Artin-Hasse Functions and their Invertions in Local Fields

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Abstract. The extension of Artin-Hasse functions and the inverse maps on formal modules over local fields is presented.

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Андрею Суслину, прекрасному человеку и ученому, на 60-летие

1 Introduction

The classical Artin-Hasse function was defined in 1928 in [1] in order to find Ergängungsätze for the global reciprocity law in the cyclotomic field $\mathbb{Q}(\zeta)$, where ζ is a primitive root of unity of order p^n. The Artin-Hasse functions turned out to be a convenient tool for describing the arithmetic of the multiplicative group of a local field. Using them, I.R. Shafarevich constructed in [2] a canonical basis, which gives the decomposition of elements up to p^nth powers. When in the early 1960s Lubin and Tate constructed formal groups with complex multiplication in a local field to define the reciprocity relation explicitly, the role of Artin-Hasse functions became clear. They turned out to be the isomorphism between the canonical formal group with logarithm $x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + ...$ and the multiplicative formal group (see [3]).
S.V. Vostokov in [4] generalized Artin-Hasse functions in the multiplicative case using the Frobenius operator Δ, defined on the ring of Laurent series over the ring of Witt vectors, to the function E_Δ:

$$E_\Delta(f(x)) = \exp\left(1 + \frac{\Delta}{p} + \frac{\Delta^2}{p^2} + ...\right)(f(x)),$$

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where Δ acts on x as raising to the power p, and on the coefficients of the ring of Witt vectors as the usual Frobenius. The map E_Δ induces a homomorphism from the additive group of the ring of power series to the multiplicative group. In addition, [4] contains the definitions of the inverse operator function l_Δ:

$$l_\Delta(f(x)) = \left(1 - \frac{\Delta}{p}\right) \log f(x), \qquad f(x) \equiv 1 \pmod{x},$$

which induces the inverse homomorphism.

To find explicit formulas for the Hilbert pairing on the Lubin-Tate formal groups, the work [5] generalized Artin-Hasse functions $E_\Delta(f(x))$ and their inverse functions $l_\Delta(f(x))$ to the Lubin-Tate formal groups. Using them, explicit formulas for the Hilbert pairing for formal Lubin-Tate modules were obtained in [4], [5]. They played a key role in the construction of the arithmetic of formal modules (see [5]).

The Artin-Hasse functions for Honda formal groups were defined in the work [6]. In the construction of generalised Artin-Hasse functions one uses the existence of a classification of this type of formal groups. Until 2002 such a classification was known in two cases only:

1. Lubin-Tate formal groups, i.e. formal groups of minimal height defined over the ring of integers of a local field isomorphic to the endomorphism ring of a formal group;

2. Honda formal groups, i.e. formal groups over the ring of Witt vectors of a finite field.

In 2002 M. Bondarko and S. Vostokov [6] obtained an explicit classification of formal groups in arbitrary local fields.

To study the arithmetic of the formal modules in an arbitrary local field and to derive explicit formulas for the generalized Hilbert pairing on these modules we should extend Artin-Hasse functions and the inverse maps on these formal modules. This paper contains a solution of this problem.

2 Notation

Suppose that $K/\mathbb{Q}$ is a local field with ramification index e and ring of integers O_K. By N we will denote its inertia subfield, and by O the ring of integers of N; σ will denote the Frobenius map on N.

Define a linear operator $\Delta : O[[x]] \to O[[x]]$ which acts as follows: $\Delta(\sum a_i x^i) = \sum \sigma(a_i) x^{pi}$.

Consider the ring $W = O[[t]]$. We can extend the action of σ on $O[[t]]$ by defining $\sigma(t) = t^p$. Then we can define Δ on $W[[x]]$ just as on $O[[x]]$. Let $\psi : W \to O_K$ be a homomorphism satisfying $\psi(t) = \pi$ and $\psi|_O = id_O$. Define a map $\phi_0 : O_K \to W$ by the formula: $\phi_0(\sum_{i=0}^{e-1} w_i \pi^i) = \sum_{i=0}^{e-1} w_i t^i$, where $w_i \in O$.

$V = N((t))$ will denote the field of fractions of W. We can easily extend ψ on it. We can also extend Δ on $V[[x]]$.
Denote by R_O the set of Teichmüller representatives in the ring O.
Let F_1 and F_2 be isomorphic formal groups over O_K with logarithms λ_1 and λ_2 respectively. Suppose that F_1 is a p-typical formal group. We will denote the isomorphism between them by $f(x) = \lambda_1^{-1}(\lambda_2(x)) \in O_K[[x]]$. Since F_1 is p-typical, we get $\lambda_1(x) = \Lambda_1(\Delta)x$, where $\Lambda_1(\Delta) \in O_K[[\Delta]]\Delta$.
We use the following convention: sometimes we write λ or Λ or F without an index if the index is equal to 1.

Lemma 1. *There exists a formal group $\overline{F} \in W[[x, y]]$ with p-typical logarithm $\overline{\lambda} \in xV[[x]]$ such that the following two statements are true:*

1. $\psi(\overline{\lambda}) = \lambda$.
2. $\psi(\overline{F}) = F$.

Proof. Let $G(R)$ be a universal formal group. Then the homomorphism between G and F can be factored through W. This is what we wanted. □

From now on we fix some $\overline{F}$ and $\overline{\lambda}$ from the lemma above. Since $\overline{\lambda}$ is p-typical, we can write it in the form $\overline{\lambda}(x) = \overline{\Lambda}(\Delta)x$, where $\overline{\Lambda}(\Delta) \in V[[\Delta]]\Delta$.

3 Definition of $\overline{l}$

Define the function $\overline{l} : xW[[x]] \to xV[[x]]$ by formula

$$\overline{l}(g(x)) = \overline{\Lambda}^{-1}(\Delta)\overline{\lambda}(g(x)). \tag{1}$$

This is an analog of the inverted Artin-Hasse function.

Theorem 1. $\overline{l}(xW[[x]]) \subset xW[[x]]$. *Moreover, $\overline{l}$ is a bijection from $xW[[x]]$ onto $xW[[x]]$.*

Proof. We begin by proving several lemmas.

Lemma 2. *Let $f_1, f_2 \in xW[[x]]$ be such that $\overline{l}(f_1), \overline{l}(f_2) \in W[[x]]$. Then $\overline{l}(f_1 +_{\overline{F}} f_2) = \overline{l}(f_1) + \overline{l}(f_2) \in xW[[x]]$.*

Proof. $\overline{l}(f_1 +_{\overline{F}} f_2) = \overline{\Lambda}^{-1}(\Delta)(\lambda(f_1 +_{\overline{F}} f_2)) = \overline{\Lambda}^{-1}(\Delta)(\lambda(f_1) + \lambda(f_2)) = \overline{l}(f_1) + \overline{l}(f_2)$. □

Lemma 3. *Suppose that $f \in xW[[x]]$ is such that $\overline{l}(f) \in xW[[x]]$ and $a \in W$ is such that $a = \theta t^n$, where $\theta \in R_O$. Suppose that $m \in N$ and $g(x) = f(ax^m)$. Then $\overline{l}(g(x)) = \overline{l}(f(x))(ax^m) \in xW[[x]]$.*

Proof. Denote $y = ax^m$. We know that $\sigma(\theta) = \theta^p$ and $\sigma(t) = t^p$, so $\Delta(y) = y^p$. So Δ acts on $f(x)$ just as on $f(y)$. This implies the statement of the lemma. □

By definition $\overline{\lambda}(x) = \overline{\Lambda}(\Delta)x$, so $\overline{l}(x) = \overline{\Lambda}(\Delta)^{-1}\overline{\lambda}(x) = \overline{\Lambda}(\Delta)^{-1}\overline{\Lambda}(\Delta)x = x$. So $\overline{l}(x) = x \in W[[x]]$. From the previous lemma we get $\overline{l}(x^n) = x^n \in W[[x]]$.
Then, $\overline{l}([p^m]x^n) = \overline{l}(x^n +_{\overline{F}} x^n +_{\overline{F}} + \cdots +_{\overline{F}} x^n) = p^m\overline{l}(x^n) = p^m x^n$.
Now, suppose that $f(x) = a_k x^k + a_{k+1}x^{k+1} + \dots$ is an element of $x^k W[[x]]$, where $k > 0$. Then a_k (as any element of W) can be written in the form

$$a_k = \sum_{i=0}^{\infty} p^i \sum_{j=0}^{\infty} t^j r_{i,j}, \tag{2}$$

where $r_{i,j} \in R_O$. Define g_k by the formula:

$$g_k(x) = \overline{\sum_{i=0}^{\infty}}[p^i]\overline{\sum_{j=0}^{\infty}} t^j r_{i,j} x^k, \tag{3}$$

where overlined sums are taken using the $\overline{F}$ formal group law.

Lemma 4. *$g_k(x)$ is a convergent series and $g_k(x) \in x^k W[[x]]$.*

Proof. The second part of the lemma follows from the definition, so we will prove only the first part.
Denote $U_l = \overline{\sum_{i=0}^{l}}[p^i]\overline{\sum_{j=0}^{\infty}} t^j r_{i,j} x^k$. We will increase l and look at the coefficient at x^t. Consider the cooefficients of x^1, x^2 $\dots x^l$ of $\overline{\lambda}^{-1}$ and $\overline{\lambda}$. Denote by Q the maximal degree of p in the denominators of these coefficients. Then $U_l = \overline{\lambda}(\sum_{i=0}^{l} p^i \sum_{j=0}^{\infty} \overline{\lambda}^{-1}(t^j r_{i,j} x^k))$. If we add a term with $l \geq 2Q$, the coefficient will increase by a multiple of p^{l-2Q}. Therefore, the coefficient will converge. ☐

We now prove Theorem 1.
We have $f_{k+1} = f -_{\overline{F}} g_k \in x^{k+1}W[[x]]$, since we have constructed g_k so that it starts with the same term as f. Then we can construct $g_{k+1} \in x^{k+1}W[[x]]$ and so on. So any $f \in xW[[x]]$ can be written as $f(x) = \overline{\sum_{k=1}^{\infty}} g_k$. (As above, the line over the sum symbol denotes that we use $\overline{F}$ group law.)
Then

$$\overline{l}(f(x)) = \overline{l}\left(\overline{\sum_{k=1}^{\infty}} g_k(x)\right) = \sum_{k=1}^{\infty} \overline{l}(g_k(x)) = \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p^j \overline{l}(r_{i,j,k} t^i x^k) = \\ = \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p^j r_{i,j,k} t^i x^k \in xW[[x]]. \tag{4}$$

This proves the first part of the theorem. The bijectivity of $\overline{l}$ follows from formulas (3) and (4): $\overline{l}$ is a bijection between $xW[[x]]$ and all $\{(r_{i,j,k})\}$. ☐

Now we can introduce the function $\overline{E} : xW[[x]] \to xW[[x]]$ as $\overline{l}^{-1}$. Immediately from the definition we get the formula

$$\overline{E}(f(x)) = \overline{\lambda}^{-1}(\overline{\Lambda}(\Delta)f(x)). \tag{5}$$

Theorem 2. *$\overline{l}$ is an isomorphism $xW[[x]]_{\overline{F}}$ into $xW[[x]]_+$ and $\overline{E}$ is its inverse.*

Proof. From Lemma (2) we get that $\overline{l}$ is a homomorphism. And from Theorem (1) it is a bijection. □

4 Definition of E

Suppose that $\phi : O_K \to W$ is any map such that the following statements hold:

1. $\phi\psi = id_{O_K}$.
2. $\phi(x+y) = \phi(x) + \phi(y)$.
3. $\phi|_{O_N} = id_{O_N}$.

Such a map exists, because we can take $\phi = \phi_0$. Then we can extend ϕ on $W[[x]]$ by taking $\phi(ax^n) = \phi(a)\phi(x^n)$.
Now define the map $E : xO_K[[x]] \to xO_K[[x]]$ by the following formula:

$$E(f(x)) = \psi(\overline{E}(\phi(x))). \tag{6}$$

Note that it is possible that we will get different E, if we take different ϕ.

Lemma 5. *Suppose that $f(x) \in xO_K[[x]]$, and $f(x) - a_k x^k \in x^{k+1}O_K[[x]]$. Then $E(f(x)) - a_k x^k \in x^{k+1}O_K[[x]]$.*

Proof. Denote $\phi(f(x))$ by $h(x)$. Then $h(x) - \phi(a_k)x^k \in x^{k+1}W[[x]]$. Now, from formula (4) we derive $\overline{E}(h(x)) - \psi(a_k)x^k \in x^{k+1}W[[x]]$. Therefore, $\psi(\overline{E}(h(x)))$ starts with $\psi(\phi(a_k))x^k = a_k x^k$. □

Theorem 3. *E is an isomorphism $xO_K[[x]]_+$ into $xO_K[[x]]_F$.*

Proof. 1. E is a homomorphism: $E(f_1 + f_2) = \psi(\overline{E}(\phi(f_1 + f_2))) = \psi(\overline{E}(\phi(f_1) + \phi(f_2))) = \psi(\overline{E}(\phi(f_1)) +_{\overline{F}} \overline{E}(\phi(f_2))) = \psi(\overline{E}(\phi(f_1))) +_F \psi(\overline{E}(\phi(f_2))) = E(f_1) +_F E(f_2)$.

2. Injectivity is obvious from Lemma 5.

3. It remains to prove that E is a surjection. Let $f(x) \in x^k O_K[[x]]$. Suppose that it starts with $a_k x^k$. Then we can take $g_k(x) = a_k x^k$: $E(g_k(x))$ then also starts with $a_k x^k$. Then denote $f_{k+1}(x) = f(x) -_F g_k(x)$. It starts with $b_{k+1}x^{k+1}$, so we can construct g_{k+1} and so on. Then put $g(x) = \sum g_k(x)$, so $E(g(x)) = \overline{\sum} g_k(x) = f(x)$. □

Moreover, we can define an isomorphism $xO_K[[x]]_+$ into $xO_K[[x]]_{F_2}$ by the formula

$$E(\hat{g}(x)) = f(E(g(x))) \tag{7}$$

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