mathcal{M}/u\mathcal{M}$. By (2.2.2), ξ induces an isomorphism $D(\mathcal{M}) \otimes_{K_{0,L}} \mathcal{O} \xrightarrow{\sim} \varphi^*(\mathcal{M})$ near the point $u = v_m$. Hence we obtain an isomorphism

$$D(\mathcal{M}) \otimes_{K_{0,L}} K(v_m) \xrightarrow{\sim} \varphi_q^*(\mathcal{M})/Q\varphi_q^*(\mathcal{M}). \tag{2.2.4}$$

Give the right hand side of (2.2.4) the filtration induced by that on $\varphi_q^*\mathcal{M}$, and pull this filtration back to $D(\mathcal{M}) \otimes_{K_{0,L}} K(v_m)$. This gives rise to a Γ-stable filtration on $D(\mathcal{M}) \otimes_{K_{0,L}} K(v_m)$, which necessarily descends to a filtration on $D(\mathcal{M})_K$. This gives $D(\mathcal{M})$ the structure of an object in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$.

Lemma (2.2.5). *Let $\mathcal{M}$ be in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$ and $D = D(\mathcal{M})$. Then for all $i \in \mathbb{Z}$ the map ξ induces an isomorphism*

$$\sum_{j\geq 0} Q^j \widehat{\mathfrak{S}}_0 \otimes_K \mathrm{Fil}^{i-j} D(\mathcal{M})_K \xrightarrow{\sim} \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathrm{Fil}^i \varphi^*(\mathcal{M}).$$

Proof. This is the analogue of [Ki, 1.2.12(4)] in our situation, and the proof is identical, so we only sketch it here. Since $D(\mathcal{M})_K \otimes \widehat{\mathfrak{S}}_0$ and $\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \varphi_q^*(\mathcal{M})$ induce the same filtration on their common quotient $D(\mathcal{M})_K$, one sees easily that it suffices to check that for all $i \in \mathbb{Z}$,

$$\xi(\mathrm{Fil}^i D(\mathcal{M})_K) \subset \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} (1 \otimes \varphi_q)(\mathrm{Fil}^i \varphi^*(\mathcal{M})).$$

We will identify $\varphi_q^*\mathcal{M}$ with its image $(1 \otimes \varphi_q)(\varphi_q^*\mathcal{M})$ in $\mathcal{M}[1/Q]$. An element $d \in \xi(\mathrm{Fil}^i D(\mathcal{M})_K)$ can be written as $d = d_0 + d_1$ with $d_0 \in \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathrm{Fil}^i \varphi_q^*(\mathcal{M})$ and $d_1 \in Q\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \varphi_q^*(\mathcal{M})$. As $N_\nabla(d) = 0$, we have

$$N_\nabla(d_1) = -N_\nabla(d_0) \in \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} (\mathrm{Fil}^i \varphi_q^*(\mathcal{M}) \cap Q\varphi_q^*(\mathcal{M})) =: M_i.$$

Thus it suffices to show that for all $i \in \mathbb{Z}$, N_∇ induces a bijection on M_i, for then $d_1 \in M_i \subset \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathrm{Fil}^i \varphi_q^*(\mathcal{M})$. For i sufficiently small this follows from the isomorphism $D(\mathcal{M}) \otimes_{\mathcal{O}} \widehat{\mathfrak{S}}_0 \xrightarrow{\sim} \varphi_q^*\mathcal{M} \otimes_{\mathcal{O}} \widehat{\mathfrak{S}}_0$. The general case follows by descending induction on i and an application of the snake lemma. □

Proposition (2.2.6). *The functors $\mathcal{M}$ and D between $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$ and $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$ are quasi-inverse, exact, $\otimes$-equivalences.*

Proof. Let D be in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$. From the definition of $\mathcal{M}(D)$, there is a natural Γ-equivariant inclusion $D \subset \mathcal{M}(D)[1/Q]$, which induces an isomorphism of D with $D(\mathcal{M}(D)) = \mathcal{M}(D)/u\mathcal{M}(D)$. Hence the image of this inclusion coincides with $\xi(D(\mathcal{M}(D)))$, and one sees from the definitions that the filtration on D_K coincides with the one on $D(\mathcal{M}(D))_K$. This produces a natural isomorphism $D \xrightarrow{\sim} D(\mathcal{M}(D))$.

Conversely, let $\mathcal{M}$ be in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$. Then both $\mathcal{M}$ and $\mathcal{M}(D(\mathcal{M}))$ may be identified with $\mathcal{O}$-submodules of $D(\mathcal{M}) \otimes_{K_{0,L}} \mathcal{O}[1/\lambda]$. At any point of $D[0,1)$ other $u = v_n$, $n \geq m$, both submodules coincide with $D(\mathcal{M}) \otimes_{K_{0,L}} \mathcal{O}[1/\lambda]$. Since both $\mathcal{M}$ and $\mathcal{M}(D(\mathcal{M}))$ are in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma}$, to show these submodules are equal it suffices to check that these two submodules coincide at $u = v_m$. This follows from (2.2.5). (cf. [Ki, 1.2.13]). Hence we have a natural isomorphism $\mathcal{M}(D(\mathcal{M})) \xrightarrow{\sim} \mathcal{M}$.

That $\mathcal{M}$ and D are exact follows from (2.2.5). One checks easily that $\mathcal{M}$ and D respect $\otimes$-products (cf. [Ki, 1.2.15]). □

(2.3) We now apply Kedlaya's slope filtration as in [Be 2] and [Ki, §1.3] to show that an object $\mathcal{M}$ of $\mathrm{Mod}_{/\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$ can be descended to $\mathfrak{S}_L$ if and only if $D(\mathcal{M})$ is weakly admissible.[4] Again, as many of the arguments are identical to those of [Ki] we sometimes only sketch the proofs.

Let $\mathcal{R} = \lim_{r\to 1^-} \mathcal{O}_{(r,1)}$ denote the Robba ring, and $\mathcal{R}^b = \lim_{r\to 1^-} \mathcal{O}^b_{(r,1)}$, where $\mathcal{O}^b_{(r,1)} \subset \mathcal{O}_{(r,1)}$ denotes the subring of bounded functions. Then $\mathcal{R}^b$ is a discrete valuation field, with valuation

$$v_{\mathcal{R}^b}(f) = -\log_{\pi_L} \lim_{r\to 1^-} \sup_{x\in D(r,1)} |f(x)|$$

and uniformizer π_L. The endomorphism φ_q and the derivation N_∇ of $\mathcal{O}$ induce an automorphism and a derivation respectively of $\mathcal{R}$ and $\mathcal{R}^b$, which we will again denote by φ_q and N_∇.

Denote by $\mathrm{Mod}_{/\mathcal{R}}^{\varphi_q}$ (resp. $\mathrm{Mod}_{/\mathcal{R}^b}^{\varphi_q}$) the category of finite free $\mathcal{R}$-modules (resp. $\mathcal{R}^b$-modules) $\mathcal{M}$ equipped with an isomorphism $\varphi_q^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$. For an $\mathcal{M}$ in $\mathrm{Mod}_{/\mathcal{R}}^{\varphi_q}$, we denote by

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}$$

Kedlaya's slope filtration [Ke 1], [Ke 2]. We write s_i for the slope of the pure slope quotient $\mathcal{M}_i/\mathcal{M}_{i-1}$, which is finite free over $\mathcal{R}$. The filtration is functorial for maps in $\mathrm{Mod}_{/\mathcal{R}}^{\varphi_q}$. One of Kedlaya's results about the filtration says that a module of pure slope s has a canonical descent to a module $\mathcal{M}^b$ in $\mathrm{Mod}_{/\mathcal{R}^b}^{\varphi_q}$ which has slope s in the sense of Dieudonné-Manin theory (and the valuation on $\mathcal{R}^b$ normalized so that $v(\pi_L) = 1$).

For $\mathcal{M}$ in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,\Gamma,\mathrm{an}}$ we write $\mathcal{M}_\mathcal{R} = \mathcal{M}\otimes_\mathcal{O}\mathcal{R}$. The operators φ_q and N_∇ on $\mathcal{M}$ induce operators φ_q and N_∇ on $\mathcal{M}_\mathcal{R}$.

Lemma (2.3.1). *Let $\mathcal{M}$ be in $\mathrm{Mod}_{\mathcal{O}}^{\varphi_q,\Gamma,\mathrm{an}}$. The slope filtration on $\mathcal{M}_\mathcal{R}$ is induced by a unique filtration on $\mathcal{M}$ by saturated, finite free $\mathcal{O}$-submodules. This filtration on $\mathcal{M}$ is stable by φ_q and the action of Γ.*

Proof. It is clear that a such filtration on $\mathcal{M}$, if it exists is unique and stable by φ_q. The functoriality of the slope filtration on $\mathcal{M}_\mathcal{R}$ implies that it is stable by Γ, and hence so is the filtration on $\mathcal{M}$.

It remains to show the existence of such a filtration. As $\varphi_q(\lambda) = \pi_L[\pi_L^{m-1}]/[\pi_L^m]\lambda$, for any integer s, the slope filtration on $\lambda^{-s}\mathcal{M}_\mathcal{R}$ is given by $\lambda^{-s}\mathcal{M}_{\mathcal{R},i}$, and the slopes of $\lambda^{-s}\mathcal{M}_\mathcal{R}$ are those of $\mathcal{M}_\mathcal{R}$ shifted by $-s$. Since $[\pi_L^m]/[\pi_L^{m-1}]\pi_L$ has a unique, simple zero on $D[0,1)$ at x_0, we may replace $\mathcal{M}$ by $\lambda^{-s}\mathcal{M}$ for s sufficiently large, and assume that φ_q induces a map $\varphi_q^*\mathcal{M} \to \mathcal{M}$. We first show that the slope filtration is induced by a filtration on $\mathcal{M}|_{D(0,1)}$ by saturated $\mathcal{O}_{(0,1)}$-submodules. For some r_0 sufficiently close to 1, the slope

[4] The idea of relating Kedlaya's slope filtration to the condition of weak admissibility comes from Berger's beautiful paper [Be 2], however our treatment here is closer to that of [Ki].

filtration on $\mathcal{M}_{\mathcal{R}}$ is induced by a filtration on $\mathcal{M}|_{D(r_0,1)}$ by saturated $\mathcal{O}_{(r_0,1)}$-submodules. Let $r_0 > r_1 > \dots$ be a sequence approaching 0, and such that $\varphi_q^{-1}(D(r_i,1)) \subset D(r_{i-1},1)$ for $i \geq 1$. The same argument as in [Ki, 1.3.4] shows that for $j \geq 0$, $\mathcal{M}_{\mathcal{R},i}$ is induced by a filtration on $\mathcal{M}|_{D(r_j,1)}$ by closed, saturated $\mathcal{O}_{(r_j,1)}$-submodules, and hence by a filtration on $\mathcal{M}|_{D(0,1)}$ by closed, saturated $\mathcal{O}_{(0,1)}$-submodules.

The filtration on $\mathcal{M}_{D(0,1)}$ is stable by Γ, by uniqueness, and hence it is stable by N_∇. Consider the operator $\partial = \langle \nabla, -u\frac{d}{du} \rangle$ on $\mathcal{M}$. This is well defined in a neighbourhood of 0, and preserves the filtration on $\mathcal{M}|_{D(0,1)}$ over this neighbourhood as N_∇ does. Hence the filtration on $\mathcal{M}|_{D(0,1)}$ is induced by a filtration on $\mathcal{M}$ by closed, saturated $\mathcal{O}$-submodules by [Ki, 1.3.5]. □

(2.3.2) Let D in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ be 1-dimensional over $K_{0,L}$. Choose a basis vector v for D, and set $t_{N,L}(D) = v_{\pi_L}(\alpha)$ where $\alpha \in K_{0,L}$ satisfies $\varphi_q(v) = \alpha v$. We write $t_{H,L}(D)$ for the unique integer i such that $\mathrm{gr}^i D_K$ is non-zero. For D of arbitrary dimension d, we set $t_{N,L}(D) = t_{N,L}(\wedge^d D)$ and $t_{H,L}(D) = t_{H,L}(\wedge^d D)$. We will say that D is *weakly admissible* if the usual conditions of weak admissibility are satisfied with these invariants in place of the usual ones. That is if $t_{H,L}(D) = t_{N,L}(D)$ and $t_{H,L}(D') \leqslant t_{N,L}(D')$ for all φ_q-stable submodules $D' \subset D$.

Proposition (2.3.3). *Let D be in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ and $\mathcal{M} = \mathcal{M}(D)$. Then D is weakly admissible if and only if $\mathcal{M}_{\mathcal{R}}$ is pure of slope* 0.

Proof. Suppose first that $\dim_{K_{0,L}} D = 1$. Let v be a basis vector for D, and write $\varphi(v) = \alpha v$ for some $\alpha \in K_{0,L}^\times$. From the definition of $\mathcal{M}(D)$ one finds that $\mathcal{M}(D) = \lambda^{-t_{H,L}(D)}(D \otimes_{K_{0,L}} \mathcal{O})$, so

$$\varphi_q(\lambda^{-t_{H,L}(D)}e) = ([\pi_L^{m-1}]\pi_L/[\pi_L^m])^{-t_{H,L}(D)}\alpha\lambda^{-t_{H,L}(D)}e$$

As, $[\pi_L^m] = [\pi_L] \circ [\pi_L^{m-1}]$, we have that $[\pi_L^m]/[\pi_L^{m-1}] \in \mathfrak{S}_L$, is an element whose reduction modulo π_L is $u^{q^m - q^{m-1}}$. Hence $[\pi_L^m]/[\pi_L^{m-1}]$ is a unit in $\mathcal{R}^b$. It follows that $\mathcal{M}(D)$ has slope

$$v_{\pi_L}(\alpha) - t_{H,L}(D) = t_{N,L}(D) - t_{H,L}(D).$$

This proves the proposition when D has dimension 1. The general case follows from exactly the same argument as in [Ki, 1.3.8], using the equivalence (2.2.6) and (2.3.1). □

(2.4) Denote by $\mathrm{Mod}^{\varphi_q}_{/\mathfrak{S}_L}$ the category consisting of finite free $\mathfrak{S}_L$-modules $\mathfrak{M}$ equipped with an isomorphisms $1 \otimes \varphi_q : \varphi_q^*\mathfrak{M}[1/Q] \xrightarrow{\sim} \mathfrak{M}[1/Q]$. We denote by $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathfrak{S}_L}$ the category whose objects consist of an object of $\mathrm{Mod}^{\varphi_q}_{/\mathfrak{S}_L}$ equipped a semi-linear action of Γ on $\mathfrak{M}$ which commutes with the action of φ_q, and such that Γ acts trivially on $\mathfrak{M}/u\mathfrak{M}$.

We denote by $\mathrm{Mod}^{\varphi_q,0}_{/\mathcal{O}}$ (resp. $\mathrm{Mod}^{\varphi_q,\Gamma,0}_{/\mathcal{O}}$) the full subcategory of $\mathrm{Mod}^{\varphi_q}_{/\mathcal{O}}$ (resp. $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$) consisting of objects $\mathcal{M}$ such that $\mathcal{M}_{\mathcal{R}} = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ is pure of slope 0.

LEMMA (2.4.1). *There is an equivalence of* $\otimes$*-categories*

$$\mathrm{Mod}^{\varphi_q}_{/\mathfrak{S}_L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Mod}^{\varphi_q,0}_{/\mathcal{O}}; \quad \mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathrm{W}_L[\![u]\!]} \mathcal{O}$$

where the left hand side means the category obtained from $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathfrak{S}_L}$ *by applying* $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ *to the* Hom *groups.*

Proof. This is identical to [Ki, 1.3.13]. □

COROLLARY (2.4.2). *There is an equivalence of exact* $\otimes$*-categories*

$$\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathfrak{S}_L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Mod}^{\varphi_q,\Gamma,0}_{/\mathcal{O}}; \quad \mathfrak{M} \mapsto \mathfrak{M}/u\mathfrak{M}.$$

Proof. This follows from (2.4.1), since the action of $\gamma \in \Gamma$ can be thought of as an isomorphism $\gamma^*(\mathfrak{M}) \xrightarrow{\sim} \mathfrak{M}$ for $\mathfrak{M}$ in $\mathrm{Mod}^{\varphi_q}_{/\mathfrak{S}_L}$ or $\mathrm{Mod}^{\varphi_q,\Gamma,0}_{/\mathcal{O}}$. □

(2.4.3) We denote by $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathfrak{S}_L}$ the full subcategory of $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathfrak{S}_L}$ such that the corresponding object in $\mathrm{Mod}^{\varphi_q,\Gamma}_{/\mathcal{O}}$ is in $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathcal{O}}$

COROLLARY (2.4.4). *There is an exact, fully faithful* $\otimes$*-functor*

$$\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{/\mathfrak{S}_L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$$

whose essential image consists of the weakly admissible modules in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$*.*

Proof. This follows by combining (2.4.2), (2.2.6) and (2.3.3). □

§3 (φ_q,Γ)-MODULES AND CRYSTALLINE REPRESENTATIONS

(3.1) Recall the ring $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ introduced in (1.1). Denote by $B_{\mathrm{dR}}, B_{\mathrm{cris}} \supset W(R)$ the usual rings introduced by Fontaine [Fo 2]. Write $B_{\mathrm{cris},L} = B_{\mathrm{cris}} \otimes_{L_0} L$. We write φ_q for the L-linear extension of the operator φ^r on B_{cris}. Note that we have an embedding

$$B_{\mathrm{cris},L} \otimes_{K_{0,L}} K \xrightarrow{\sim} B_{\mathrm{cris}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}. \tag{3.1.1}$$

For D in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ this gives rise to an embedding

$$B_{\mathrm{cris},L} \otimes_{K_{0,L}} D_K \xrightarrow{\sim} B_{\mathrm{cris}} \otimes_{K_0} D_K \hookrightarrow B_{\mathrm{dR}} \otimes_K D_K. \tag{3.1.2}$$

We say an element in $B_{\mathrm{cris},L} \otimes_{K_{0,L}} D$ is in Fil^0 if its image in $B_{\mathrm{dR}} \otimes_K D_K$ under (3.1.2) is. Dually, we say $K_{0,L}$-linear map $D \to B_{\mathrm{cris},L}$ is compatible with filtrations if the map $D_K \to B_{\mathrm{dR}}$ induced by (3.1.1) is compatible with filtrations.

For D in $\mathrm{Mod}_{K_{0,L}}^{F,\varphi_q}$ let

$$V_L(D) = \mathrm{Fil}^0(B_{\mathrm{cris},L} \otimes_{K_{0,L}} D)^{\varphi_q=1}$$

and

$$V_L^*(D) = \mathrm{Hom}_{\varphi_q,\mathrm{Fil}}(D, B_{\mathrm{cris},L}).$$

There is a canonical isomorphism $V_L^*(D) \xrightarrow{\sim} V_L(D^*)$, where D^* denotes the dual of D in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$.
We will prove an analogue for the category $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$ of the result that weakly admissible modules are admissible.

Lemma (3.1.3). *Let D in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$ be of $K_{0,L}$-dimension 1 such that $t_{H,L}(D) \leqslant t_{N,L}(D)$. Then $V_L(D) = 0$ unless $t_{H,L}(D) = t_{N,L}(D)$, in which case $\dim_L V_L(D) = 1$.*

Proof. Give $D(\mathcal{G}) = K_{0,L}$ the structure of an object of $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$ by equipping it with a φ_q semi-linear automorphism given by sending 1 to π_L^{-1} and defining $\mathrm{Fil}^i D(\mathcal{G})_K = D(\mathcal{G})_K$ if $i \leqslant -1$ and 0 otherwise. Then by [Col 1, Prop. 9.19]

$$V_L(D(\mathcal{G})) = \mathrm{Fil}^0(D(\mathcal{G}) \otimes_{K_{0,L}} B_{\mathrm{cris},L})^{\varphi_q=1} = \mathrm{Fil}^1 B_{\mathrm{cris},L}^{\varphi_q=\pi_L} = t_L \cdot L,$$

where $t_L = \log_{\mathcal{G}} u$, is a unit in $B_{\mathrm{cris},L}$ [Col 1, Prop. 9.10, 9.17].
Let D in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$ be of $K_{0,L}$-dimension 1 such that $t_{H,L}(D) \leqslant t_{N,L}(D)$. Let $d \in D$ be a $K_{0,L}$-basis vector. Since multiplication by t_L^j induces a bijection of $V_L(D)$ with $V_L(D \otimes D(\mathcal{G})^j)$, we may assume that $\varphi_q(d) = \alpha d$ with $\alpha \in W_L^\times$. Furthermore, if $\bar{k}$ denotes the residue field of $\bar{K}$, then there exists $\beta \in W(\bar{k})_L^\times$ such that $\varphi_q(\beta) = \alpha\beta$, so we may assume that $\alpha = 1$. Then $t_{N,L}(D) = 0 \geq t_{H,L}(D)$,

$$V_L(D) = \mathrm{Fil}^{-t_{H,L}(D)} B_{\mathrm{cris},L}^{\varphi_q=1}$$

and the lemma follows from [Col 1, Lem. 9.14]. □

Lemma (3.1.4). *Suppose that D in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$ is weakly admissible. Then*

$$\dim_L V_L(D) \leqslant \dim_{K_{0,L}} D.$$

Proof. This is very similar to [CF, Prop. 4.5]. Denote by $C_{\mathrm{cris},L}$ the field of fractions of $B_{\mathrm{cris},L}$. Let $\mathcal{V}$ denote the $C_{\mathrm{cris},L}$-span of $V_L(D) \subset C_{\mathrm{cris}} \otimes_{K_{0,L}} D$. Then $\mathcal{V}$ is invariant under the action of G_K, and [CF, Lem. 4.6] implies that there exists a unique $K_{0,L}$-subspace $D' \subset D$ such that $C_{\mathrm{cris},L} \otimes_{K_{0,L}} D'$ is equal to $\mathcal{V}$.
Let $s = \dim_{K_{0,L}} D'$ and $d_1, \dots, d_s \in D'$ a $K_{0,L}$-basis. We also choose $v_1, \dots, v_s \in V_L(D)$ which span $\mathcal{V}$. Then

$$w := v_1 \wedge \cdots \wedge v_s = b d_1 \wedge \cdots \wedge d_s$$

for some $b \in C_{\mathrm{cris},L}$. As $0 \neq w \in V_L(\wedge^s D')$, $t_{H,L}(D') = t_{N,L}(D')$ by (3.1.3) and $\dim_L V_L(\wedge^s D') = 1$. Moreover, (3.1.3) implies that there is a perfect pairing

$$V_L(\wedge^s D') \otimes_L V_L((\wedge^s D')^*) \to V_L(K_{0,L}) \xrightarrow{\sim} L,$$

so $V_L((\wedge^s D')^*) = b^{-1} d_1 \wedge \cdots \wedge d_s$ and b is a unit in $B_{\mathrm{cris},L}$.
Finally, if $v \in V_L(D)$, write $v = \sum_{i=1}^s b_i v_i$ with $b_i \in C_{\mathrm{cris}}$. Then for $1 \leqslant i \leqslant s$

$$v_1 \wedge \cdots \wedge v_{i-1} \wedge v \wedge v_{i+1} \cdots \wedge v_s = b_i w \in V_L(\wedge^s D').$$

Hence $b_i \in L$, which shows that $\dim_L V_L(D) = s \leqslant \dim_{K_{0,L}} D$. □

(3.2) Write $\mathfrak{S}_L^{\mathrm{ur}} = \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \cap W(R)_L \subset W(\mathrm{Fr}\, R)_L$. We set $P = [\pi_L^{m-1}](u)$. For $\gamma \in \Gamma$, we have $\gamma(P) = [\chi(\gamma)] \circ [\pi_L^{m-1}] = [\chi(\gamma)\pi_L^{m-1}]$, so $\gamma(P)/P$ is a unit in $\mathfrak{S}_L$. In particular $\mathfrak{S}_L^{\mathrm{ur}}[1/P]$ is G_K-stable.
Note that the embedding $\mathfrak{S}_L \hookrightarrow W(R)_L$, extends uniquely to a continuous embedding $\mathcal{O} \hookrightarrow B^+_{\mathrm{cris},L}$, where $B^+_{\mathrm{cris},L} = B^+_{\mathrm{cris}} \otimes_{\mathcal{O}_{L_0}} \mathcal{O}_L$, as usual.

LEMMA (3.2.1). *Let $\mathfrak{M}$ be in $\mathrm{Mod}^{\varphi_q}_{/\mathfrak{S}_L}$. The natural map*

$$(3.2.2)\quad V_{\mathfrak{S}_L}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M}, \mathfrak{S}_L^{\mathrm{ur}}[1/P]) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi_q}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}_L} \mathfrak{M}, \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{ur}}})$$

is an isomorphism, and both sides are free $\mathcal{O}_L$-modules of rank $\mathrm{rk}_{\mathfrak{S}_L}\mathfrak{M}$. Moreover, if φ_q on $\mathfrak{M}$ induces a map $\varphi_q^*\mathfrak{M} \to \mathfrak{M}$ then the left hand side of (3.2.2) is equal to $\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M}, \mathfrak{S}_L^{\mathrm{ur}})$.

Proof. Suppose first that φ_q induces a map $\varphi_q^*\mathfrak{M} \to \mathfrak{M}$. In this case the proof of the lemma is identical to the proof of [Ki, 2.1.4], using [Fo 1, §A.1.2].
Next let $t_L = \log_{\mathcal{G}} u \in \mathcal{O}$ as in the proof of (3.1.3). Then $\varphi_q(\lambda^{-1}t_L) = Q(u)\lambda^{-1}t_L$, and the zeroes of $\lambda^{-1}t_L$ on $D[0,1)$ coincide with those of P. Hence $\lambda^{-1}t_L P^{-1} \in \mathfrak{S}_L[1/p]^\times$, and there exists $w \in \mathfrak{S}_L[1/P]^\times$ such that $\varphi_q(w) = Q(u)w$. Let $\mathfrak{M}(\mathcal{G}) = \mathfrak{S}_L$ equipped with a semi-linear action of φ_q which takes 1 to $Q(u)$. Then multiplication by w^i induces a bijection

$$V_{\mathfrak{S}_L}(\mathfrak{M}) \to V_{\mathfrak{S}_L}(\mathfrak{M} \otimes_{\mathfrak{S}_L} \mathfrak{M}(\mathcal{G})^{\otimes i}).$$

Hence the lemma for general $\mathfrak{M}$ in $\mathrm{Mod}^{\varphi_q}_{/\mathfrak{S}_L}$ follows from the case considered above. □

PROPOSITION (3.2.3). *Let $\mathfrak{M}$ be in $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{an}}_{\mathfrak{S}_L}$ and D in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ the weakly admissible module associated to $\mathfrak{M}$ by (2.4.4). Then there is a a canonical G_K-equivariant bijection*

$$\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M}, \mathfrak{S}_L^{\mathrm{ur}}[1/P]) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi_q}(D, B_{\mathrm{cris},L})$$

where the right hand side means maps compatible with filtrations in the sense explained in (3.1). In particular, both sides of the above isomorphism have dimension $\dim_{K_{0,L}} D$ over L.

Proof. The argument is similar to [Ki, 2.1.5]. Let $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}_L} \mathcal{O}$. Note that $P^{-1} \in \lambda t_L^{-1}\mathfrak{S}_L[1/p]^\times \subset B_{\mathrm{cris},L}$. Similarly, $\lambda^{-1} \in P t_L^{-1}\mathfrak{S}_L[1/p]^\times \subset B_{\mathrm{cris},L}$.

Consider the composite
(3.2.4)
$$\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M},\mathfrak{S}_L^{\mathrm{ur}}[1/P]) \to \mathrm{Hom}_{\mathcal{O},\varphi_q}(\mathcal{M},B_{\mathrm{cris},L}) \to \mathrm{Hom}_{\mathcal{O},\varphi_q}(\varphi_q^*\mathcal{M},B_{\mathrm{cris},L}).$$

We claim the image of composite map consists of morphisms respecting filtrations. To see this suppose first that φ_q on $\mathcal{M}$ induces a map $\varphi_q^*\mathcal{M} \to \mathcal{M}$. Then by (3.2.1), the left hand side of (3.2.4) is equal to $\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M},\mathfrak{S}_L^{\mathrm{ur}})$. A $\mathfrak{S}_L$-linear map $f:\mathfrak{M}\to\mathfrak{S}_L^{\mathrm{ur}}$ induces an $\mathcal{O}$-linear map $f:\mathcal{M}\to B^+_{\mathrm{cris},L}$. If $m\in\varphi_q^*\mathcal{M}$ satisfies $(1\otimes\varphi_q)(m)\in Q^i\mathcal{M}$ for some integer i, then $f\circ(1\otimes\varphi_q)(m)\in Q^iB^+_{\mathrm{cris},L}\subset \mathrm{Fil}^iB_{\mathrm{dR}}$, as $Q(u)\in\mathrm{Fil}^1B_{\mathrm{dR}}$ [Col 1, Lem. 9.3]. This proves the claim when $\varphi_q^*\mathcal{M}$ maps to $\mathcal{M}$.
To prove the claim for general $\mathcal{M}$ we use the notation of the proof of (3.2.1). Let $\mathfrak{M}(i)=\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{M}(\mathcal{G})^{\otimes i}$, and $\mathcal{M}(i)=\mathfrak{M}(i)\otimes_{\mathfrak{S}_L}\mathcal{O}$, where i is an integer which is large enough that φ_q induces a map $\varphi_q^*\mathfrak{M}(i)\to\mathfrak{M}(i)$. The underlying $\mathcal{O}$-module of $\mathcal{M}(i)$ may be identified with $\mathcal{M}$, and the induced identification $\varphi_q^*\mathcal{M}=\varphi_q^*\mathcal{M}(i)$ identifies $\mathrm{Fil}^j\varphi_q^*\mathcal{M}$ with $\mathrm{Fil}^{i+j}\varphi_q^*\mathcal{M}$ for all j. As we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M}(i),\mathfrak{S}_L^{\mathrm{ur}}[1/P]) & \longrightarrow & \mathrm{Hom}_{\mathcal{O},\mathrm{Fil},\varphi_q}(\varphi_q^*\mathcal{M}(i),B_{\mathrm{cris},L}) \\
\sim\downarrow w^{-i} & & \sim\downarrow w^{-i} \\
\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M},\mathfrak{S}_L^{\mathrm{ur}}[1/P]) & \longrightarrow & \mathrm{Hom}_{\mathcal{O},\varphi_q}(\varphi_q^*\mathcal{M},B_{\mathrm{cris},L})
\end{array}$$

the claim follows for general $\mathfrak{M}$.
We now compose (3.2.4) with the map

(3.2.5) $$\mathrm{Hom}_{\mathcal{O},\mathrm{Fil},\varphi_q}(\varphi_q^*\mathcal{M},B_{\mathrm{cris},L})\to\mathrm{Hom}_{K_{0,L},\mathrm{Fil},\varphi_q}(D,B_{\mathrm{cris},L}).$$

induced by extending a map in the left hand side to $\varphi_q^*\mathcal{M}[1/\lambda]$ and composing with $\varphi_q^*(\xi):D\to\varphi_q^*\mathcal{M}[1/\lambda]$, where ξ is the map of (2.2.2). Note that (3.2.5) respects filtrations as $\varphi_q^*(\xi)$ is compatible with filtrations by (2.2.5), and λ vanishes to order 1 at $u=v_m$. Combining (3.2.4) and (3.2.5) we obtain a canonical G_K-equivariant map

(3.2.6) $$\mathrm{Hom}_{\mathfrak{S}_L,\varphi_q}(\mathfrak{M},\mathfrak{S}_L^{\mathrm{ur}}[1/P])\to\mathrm{Hom}_{K_{0,L},\mathrm{Fil},\varphi_q}(D,B_{\mathrm{cris},L}).$$

It is easy to see that (3.2.6) is injective. By (3.2.1) the left hand side of (3.2.6) has L-dimension $d=\mathrm{rk}_{\mathfrak{S}_L}\mathfrak{M}$, while by (3.1.4) the right hand side has dimension $\leqslant d$. Hence (3.2.6) is an isomorphism. □

(3.3) We now explain how to pass from φ_q-modules to φ-modules, using an induction procedure which was explained to us by Fontaine (cf. [FY, §7.3]). Suppose that D is a finite dimensional $K_{0,L}$-vector space. We set

$$\tilde{D}:=\oplus_{i=0}^{r-1}\varphi^{i*}(D)=\oplus_{\sigma:L_0\hookrightarrow L}\sigma^*(D),$$

so that $\tilde{D}$ is a finite free $K_0 \otimes_{\mathbb{Q}_p} L$-module. Here we have denoted by φ^i the map $K_0 \otimes_{L_0} L \stackrel{\varphi^i\otimes 1}{\to} K_0 \otimes_{\varphi^i, L_0} L$. We put $\tilde{D}_K = \tilde{D} \otimes_{K_0} K$.

We denote by $\mathrm{Mod}^{F,\varphi}_{/K_0\otimes_{\mathbb{Q}_p} L}$ the category of finite free $K \otimes_{\mathbb{Q}_p} L$-modules $\tilde{D}$ equipped with an isomorphism $\varphi^*(\tilde{D}) \xrightarrow{\sim} \tilde{D}$ and a decreasing separated filtration on $\tilde{D}_K = \tilde{D} \otimes_{K_0} K$, indexed by $\mathbb{Z}$, by $K \otimes_{\mathbb{Q}_p} L$-submodules. For any $\tilde{D}$ in $\mathrm{Mod}^{F,\varphi}_{K_O\otimes_{\mathbb{Q}_p} L}$ we denote by $t_N(\tilde{D})$ and $t_H(\tilde{D})$ the usual invariants when $\tilde{D}$ is considered as a filtered φ-module over K_0.

For any $\tilde{D}$ in $\mathrm{Mod}^{F,\varphi}_{/K_0\otimes_{\mathbb{Q}_p} L}$ we may write $\tilde{D}_K = \oplus_{\mathfrak{m}}(\tilde{D}_K)_{\mathfrak{m}}$ where $\mathfrak{m}$ runs over the maximal ideals of $K \otimes_{\mathbb{Q}_p} L$. Denote by $\mathfrak{m}_0$ the kernel of the natural map $K \otimes_{\mathbb{Q}_p} L \to K$. Given D in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$ we give $\tilde{D}$ the structure of an object in $\mathrm{Mod}^{F,\varphi}_{/K_0\otimes_{\mathbb{Q}_p} L}$ by noting that $(\tilde{D}_K)_{\mathfrak{m}_0}$ may be identified with D_K, and giving $\tilde{D}_K$ the direct sum of the filtration on D_K and the trivial filtration on $\oplus_{\mathfrak{m}\neq\mathfrak{m}_0}(\tilde{D}_K)_{\mathfrak{m}}$.

Lemma (3.3.1). *The functor $D \mapsto \tilde{D}$ induces an fully faithful $\otimes$-functor*

$$\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}} \xrightarrow{\sim} \mathrm{Mod}^{F,\varphi}_{K_0\otimes_{\mathbb{Q}_p} L}$$

The essential image of this functor consists of those objects such that the filtration on $\oplus_{\mathfrak{m}\neq\mathfrak{m}_0}\tilde{D}_K$ is trivial.

Proof. Given D in $\mathrm{Mod}^{F,\varphi_q}_{K_{0,L}}$ there is a natural isomorphism

$$\varphi^*(\tilde{D}) = \oplus_{i=1}^r \varphi^{i*}(D) \xrightarrow{\sim} \oplus_{i=0}^{r-1}\varphi^{i*}(D) = \tilde{D},$$

which sends $\varphi^{i*}(D)$ identically to $\varphi^{i*}(D)$ for $i \neq r$ and maps $\varphi^{r*}(D) = \varphi_q^*(D)$ to D using the map φ_q on D. This defines the functor of the lemma.

To define a quasi-inverse on the essential image of the functor, let $\tilde{D}$ be in $\mathrm{Mod}^{F,\varphi}_{/K_0\otimes_{\mathbb{Q}_p} L}$ be such that the filtration on $\oplus_{\mathfrak{m}\neq\mathfrak{m}_0}\tilde{D}_K$ is trivial, and set $D' = \tilde{D} \otimes_{K_0\otimes_{\mathbb{Q}_p} L} (K_0 \otimes_{L_0} L)$. There is an isomorphism $\varphi^{*r}(D') \xrightarrow{\sim} D'$ induced by $\varphi^{*r}(\tilde{D}) \xrightarrow{\sim} \tilde{D}$. Using the decomposition

$$K_0 \otimes_{\mathbb{Q}_p} L \xrightarrow{\sim} \oplus_{i=0}^{r-1} K_0 \otimes_{\varphi^i, L_0} L$$

one sees that there is a canonical isomorphism $\tilde{D}' \xrightarrow{\sim} \tilde{D}$. In particular this makes D' into an object of $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$.

One checks immediately that these two functors are quasi-inverse. □

Lemma (3.3.2). *Let D be in $\mathrm{Mod}^{F,\varphi_q}_{/K_{0,L}}$. Then*

$$t_{N,L}(D) = t_N(\tilde{D}) \text{ and } t_{H,L}(D) = t_H(\tilde{D}) \tag{3.3.3}$$

and D is weakly admissible if and only if $\tilde{D}$ is weakly admissible.

Proof. Since the functor of (3.3.1) respects $\otimes$-products, it suffices to prove (3.3.3) when D is 1-dimensional over $K_{0,L}$. Moreover, as the essential image of the functor in (3.3.1) is stable under subobjects, (3.3.3) implies the claim regarding weak admissibility.
Let D be 1-dimensional over $K_{0,L}$ with basis vector v, and $\varphi_q(v) = \alpha v$ for some $\alpha \in K_{0,L}$. Then for $i = 0, \dots, r-1$ the K_0-vector space $\wedge_{K_0}^{[L:L_0]} \varphi^{i*}(D)$ has a basis vector e_i such that $\varphi(e_i) = e_{i+1}$ if $i < r-1$, and $\varphi(e_{r-1}) = N_{K_{0,L}/K_0}(\alpha)e_0$. Hence φ takes

$$e_0 \wedge \dots \wedge e_{r-1} \in \wedge_{K_0}^{[L:\mathbb{Q}_p]} \tilde{D} \xrightarrow{\sim} \otimes_{i=0}^{r-1} \wedge_{K_0}^{[L:L_0]} \varphi^{i*}(D)$$

to $(-1)^r N_{K_{0,L}/K_0}(\alpha) e_0 \wedge \dots \wedge e_{r-1}$, and

$$t_N(\tilde{D}) = v_p(N_{K_{0,L}/K_0}(\alpha)) = [L : L_0] v_p(\alpha) = v_{\pi_L}(\alpha) = t_{N,L}(D).$$

On the other hand, $t_{H,L}(D) = t_H(\tilde{D})$, from the definition of the filtration on $\tilde{D}$. □

Proposition (3.3.4). *Let D be a weakly admissible object in $\mathrm{Mod}_{/K_{0,L}}^{F,\varphi_q}$. Then there is a canonical G_K-equivariant isomorphism*

$$V_L^*(D) \xrightarrow{\sim} V^*(\tilde{D}) := \mathrm{Hom}_{\mathrm{Fil},\varphi}(\tilde{D}, B_{\mathrm{cris}}).$$

Proof. Suppose $\tilde{f} : \tilde{D} \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L$ is a $K_0 \otimes_{\mathbb{Q}_p} L$-linear, φ-compatible map, such that

$$f_K : \tilde{D}_K \to K \otimes_{K_0} B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L \hookrightarrow B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$$

is compatible with filtrations. Consider the composite

$$\theta(f) : D \hookrightarrow \tilde{D} \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L \to B_{\mathrm{cris},L}.$$

This is a φ_q-compatible map, such that the composite

$$\theta(f)_K : D_K \to K \otimes_{K_{0,L}} B_{\mathrm{cris},L} \hookrightarrow B_{\mathrm{dR}}.$$

is obtained from f_K by localizing at $\mathfrak{m}_0$. In particular, $\theta(f)_K$ is compatible with filtrations. Note also that f can be recovered from $\theta(f)$: The decomposition

$$K_0 \otimes_{\mathbb{Q}_p} L \xrightarrow{\sim} \oplus_{i=0}^{r-1} K_0 \otimes_{\varphi^i, L_0} L$$

allows us to view $B_{\mathrm{cris},L}$ as a direct summand in $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L$. Then f is the unique φ-linear extension of the φ_q-linear map

$$D \overset{\theta(f)}{\to} B_{\mathrm{cris},L} \hookrightarrow B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L.$$

Hence we have an injective map

$$\text{(3.3.5)} \qquad \mathrm{Hom}_{\mathrm{Fil},\varphi,K_0\otimes_{\mathbb{Q}_p}L}(\tilde{D}, B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p} L) \hookrightarrow \mathrm{Hom}_{\mathrm{Fil},\varphi_q,K_{0,L}}(D, B_{\mathrm{cris},L})$$

On the other hand, the trace map $L \to \mathbb{Q}_p$ induces an isomorphism

$$\text{(3.3.6)} \qquad \mathrm{Hom}_{\mathrm{Fil},\varphi,K_0\otimes_{\mathbb{Q}_p}L}(\tilde{D}, B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p} L) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi,K_0}(\tilde{D}, B_{\mathrm{cris}}).$$

Composing the inverse of (3.3.6) with (3.3.5) gives an injective map $V^*(\tilde{D}) \to V_L^*(D)$. As $\tilde{D}$ is admissible by (3.3.2), $\dim_L V^*(\tilde{D}) = \dim_{K_{0,L}} D$, and this is equal to $\dim_L V_L^*(D)$ by (3.3.2). Hence we have $V^*(\tilde{D}) \xrightarrow{\sim} V_L^*(D)$. □

(3.3.7) Denote by $\mathrm{Rep}_{G_K}^{L\text{-cris}}$ the full subcategory of Rep_{G_K} consisting of those objects V such that $V\otimes_{\mathcal{O}_L} L$ is a crystalline representation and, if $D_{\mathrm{dR}}(V) = (V\otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$, then the filtration on $D_{\mathrm{dR}}(V)_{\mathfrak{m}}$ is trivial for $\mathfrak{m} \neq \mathfrak{m}_0$ a maximal ideal of $K\otimes_{\mathbb{Q}_p} L$.

Corollary (3.3.8). *There is an exact equivalence of $\otimes$-categories*

$$\mathrm{Mod}_{/\mathfrak{S}_L}^{\varphi_q,\Gamma,\mathrm{an}} \xrightarrow{\sim} \mathrm{Rep}_{G_K}^{L\text{-cris}}; \quad \mathfrak{M} \mapsto V(\mathcal{O}_{\mathcal{E}}\otimes_{\mathfrak{S}_L}\mathfrak{M})$$

where V is the functor introduced in (1.5).

Proof. Using (3.2.3) and (3.3.4) one sees that the functor of (1.6) induces a fully faithful, exact $\otimes$-functor as in the corollary. To show that this functor is essentially surjective let V be in $\mathrm{Rep}_{G_K}^{L\text{-cris}}$, and let $M = M_{\mathcal{O}_{\mathcal{E}}}(V)$. By (2.4.4) and (3.3.1), there exists an $\mathfrak{M}'$ in $\mathrm{Mod}_{/\mathfrak{S}_L}^{\varphi_q,\Gamma,\mathrm{an}}$ such that $V(\mathcal{O}_{\mathcal{E}}\otimes_{\mathfrak{S}_L}\mathfrak{M}')$ is isomorphic to a G_K-stable $\mathcal{O}_L$-lattice $V' \subset V\otimes_{\mathcal{O}_L} L$. Thus, by the equivalence of (1.6) there is an isomorphism $\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{E}}} M \xrightarrow{\sim} \mathcal{E}\otimes_{\mathfrak{S}_L}\mathfrak{M}'$. Then $\mathfrak{M} = M\cap\mathfrak{M}'[1/p] \subset M[1/p]$ is in $\mathrm{Mod}_{/\mathfrak{S}_L}^{\varphi_q,\Gamma,\mathrm{an}}$ and satisfies $\mathcal{O}_{\mathcal{E}}\otimes_{\mathfrak{S}_L}\mathfrak{M} \xrightarrow{\sim} M$. Hence $V(\mathcal{O}_{\mathcal{E}}\otimes_{\mathfrak{S}_L}\mathfrak{M}) \xrightarrow{\sim} V(M) \xrightarrow{\sim} V$. □

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