

# On a Unimodality Conjecture in Matroid Theory

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A certain unimodal conjecture in matroid theory states the number of rank- $r$  matroids on a set of size  $n$  is unimodal in  $r$  and attains its maximum at  $r = \lfloor n/2 \rfloor$ . We show that this conjecture holds up to  $r = 3$  by constructing a map from a class of rank-2 matroids into the class of loopless rank-3 matroids. Similar inequalities are proven for the number of non-isomorphic loopless matroids, loopless matroids and matroids.

**Keywords:** Matroid Theory, Unimodal Conjecture, Rank-2 matroids, Rank-3 matroids

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## 1 Introduction

Let us first recall some elementary definitions (further details may be found in Oxley (1), Welsh (2) and the excellent appendix of White (3)). Let  $S_n$  be a finite set of size  $n$ . A matroid  $M$  on the ground set  $S_n$  is a collection of subsets  $I(M)$  of  $S_n$  satisfying

- $I(M) \neq \emptyset$ ,
- if  $X \in I(M)$  and  $Y \subseteq X$  then  $Y \in I(M)$ ,
- if  $X, Y \in I(M)$  with  $|X| = |Y| + 1$ , then there exists  $x \in X \setminus Y$  such that  $Y \cup \{x\} \in I(M)$ .

Sets in  $I(M)$  are called *independent sets*. The *rank* of a set  $X \subseteq S_n$ , denoted  $r(X)$ , is the size of the largest independent set which it contains. The rank of the matroid  $r(M) := r(S_n)$ . A set  $X$  is *closed* (or termed a *flat*) if  $r(X \cup \{x\}) = r(X) + 1$  for all  $x \in S_n \setminus X$ . We denote by  $\mathcal{F}(M)$  the closed sets of  $M$ . The *loops* of  $M$  are the elements of the rank-0 flat. Also note that  $x \in S_n$  is a loop if it is not contained in any of the independent sets of  $M$ .

A certain unimodal conjecture in matroid theory states that the sequence of the number of non-isomorphic rank- $r$  matroids on  $S_n$ ,  $\{f_r(n) : 1 \leq r \leq n\}$ , is unimodal in  $r$  and attains its maximum at  $r = \lfloor n/2 \rfloor$  (see Oxley (1) or Welsh (4) p.300). It is easily seen that  $f_1(n) \leq f_2(n)$  holds since  $f_1(n) = n$  and  $f_2(n) = p(1) + \dots + p(n) - n$ , where  $p(n)$  is the number of integer partitions of  $n$ . The step between rank-2 and rank-3 is not as clear since the exact value of  $f_3(n)$  remains unknown. We show, through construction of a map between a class of rank-2 matroids and loopless rank-3 matroids and known values of these numbers from the On-line Encyclopedia of Integer Sequences, that this unimodal conjecture holds for

these rank-2 versus rank-3 matroids. Furthermore, we show the corresponding inequalities hold for the number of rank-2, 3 non-isomorphic loopless matroids,  $g_2(n) \leq g_3(n)$ , loopless matroids,  $c_2(n) \leq c_3(n)$ , and matroids,  $m_2(n) < m_3(n)$ .

Let  $b_i(n)$  be the number of partitions of the set  $S_n$  into  $i$  parts and  $b(n)$  be the  $n^{\text{th}}$  Bell number. Let  $p_i(n)$  be the number of partitions of the integer  $n$  into  $i$  parts. The number of rank-2 matroids can be enumerated through considering the points and lines of the associated geometry. We have  $c_2(n) = b(n) - 1$ ,  $g_2(n) = p(n) - 1$  and  $m_2(n) = b(n+1) - 2^n$  (for proofs see Dukes (5)). The main results of this paper are given in Theorems 2.5, 2.6, 2.11 and 2.12.

## 2 Mapping rank-2 to rank-3 matroids

Let  $\mathcal{M}_r(n)$  be the collection of rank- $r$  matroids on  $S_n$ . Let  $\mathcal{A}_r(n)$  be the collection of rank- $r$  matroids on  $S_n$  with at least one loop and  $\mathcal{B}_r(n) := \mathcal{M}_r(n) \setminus \mathcal{A}_r(n)$ . We define the map  $\sigma : \mathcal{A}_2(n) \rightarrow \mathcal{B}_3(n)$  as follows: given  $M \in \mathcal{A}_2(n)$  with loops  $F_0$  and rank-1 flats  $\mathcal{F}_1(M) = \{F_0 \cup F_1, \dots, F_0 \cup F_m\}$  define  $M' = \sigma(M)$  as:

$$\begin{aligned} \mathcal{F}_0(M') &:= \{\emptyset\} \\ \mathcal{F}_1(M') &:= \{F_0, F_1, \dots, F_m\} \\ \mathcal{F}_2(M') &:= \{F_0 \cup F_i \mid 1 \leq i \leq m\} \cup \{F_1 \cup \dots \cup F_m\} \\ \mathcal{F}_3(M') &:= \{S_n\}. \end{aligned}$$

It is easily checked that these collections of flats satisfy the axioms for a loopless rank-3 matroid. For  $M \in \mathcal{M}_r(n)$ , let us write  $d(M)$  for the number of rank-1 flats of  $M$  (which we will refer to as the *degree* of  $M$ ). Let us mention that for any loopless matroid  $M$ , the rank-1 flats of  $M$  partition the ground set. Similarly, for any matroid, the rank-1 flats partition the ground set less the set of loops. Also note that in the collection  $\mathcal{F}_2(M')$ , there are precisely  $d(M)$  sets containing  $F_0$ , 2 sets containing  $F_i$  (for any  $1 \leq i \leq d(M)$ ) and one set containing  $F_i \cup F_j$  (for all  $0 \leq i < j \leq d(M)$ ).

The following lemma shows that to each rank-2 matroid with at least one loop, there corresponds a rank-3 loopless matroid (although not necessarily unique). The following lemma classifies those matroids which map to a unique loopless matroid in  $\mathcal{B}_3(n)$  and those which do not.

**Lemma 2.1** *Let  $M_1, M_2 \in \mathcal{A}_2(n)$  be such that  $\mathcal{F}_0(M_1) = \{F_0^{(1)}\}$ ,  $\mathcal{F}_0(M_2) = \{F_0^{(2)}\}$ ,  $\mathcal{F}_1(M_1) = \{F_0^{(1)} \cup F_1^{(1)}, \dots, F_0^{(1)} \cup F_{d(M_1)}^{(1)}\}$  and  $\mathcal{F}_1(M_2) = \{F_0^{(2)} \cup F_1^{(2)}, \dots, F_0^{(2)} \cup F_{d(M_2)}^{(2)}\}$ . Then  $\sigma(M_1) = \sigma(M_2)$  if and only if  $d(M_1) = d(M_2) = 2$  and*

$$\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}.$$

**Proof:** ONLY IF: Let  $M_1, M_2 \in \mathcal{A}_2(n)$  be such that  $M_1 \neq M_2$  and  $\sigma(M_1) = \sigma(M_2)$ . Let  $M'_1 := \sigma(M_1)$  and  $M'_2 := \sigma(M_2)$ . Then we must have  $\mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2)$  and  $\mathcal{F}_2(M'_1) = \mathcal{F}_2(M'_2)$ . Now  $\mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2) \Rightarrow d(M_1) = d(M_2)$  and  $\{F_i^{(1)}\}_{i=0}^{d(M_1)} = \{F_i^{(2)}\}_{i=0}^{d(M_2)}$ . If  $d(M_1) > 2$  then we must have  $F_0^{(1)} = F_0^{(2)}$  which would imply  $M_1 = M_2$ . Hence  $d(M_1) = 2 = d(M_2)$ . This gives  $\mathcal{F}_2(M'_1) = \{F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_2^{(1)}, F_1^{(1)} \cup F_2^{(1)}\} = \mathcal{F}_2(M'_2)$  only if  $\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}$ .

IF: This is trivial as  $\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}$  gives  $\mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2)$  and

$$\begin{aligned}\mathcal{F}_2(M'_1) &:= \{F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}\} \\ &= \{F_0^{(2)} \cup F_1^{(2)}, F_0^{(2)} \cup F_1^{(2)}, F_0^{(2)} \cup F_1^{(2)}\} =: \mathcal{F}_2(M'_2).\end{aligned}$$

□

Thus it is seen for each matroid  $M \in \sigma(\mathcal{A}_2(n))$  such that  $d(M) = 2$ , there are precisely three different matroids  $M_1, M_2, M_3 \in \mathcal{A}_2(n)$  such that  $\sigma(M_1) = \sigma(M_2) = \sigma(M_3) = M$ .

**Lemma 2.2** For all  $n \geq 3$ ,  $c_3(n) \geq b(n+1) - b(n) - 3^{n-1}$ .

**Proof:** We show that the number of unique matroids in the image of  $\mathcal{A}_2(n)$  under  $\sigma$  is given by  $b(n+1) - b(n) - 3^{n-1}$ , thereby lower-bounding  $c_3(n)$ . In the enumeration below, we divide the matroids to be counted in the image into two classes, those matroids  $M$  with  $d(M) = 2$  and those with  $d(M) > 2$ . The former class projects different matroids to the same matroid in  $\mathcal{B}_3(n)$  and through the use of the previous lemma we take care of this over-counting, hence

$$\begin{aligned}\#\{\sigma(M) | M \in \mathcal{A}_2(n)\} &= \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} + \sum_{i=3}^n \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} \\ &= \frac{1}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} + \sum_{i=3}^n \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} \\ &= \sum_{i=2}^n \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} - \frac{2}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} \\ &= \#\mathcal{A}_2(n) - \frac{2}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} \\ &= b(n+1) - 2^n - (b(n) - 1) - \frac{2}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\}.\end{aligned}$$

Note that  $\#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} = \sum_{l=2}^{n-1} \binom{n}{l} b_2(l)$  and  $b_2(l) = \frac{1}{2} \sum_{j=1}^{l-1} \binom{l}{j} = 2^{l-1} - 1$ , giving:

$$\begin{aligned}\#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} &= \sum_{l=2}^{n-1} \binom{n}{l} (2^{l-1} - 1) \\ &= \frac{1}{2} (3^n - 2^n - 2n - 1) - (2^n - n - 2) \\ &= \frac{3}{2} (3^{n-1} - 2^n + 1).\end{aligned}$$

Thus

$$\begin{aligned}\#\{\sigma(M) | M \in \mathcal{A}_2(n)\} &= b(n+1) - 2^n - b(n) + 1 - \frac{2}{3} \frac{3}{2} (3^{n-1} - 2^n + 1) \\ &= b(n+1) - b(n) - 3^{n-1}.\end{aligned}$$

□

The corresponding inequality for the number of non-isomorphic loopless matroids is proved in Lemma 2.3. We do this in a similar manner as before, by showing that each rank-2 matroid (which is not a loopless matroid) of degree greater than 3 corresponds uniquely to a rank-3 loopless matroid.

**Lemma 2.3** For all  $n \geq 4$ ,  $g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$ .

**Proof:** We show the number of non-isomorphic matroids in the image of  $\mathcal{A}_2(n)$  under  $\sigma$  is given by  $\sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$  which lower bounds  $g_3(n)$ .

Let us identify  $\mathcal{A}_2^*(n) \subseteq \mathcal{A}_2(n)$  by placing an ordering on the elements of  $S_n = \{x_1, \dots, x_n\}$ . Given  $M \in \mathcal{A}_2(n)$  with  $d(M) = m$ , loops  $F_0$  and rank-1 flats  $\{F_0 \cup F_1, \dots, F_0 \cup F_m\}$ , let  $M \in \mathcal{A}_2^*(n)$  if and only if  $F_0 = \{x_1, \dots, x_{|F_0|}\}$ ,  $F_1$  contains the next  $|F_1|$  elements of  $S_n$ , i.e.  $\{x_{|F_0|+1}, \dots, x_{|F_0|+|F_1|}\}$  and so forth. Define

$$\mathcal{T}(n) := \{M \in \mathcal{A}_2^*(n) \mid d(M) = 2 \text{ and } |F_0| \leq |F_1| \leq |F_2|\}$$

and for  $3 \leq i \leq n-1$ ,  $3 \leq j \leq i$ , define

$$\Omega_{i,j}(n) := \{M \in \mathcal{A}_2^*(n) \mid d(M) = j, |F_0| = n-i \text{ and } |F_1| \leq \dots \leq |F_j|\}.$$

Let us now write

$$\mathcal{A}_2^{**}(n) := \mathcal{T}(n) \cup \bigcup_{i=3}^{n-1} \bigcup_{j=3}^i \Omega_{i,j}(n) \subseteq \mathcal{A}_2^*(n).$$

It is obvious that no two matroids in  $\mathcal{T}(n)$  are isomorphic to one-another. Similarly with  $\Omega_{k,l}(n)$ . We have simply reduced our class of matroids from  $\mathcal{A}_2(n)$  to  $\mathcal{A}_2^{**}(n)$  in the same manner as one moves from the set of partitions of a finite set of size  $n$  to the set of integer partitions of  $n$ .

The unions in the definition of  $\mathcal{A}_2^{**}(n)$  are strictly disjoint and no isomorphisms may occur between matroids in different classes or matroids in the same class. The same is true of the image of  $\mathcal{A}_2^{**}(n)$  under the map  $\sigma$ . We may directly enumerate the number of non-isomorphic matroids in  $\mathcal{B}_3(n)$  in the image of  $\mathcal{A}_2^{**}(n)$  under  $\sigma$  as

$$p_3(n) + \sum_{i=3}^{n-1} \sum_{j=3}^i p_j(i).$$

The rightmost term is bounded below;

$$\begin{aligned}
\sum_{i=3}^{n-1} \sum_{j=3}^i p_j(i) &= \sum_{i=2}^{n-1} \{p(i) - p_1(i) - p_2(i)\} \\
&= \sum_{i=3}^{n-1} \{p(i) - 1 - \lfloor i/2 \rfloor\} \\
&= -(n-3) + \sum_{i=3}^{n-1} p(i) - \sum_{i=3}^{n-1} \lfloor i/2 \rfloor \\
&= \begin{cases} \sum_{i=1}^{n-1} p(i) - \frac{n(n+2)}{4} + 1, & n \text{ even,} \\ \sum_{i=1}^{n-1} p(i) - \frac{(n+1)^2}{4} + 1, & n \text{ odd,} \end{cases} \\
&\geq \sum_{i=1}^{n-1} p(i) - \frac{(n+1)^2}{4},
\end{aligned}$$

for all  $n \geq 2$ . As for  $p_3(n)$ , from Hall (6) [p.32], we have

$$\begin{aligned}
p_3(n) &= \begin{cases} \lfloor n^2/12 \rfloor, & \text{for } n \not\equiv 3 \pmod{6}, \\ \lceil n^2/12 \rceil, & \text{for } n \equiv 3 \pmod{6}, \end{cases} \\
&\geq \frac{n^2}{12} - 1,
\end{aligned}$$

and so

$$\begin{aligned}
g_3(n) &\geq \sum_{i=1}^{n-1} p(i) + \frac{n^2}{12} - \frac{(n+1)^2}{4} - 1 \\
&= \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1.
\end{aligned}$$

□

## 2.1 Matroids

The following lemma is needed in order to support the theorem which follows it.

**Lemma 2.4** For all  $n \geq 1$ ,  $b(n+1) - 2^n \geq 2^n - (1+n)$ .

**Proof:** We have that  $b(i) \geq 2$  for all  $i \geq 2$ . Since  $n \geq 2$ , it follows that

$$\begin{aligned}
b(n+1) - 2^n &= \sum_{i=0}^n \binom{n}{i} (b(i) - 1) \\
&\geq \sum_{i=2}^n \binom{n}{i} 1 \\
&= 2^n - (1+n).
\end{aligned}$$

□

**Theorem 2.5** For all  $n > 4$ ,  $c_3(n) \geq c_2(n)$ .

**Proof:** From Lemma 2.2 we have that  $c_3(n) \geq b(n+1) - b(n) - 3^{n-1}$ . We know  $c_2(n) = b(n) - 1$ . It suffices to show that

$$b(n+1) - b(n) - 3^{n-1} \geq b(n) - 1$$

for all  $n > 4$ . Let us look at the value  $b(n+1) - 2b(n) - 3^{n-1} + 2^{n-1}$ :

$$\begin{aligned} b(n+1) - 2b(n) - 3^{n-1} + 2^{n-1} &= \sum_{i=0}^{n-1} \binom{n}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} 2^i + \sum_{i=0}^{n-1} \binom{n-1}{i} \\ &= \sum_{i=0}^{n-2} \binom{n-1}{i} (b(i+1) - 2^i) \end{aligned}$$

and using Lemma 2.4,

$$\begin{aligned} &\geq \sum_{i=1}^{n-2} \binom{n-1}{i} (2^i - (1+i)) \\ &= 3^{n-1} - 2^n + 1 - \sum_{i=1}^{n-2} \binom{n-1}{i} i \\ &= 3^{n-1} - (n+3)2^{n-2} + n. \end{aligned}$$

The problem has been reduced to showing  $3^{n-1} - (n+3)2^{n-2} + n \geq 2^{n-1} - 1$  for all  $n \geq 5$ , which is easily shown by induction.  $\square$

We now show the number of rank-3 matroids dominates the number of rank-2 matroids by using two things: the first is the result proved previously, that the number of rank-3 loopless matroids is at least as large as the number of rank-2 loopless matroids; the second is the first few known values of the numbers  $c_2(n)$  and  $c_3(n)$ . The latter knowledge makes the inequality strict.

**Theorem 2.6** For all  $n \geq 5$ ,  $m_3(n) \geq m_2(n)$ .

**Proof:** The number of rank- $r$  matroids on  $S_n$  is related to the number of loopless matroids on  $S_n$  by

$$m_r(n) = \sum_{i=r}^n \binom{n}{i} c_r(i).$$

In Theorem 2.5 we showed that  $c_3(n) \geq c_2(n)$  for all  $n \geq 5$ . Replacing  $r = 3$  in the above expression and using the first few values of  $c_3(n)$  (taken from row 3, table A058710, of Sloane (7)),

$$\begin{aligned} m_3(n) &= \sum_{i=3}^n \binom{n}{i} c_3(i) \\ &= 1 \binom{n}{3} + 11 \binom{n}{4} + 106 \binom{n}{5} + 1232 \binom{n}{6} + \sum_{i=7}^n \binom{n}{i} c_3(i) \\ &\geq 1 \binom{n}{3} + 11 \binom{n}{4} + 106 \binom{n}{5} + 1232 \binom{n}{6} + \sum_{i=7}^n \binom{n}{i} c_2(i) \\ &= 830 \binom{n}{6} + 75 \binom{n}{5} - 3 \binom{n}{4} - 3 \binom{n}{3} - \binom{n}{2} + \sum_{i=2}^n \binom{n}{i} c_2(i) \end{aligned}$$

A simple check shows that  $830\binom{n}{6} + 75\binom{n}{5} - 3\binom{n}{4} - 3\binom{n}{3} - \binom{n}{2}$  is greater than zero and increasing for all  $n \geq 7$ . From Table 1 (see Appendix), the result is also seen to hold for  $n = 5, 6$ . Equality holds only for  $n = 5$ , for all other values of  $n$  the inequality is strict.  $\square$

## 2.2 Non-isomorphic matroids

Proving the corresponding inequalities for the non-isomorphic numbers is more difficult. We first prove several lemmas related to the numbers  $p(n)$  which we will need in the proofs of the two remaining theorems.

**Lemma 2.7** For all  $n \geq 1$ ,  $p(n+1) \geq p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor$ .

**Proof:** The number of partitions of the integer  $n+1$  whose first part contains the integer 1 is precisely  $p(n)$ . The number beginning with  $i$ , for any  $2 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor$  is at least 1 since we can have the partition  $n+1 = i + (n+1-i)$ . Also, the number  $n+1$  is a partition by itself, hence,

$$\begin{aligned} p(n+1) &\geq p(n) + \left( \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \right) + 1 \\ &= p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

$\square$

**Lemma 2.8** For all  $n \geq 1$ ,  $p(n) \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \geq \frac{n^2+3}{4}$ .

**Proof:** From Lemma 2.7 we have

$$p(n+1) \geq p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor$$

for all  $n \geq 1$ . Applying this lemma recursively gives

$$\begin{aligned} p(n) &\geq p(n-1) + \left\lfloor \frac{n}{2} \right\rfloor \\ &\geq p(n-2) + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &\vdots \\ &\geq p(1) + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &\geq 1 + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned} \tag{1}$$

Now we wish to evaluate the sum  $\sum_{i=2}^n \left\lfloor \frac{i}{2} \right\rfloor$ . Let  $n = 2m+1$  for some  $m \geq 1$ , then

$$\begin{aligned} \sum_{i=2}^n \left\lfloor \frac{i}{2} \right\rfloor &= \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor \\ &= \sum_{i=1}^m \left\lfloor \frac{2i}{2} \right\rfloor + \left\lfloor \frac{2i+1}{2} \right\rfloor \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$

For the  $n = 2m$  case with  $m \geq 1$ , we simply remove the last term in the previous expression, thus

$$\begin{aligned} \sum_{i=2}^n \left\lfloor \frac{i}{2} \right\rfloor &= \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor - \left\lfloor \frac{2m+1}{2} \right\rfloor \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$

Continuing to the inequality in Equation 1 above,

$$\begin{aligned} p(n) &\geq 1 + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &= 1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil, \end{aligned}$$

for all  $n \geq 1$ . If  $n$  is even, then  $\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \frac{n^2}{4}$ . If  $n$  is odd, then  $\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \frac{n-1}{2} \frac{n+1}{2}$ . In either case,  $1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \geq 1 + \frac{n^2-1}{4}$ .  $\square$

**Lemma 2.9** For all  $n \geq 5$ ,  $p(n+1) < 2p(n) - \frac{n+2}{3}$ .

**Proof:** Let  $x_1 + x_2 + \cdots + x_k = n+1$  be a partition of  $n+1$  with  $1 \leq x_1 < \cdots < x_k$ . There are precisely  $p(n)$  partitions with  $x_1 = 1$ , since  $x_2 + \cdots + x_k = n+1-1$ .

For all those partitions with  $x_1 \geq 2$ , we see that reducing  $x_1$  by 1 will yield a partition of  $n$ . Thus an upper bound on the number beginning with  $x_2 \geq 2$  is  $p(n)$ . For all partitions starting with  $x_1 = 2$ , we see that  $x_2 \neq 1$ , thus we may remove all those sequences with  $x_2 = 1 \leq x_3 \leq \cdots \leq x_k$  such that  $2 + 1 + x_3 + \cdots + x_k = n+1$ . Reformulated, this means all those partitions with  $x_3 + \cdots + x_k = n-2$  and  $1 \leq x_3 \leq \cdots \leq x_k$  of which there are  $p(n-2)$ .

Thus we see that  $p(n+1) < p(n) + p(n) - p(n-2) = 2p(n) - p(n-2)$ . From lemma 2.8 we know that for  $n \geq 3$ ,

$$p(n-2) \geq \frac{(n-2)^2 + 3}{4} = \frac{n^2 - 4n + 7}{4}.$$

Now, we see that the simple inequality  $(3n-13)(n-1) \geq 0$  holds for all  $n \geq \frac{13}{3}$ , i.e.  $\frac{(n^2-4n+7)}{4} \geq \frac{(n+2)}{3}$ . From above, this gives

$$\begin{aligned} p(n+1) &< 2p(n) - p(n-2) \\ &\leq 2p(n) - \frac{(n^2 - 4n + 7)}{4} \\ &\leq 2p(n) - \frac{(n+2)}{3}, \end{aligned}$$

for all  $n \geq 5$  and we are done. A check of the first few values of  $p(n)$  shows the stated inequality to hold for all  $n \geq 2$ .  $\square$

**Lemma 2.10** For all  $n \geq 7$ ,  $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3)$ .



**Proof:** By simple induction. The result is true for  $n = 7$  since  $p(1) + p(2) + \dots + p(6) = 30$  and  $p(7) + \frac{1}{12}(2(7)^2 + 6(7) + 3) < 27$ . Suppose it to be true for some  $n = m \geq 7$ , then:

$$\begin{aligned} \sum_{i=1}^m p(i) &= p(m) + \sum_{i=1}^{m-1} p(i) \\ &\geq p(m) + p(m) + \frac{1}{12}(2m^2 + 6m + 3) \\ &= 2p(m) + \frac{1}{12}(2m^2 + 6m + 3) \end{aligned}$$

and using Lemma 2.9,

$$\begin{aligned} &> p(m+1) + \frac{m+2}{3} + \frac{1}{12}(2m^2 + 6m + 3) \\ &= p(m+1) + \frac{1}{12}(2(m+1)^2 + 6(m+1) + 3). \end{aligned}$$

□

**Theorem 2.11** For all  $n \geq 5$ ,  $g_3(n) \geq g_2(n)$ .

**Proof:** We have that  $g_2(n) = p(n) - 1$ . Also, we know from Theorem 2.6 that  $g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$ . From Lemma 2.10, we have  $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3)$  for all  $n \geq 7$ . Combining these facts gives

$$\begin{aligned} g_3(n) &\geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1 \\ &\geq p(n) - 1 \end{aligned}$$

which is  $g_2(n)$ . From Table 1, the result is also seen to hold for  $n = 5, 6$ .

□

**Theorem 2.12** For all  $n \geq 5$ ,  $f_3(n) \geq f_2(n)$ .

**Proof:** The number of non-isomorphic rank-3 matroids on  $S_n$  in terms of loopless non-isomorphic rank-3 matroids is given through the relation  $f_3(n) = \sum_{i=3}^n g_3(i)$  for all  $n \geq 3$ . The value  $f_2(n) = p(1) + p(2) + \dots + p(n) - n$  for all  $n \geq 2$ . From Theorem 2.11 we have  $g_3(n) \geq g_2(n)$  for all  $n \geq 7$ . Applying the above expression for  $f_3(n)$ , using the known value for  $g_3(n)$  (from Sloane (7), row 3 of A058716) and assuming  $n \geq 7$ ,

$$\begin{aligned} f_3(n) &= 38 + \sum_{i=7}^n g_3(i) \\ &> 23 + \sum_{i=7}^n g_2(i) \\ &= \sum_{i=2}^n g_2(i) \end{aligned}$$

which is precisely  $f_2(n)$ . From Table 1, the result is seen to hold for  $n = 5, 6$ . Note that the above inequality is strict for  $n \geq 6$  and equality holds only for  $n = 5$ .

□

Note that, by duality, an immediate Corollary of Theorems 2.6 and 2.12 is the following.

**Corollary 2.13** For all  $n \geq 6$ ,

$$\begin{aligned} f_n(n) &\leq f_{n-1}(n) \leq f_{n-2}(n) \leq f_{n-3}(n) \\ m_n(n) &\leq m_{n-1}(n) \leq m_{n-2}(n) \leq m_{n-3}(n). \end{aligned}$$

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## Appendix

| $n$      | 2 | 3 | 4  | 5   | 6    | 7     | 8       | OLEIS Number | Row |
|----------|---|---|----|-----|------|-------|---------|--------------|-----|
| $g_2(n)$ | 1 | 2 | 4  | 6   | 10   | 14    | 21      | A058716      | 2   |
| $g_3(n)$ |   | 1 | 3  | 9   | 25   | 70    | 217     | A058716      | 3   |
| $f_2(n)$ | 1 | 3 | 7  | 13  | 23   | 37    | 58      | A053534      | 2   |
| $f_3(n)$ |   | 1 | 4  | 13  | 38   | 108   | 325     | A053534      | 3   |
| $c_2(n)$ | 1 | 4 | 14 | 31  | 202  | 876   | 4139    | A058710      | 2   |
| $c_3(n)$ |   | 1 | 11 | 106 | 1232 | 22172 | 803583  | A058710      | 3   |
| $m_2(n)$ | 1 | 7 | 36 | 171 | 813  | 4012  | 20891   | A058669      | 2   |
| $m_3(n)$ |   | 1 | 15 | 171 | 2053 | 33442 | 1022217 | A058669      | 3   |

Table 1: Known values for the number of rank-2 and rank-3 matroids taken from Sloane (7).