

Infinite families of accelerated series for some classical constants by the Markov-WZ Method

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In this article we show the Markov-WZ Method in action as it finds rapidly converging series representations for a given hypergeometric series. We demonstrate the method by finding new representations for $\log(2)$, $\zeta(2)$ and $\zeta(3)$.

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A function $H(x, z)$, in the integer variables x and z , is called **hypergeometric** if $H(x+1, z)/H(x, z)$ and $H(x, z+1)/H(x, z)$ are rational functions of x and z . In this article we consider only those hypergeometric functions which are a ratio of products of factorials (we call such hypergeometric functions **pure-hypergeometric**). A **P-recursive** function is a function that satisfies a linear recurrence relation with polynomial coefficients. A pair (H, G) is called a Markov-WZ pair (MWZ-pair for short) if there exists a polynomial $P(x, z)$ in z of the form

$$P(x, z) = a_0(x) + a_1(x)z + \cdots + a_L(x)z^L, \quad (\text{POLY})$$

for some non-negative integer L , and P-recursive functions $a_0(x), \dots, a_L(x)$ such that

$$H(x+1, z)P(x+1, z) - H(x, z)P(x, z) = G(x, z+1) - G(x, z) \quad . \quad (\text{Markov-WZ})$$

We call $G(x, z)$ an MWZ mate of $H(x, z)$. We also require that the $a_i(x)$'s satisfy the initial conditions

$$a_0(0) = 1, a_i(0) = 0, \text{ for } 1 \leq i \leq L.$$

First we will show that given a hypergeometric function $H(x, z)$, there always exists a polynomial with minimum degree that satisfies (Markov-WZ).

1 Existence of MWZ-pair

In this section, $\deg(a)$ stands for the degree of a as a polynomial in z .

Theorem 1. *Given a hypergeometric term $H(x, z)$, there exist a non-negative integer L and a polynomial $P(x, z)$ of the form (POLY) associated with $H(x, z)$ such that $H(x, z)$ has an MWZ mate.*

Proof. We need to show that there exist $L \geq 0$, $a_i(x)$'s, $G(x, z)$, and $P(x, z)$ of the form (POLY) such that $H(x, z)P(x, z)$ and $G(x, z)$ satisfy (Markow-WZ). Moreover $G(x, z)$ has the form $G(x, z) = R(x, z)F(x, z)$, where $R(x, z)$ is a ratio of two P-recursive functions in (x, z) .

Write

$$H(x+1, z)P(x+1, z) - H(x, z)P(x, z) = POL(z) \cdot \bar{H}(x, z),$$

where

$$POL(z) := A(z) \sum_{i=0}^L a_i(x+1)z^i - B(z) \sum_{i=0}^L a_i(x)z^i,$$

$$\frac{H(x+1, z)}{H(x, z)} = \frac{A(z)}{B(z)}, \text{ and } \bar{H}(x, z) = \frac{H(x, z)}{B(z)}.$$

Since $\bar{H}(x, z)$ is a hypergeometric function divided by a polynomial, we can write the above expression as

$$H(x+1, z)P(x+1, z) - H(x, z)P(x, z) = \frac{a(z)}{b(z)} \cdot \frac{POL(z+1)}{POL(z)},$$

where

$$\frac{\bar{H}(x+1, z)}{\bar{H}(x, z)} = \frac{a(z)}{b(z)}.$$

Without loss of generality, we may assume that $\gcd(a(z), b(z+h)) = 1$ for $h \geq 0$, otherwise we re-group and incorporate additional factors into the polynomial part, $POL(z)$. Then with $a(z)$, $b(z)$ and $c(z) := POL(z)$ in parametric Gosper's algorithm [MZ], look for a polynomial $X(z)$ that satisfies

$$a(z)X(z+1) - b(z-1)X(z) = c(z). \quad (\text{Gosper})$$

We may consider only those X with

$$\deg(X) = \deg(c) - \max\{\deg(a), \deg(b)\},$$

and the degree of $c(z)$ is easily seen to be

$$\deg(c) = L + \max\{\deg(A), \deg(B)\}.$$

The unknowns are the $\deg(c) - \max\{\deg(a), \deg(b)\} + 1$ coefficients of $X(z)$ and the a_i 's (there are a total of $2(L+1)$ unknowns). Comparing coefficients on both sides of (Gosper) gives $\deg(c) + 1$ linear homogeneous equations. In order to guarantee a non-zero solution, we need

$$\# \text{ of unknowns} - \# \text{ of equations} \geq 1,$$

and this holds if

$$2(L+1) - (\deg(c) + 1) \geq 1 .$$

In particular, if we choose

$$L := \max\{\deg(a), \deg(b)\} ,$$

we are guaranteed to get a non-trivial solution(!). This gives the $P(x, z)$ and the L . $G(x, z)$ is the anti-difference outputted by parametric Gosper [MZ]. $\square$

Theorem 2. Let (H, G) be an MWZ-pair.

(a) If $\lim_{j \rightarrow \infty} G(x, j) = 0 \ \forall x \geq 0$, then

$$\sum_{z=0}^{\infty} H(0, z) = \sum_{x=0}^{\infty} G(x, 0) - \lim_{i \rightarrow \infty} \sum_{z=0}^{\infty} H(i, z) P(i, z) ,$$

whenever both sides converge.

(b) If $\lim_{i \rightarrow \infty} H(i, z) P(i, z) = 0 \ \forall z \geq 0$, then

$$\sum_{z=0}^{\infty} H(0, z) - \lim_{j \rightarrow \infty} \sum_{x=0}^{\infty} G(x, j) = \sum_{x=0}^{\infty} G(x, 0) ,$$

whenever both sides converge.

Proof. **(a)** Let $P(x, z)$ be the polynomial that features in the MWZ-pair $(H(x, z), G(x, z))$ arising from $H(x, z)$.

Then apply theorem 7 [Z] to the 1-form

$$w = H(x, z) P(x, z) \delta z + G(x, z) \delta x , \tag{1}$$

and the region

$$\Omega = \{(x, z) \mid 0 \leq z \leq \infty, 0 \leq x \leq i\} ,$$

with the discrete boundary

$$\{(0, z+1) \rightarrow (0, z) \mid z \geq 0\} \cup \{(x, 0) \rightarrow (x+1, 0) \mid 0 \leq x \leq i\} \cup \{(i, z) \rightarrow (i, z+1) \mid z \geq \infty\} \cup \{(x+1, \infty) \rightarrow (x, \infty) \mid i-1 \leq x \leq 0\} ,$$

and use the initial conditions $a_i(0) = \delta_{i0}$ for $0 \leq i \leq L$.

(b) Replace the region in (a) by

$$\Omega = \{(x, z) \mid 0 \leq x \leq \infty, 0 \leq z \leq j\}$$

with the corresponding discrete boundary in the proof of (a), and apply to (1) together with the initial conditions $a_i(0) = \delta_{i0}$ for $0 \leq i \leq L$. $\square$

Corollary 1. *If the limit in the conclusion of (a) or (b) is zero in addition to the given hypothesis, then*

$$\sum_{z=0}^{\infty} H(0, z) = \sum_{x=0}^{\infty} G(x, 0) .$$

Theorem 3. *Let N_0 be a non-negative integer and (H, G) be an MWZ-pair. Then*

$$\sum_{z=0}^{\infty} H(0, z) = \sum_{x=0}^{\infty} (H(N_0 + x, x)P(N_0 + x, x) + G(N_0 + x, x + 1)) + \sum_{x=0}^{N_0-1} G(x, 0) - \lim_{j \rightarrow \infty} \sum_{x=0}^{\infty} G(x, j) ,$$

whenever both sides converge.

Proof. Let $P(x, z)$ be the polynomial that features in the MWZ-pair $(H(x, z), G(x, z))$ arising from $H(x, z)$. Then the proof follows from theorem 7 [Z] by applying to the 1-form

$$w = H(x, z)P(x, z)\delta z + G(x, z)\delta x ,$$

and the region

$$\Omega = \{(x, z) \mid 0 \leq z \leq \infty, 0 \leq x \leq z + N_0\} ,$$

with the discrete boundary

$$\partial\Omega_{N_0} := \{(0, z + 1) \rightarrow (0, z) \mid z \geq 0\} \cup \{(x, 0) \rightarrow (x + 1, 0) \mid 0 \leq x \leq N_0\} \cup \{(N_0 + x, x) \rightarrow (N_0 + x + 1, x) \rightarrow (N_0 + x + 1, x + 1) \mid x \geq 0\} \cup \{(x + 1, \infty) \rightarrow (x, \infty) \mid x \geq 0\} ,$$

and using the initial conditions $a_i(0) = \delta_{i0}$ for $0 \leq i \leq L$. □

Corollary 2. *Let (H, G) be an MWZ-pair. If $\lim_{j \rightarrow \infty} \sum_{x=0}^{\infty} G(x, j) = 0$, then*

$$\sum_{z=0}^{\infty} H(0, z) = \sum_{x=0}^{\infty} (H(x, x)P(x, x) + G(x, x + 1)) .$$

Proof. Set $N_0 = 0$ in theorem 3, and use the initial conditions $a_i(0) = \delta_{i0}$ for $0 \leq i \leq L$. □

Remark. If $\lim_{j \rightarrow -\infty} G(x, j) = 0 \forall x$ and the hypothesis of theorem 1 (a) holds, then

$$\sum_{z=-\infty}^{\infty} H(x, z)P(x, z) ,$$

has a closed form evaluation (see example 10 below).

In the following examples, we use the Maple package MarkovWZ [MZ] which, for a given $H(x, z)$, outputs the polynomial $P(x, z)$ and the $G(x, z)$.

2 Examples of Accelerating Series

Let $H(a, b) := \frac{(ax+z)!}{(bx+z+1)!}$ in examples 1 through 9.

Example 1. Consider the hypergeometric term $(-1)^z H(0, 1)$, and corresponding to this kernel determine a polynomial $P(x, z)$ in z with a minimum degree such that $((-1)^z H(0, 1), G(x, z))$ is an MWZ-pair. Using the maple package MarkovWZ [MZ], we see that the polynomial is

$$P(x, z) = \frac{x!}{2^x},$$

and the corresponding MWZ mate of $(-1)^z H(0, 1)$ is

$$G(x, z) = \frac{(-1)^z x!}{2^{x+1}} H(0, 1).$$

It is not hard to check that $((-1)^z H(0, 1), G(x, z))$ is indeed a MWZ-pair with the corresponding polynomial $P(x, z) = x!/2^x$.

Applying corollary 2 to the MWZ-pair we get,

$$\log(2) = \frac{3}{2} \sum_{x=0}^{\infty} \frac{(-1)^x x! (x+1)!}{(2x+2)! 2^x} = 2 \operatorname{arcsinh} \left(\frac{\sqrt{2}}{4} \right).$$

Similarly, if we apply corollary 1 to the MWZ-pair, we find

$$\log(2) = \frac{1}{2} \sum_{x=0}^{\infty} \frac{1}{2^x (x+1)}.$$

In the remaining examples, we simply give the hypergeometric term $H(x, z)$, the polynomial $P(x, z)$ that features in the MWZ-pair, the corresponding $G(x, z)$, and then the identities that follow from the application of the corollaries above.

Example 2. Starting with the kernel $(-1)^z H(0, 3)$, we find

$$P(x, z) = \frac{(3x)!}{8^x},$$

and

$$G(x, z) = \frac{32 + 63x^2 + 93x + 22z + 30xz + 4z^2}{8(3x+z+2)(3x+z+3)} P(x, z) (-1)^z H(0, 3).$$

Application of corollary 1 gives

$$\log(2) = \frac{1}{8} \sum_{x=0}^{\infty} \frac{(-1)^x (x+1)! (3x)! (415x^2 + 487x + 134)}{(4x+4)! 8^x},$$

On the other hand if we apply corollary 2, we get

$$\log(2) = \sum_{x=0}^{\infty} \frac{(63x^2 + 93x + 32)}{24(3x+2)(x+1)(3x+1)8^x}.$$

Example 3. By taking the kernel $(-1)^z H(0, 6)$, we find

$$P(x, z) = \frac{(6x)!}{2^{6x}},$$

and

$$G(x, z) = \frac{Q(x, z)P(x, z)}{16(6x+z+2)(6x+z+3)(6x+z+4)(6x+z+5)(6x+z+6)} (-1)^z H(0, 6),$$

where $Q(x, z)$ is a certain polynomial in x and z .

Corollary 2 gives

$$\log(2) = \sum_{x=0}^{\infty} \frac{(-1)^x (6x)! (x+1)! P(x)}{(7x+7)! 64^x},$$

where

$$P(x) := 1648544x^5 + 4584284x^4 + 4905938x^3 + 2511703x^2 + 610829x + 55914,$$

and corollary 1 gives

$$\log(2) = \sum_{x=0}^{\infty} \frac{40824x^5 + 129924x^4 + 158814x^3 + 92655x^2 + 25605x + 2654}{384(6x+1)(3x+1)(2x+1)(3x+2)(5+6x)(x+1)64^x}.$$

Example 4. Starting with $H(0, 2)^2$, we find that

$$P(x, z) = \frac{\sqrt{\pi}((2x)!)^3}{16^x \Gamma(2x+1/2)},$$

and

$$G(x, z) = \frac{Q(x, z)}{2((1+4x)(3+4x)(2x+z+2)^2)} P(x, z) H(0, 2)^2,$$

where

$$\begin{aligned} Q(x, z) := & 120x^4 + 372x^3 + 136x^3z + 56x^2z^2 + 426x^2 + 316x^2z \\ & + 242xz + 86xz^2 + 8xz^3 + 213x + 39 + 33z^2 + 6z^3 + 61z. \end{aligned}$$

Application of corollary 2 gives

$$\zeta(2) = \frac{\sqrt{\pi}}{8} \sum_{x=0}^{\infty} \frac{(2912x^4 + 7100x^3 + 6381x^2 + 2494x + 355)((x+1)!)^2((2x)!)^3}{\Gamma(2x+5/2)((3x+3)!)^2 16^x}.$$

On the other hand, corollary 1 yields

$$\zeta(2) = \frac{3\sqrt{\pi}}{32} \sum_{x=0}^{\infty} \frac{(20x^2 + 32x + 13)(2x)!}{(2x+1)(x+1)\Gamma(2x+5/2)16^x}.$$

Example 5. By taking the kernel $H(1, 2)^2$, we get

$$P(x, z) = \frac{(x!)^3 \sqrt{\pi}}{4^x \Gamma(x + 1/2)},$$

$$G(x, z) = \frac{21x^3 + 55x^2 + 47x + 13 + 28x^2z + 48xz + 20z + 13xz^2 + 11z^2 + 2z^3}{2(2x + 1)(2x + z + 2)^2} F(x, z),$$

where $F(x, z) := P(x, z)H(1, 2)^2$.

If we apply corollary 2 we get

$$\zeta(2) = \frac{1}{9\sqrt{\pi}} \sum_{x=0}^{\infty} \frac{(145x^2 + 186x + 59)(x!)^5 \Gamma(x + 1/2) 4^x}{((3x + 2)!)^2}.$$

On the other hand, corollary 1 yields

$$\zeta(2) = \frac{\pi^{3/2}}{64} \sum_{x=0}^{\infty} \frac{(21x + 13)x!^3}{(64)^x (\Gamma(x + 3/2))^3}.$$

Example 6. Corresponding to $H(1, 3)^2$, we find that

$$P(x, z) = \frac{\sqrt{\pi}(2x)!^3}{16^x \Gamma(2x + 1/2)},$$

and

$$G(x, z) = \frac{Q(x, z)}{2(3 + 4x)(1 + 4x)(3x + z + 2)^2(3x + z + 3)^2} P(x, z)(-1)^z H(1, 3)^2,$$

where $Q(x, z)$ is a polynomial in x and z . Application of corollary 2 gives

$$\zeta(2) = \frac{\pi^{3/2}}{2048} \sum_{x=0}^{\infty} \frac{((2x)!)^3 (10920x^4 + 27908x^3 + 25962x^2 + 10275x + 1421)}{(\Gamma(2x + 5/2))^3 (4096)^x},$$

and corollary 1 gives

$$\zeta(2) = \frac{\sqrt{\pi}}{72} \sum_{x=0}^{\infty} \frac{P(x)(x!)^2((2x)!)^2}{16^x \Gamma(2x + 5/2)((3x + 2)!)^2},$$

where

$$P(x) := 2912x^4 + 7100x^3 + 6381x^2 + 2494x + 355.$$

Example 7. Corresponding to $H(1, 5)^2$, we find that

$$P(x, z) = \frac{\sqrt{2\pi}(4x)!^3}{4(256)^x \sin(1/8\pi) \sin(3/8\pi) \Gamma(4x + 1/2)},$$

and a corresponding MWZ mate $G(x, z)$. If we apply corollary 2, we find

$$\zeta(2) = \frac{\sqrt{2\pi}}{3200 \sin(3/8\pi) \sin(1/8\pi)} \sum_{x=0}^{\infty} \frac{P(x)((4x)!)^3(x!)^2}{(256)^x \Gamma(4x+9/2)((5x+4)!)^2},$$

where

$$P(x) := 3333245952x^{10} + 18842142336x^9 + 47204597136x^8 + 68964524342x^7 + 65011852179x^6 \\ + 41280848445x^5 + 17862102186x^4 + 5194331883x^3 + 970166319x^2 + 104901994x + 4974228.$$

The terms of this series are $O((\frac{256}{9765625})^j) \approx O(10^{-5j})$.

Example 8. Similarly for the kernel $H(0, 2)^3$, we get

$$P(x, z) = \frac{(-1)^x(x!(2x)!)^3}{(3x)!} \text{ and}$$

$$G(x, z) = \frac{Q(x, z)}{6(3x+1)(3x+2)(2x+z+2)^3} H(0, 2)^3 P(x, z),$$

where $Q(x, z)$ is a certain polynomial in x and z .

By using corollary 2, we get

$$\zeta(3) = \sum_{x=0}^{\infty} \frac{(-1)^x(2x)!^3(x+1)!^6 P(x)}{2(x+1)^2((3x+3)!)^4},$$

where

$$P(x) := 40885x^5 + 124346x^4 + 150160x^3 + 89888x^2 + 26629x + 3116,$$

and application of corollary 1 gives

$$\zeta(3) = \sum_{x=0}^{\infty} \frac{(-1)^x(56x^2 + 80x + 29)(x!)^3}{4(2x+1)^2(3x+3)!}.$$

Example 9. Starting with the kernel $H(1, 3)^3$, we get

$$P(x, z) = \frac{(-1)^x(x!(2x)!)^3}{(3x)!} \text{ and}$$

$$G(x, z) = \frac{Q(x, z)}{6(3x+2)(3x+1)(3x+z+2)^3(3x+z+3)^3} P(x, z) H(1, 3)^3,$$

where

$$Q(x, z) =: 448x^5 + 624zx^4 + 1760x^4 + 1932zx^3 + 2728x^3 + 348z^2x^3 + 2214x^2z + 2084x^2 + 792z^2x^2 \\ + 90z^3x^2 + 594xz^2 + 1113xz + 9z^4x + 132z^3x + 784x + 6z^4 + 207z + 48z^3 + 147z^2 + 116.$$

In this example, we show all the steps to demonstrate the application of theorem 2

Let

$$F(x, z) := H(x, z)P(x, z) .$$

Define $M(n)$, for $n = 0, 1, 2, 3, 4, \dots$, by

$$M(n) := \sum_{x=0}^{n-1} G(x, 0) + \sum_{x=0}^{\infty} (F(x+n, x) + G(x+n, x+1)) .$$

Then theorem 2 says that $\zeta(3) = M(n)$, $\forall n = 0, 1, 2, 3, 4, \dots$.

In particular

$$\zeta(3) = M(0) = \frac{1}{24} \sum_{x=0}^{\infty} \frac{(x!)^3 (2x)!^6 (-1)^x P(x)}{(3x+2)!((4x+3)!)^3} , \quad (2)$$

where

$$P(x) := 126392x^5 + 412708x^4 + 531578x^3 + 336367x^2 + 104000x + 12463 .$$

On the other hand, application of corollary 1 gives

$$\zeta(3) = \frac{1}{162} \sum_{x=0}^{\infty} \frac{P(x)(x!)^6((2x)!)^3(-1)^x}{((3x+2)!)^4} ,$$

where

$$P(x) := 40885x^5 + 124346x^4 + 150160x^3 + 89888x^2 + 26629x + 3116 .$$

The series (2) was first derived in [AZ] and used by S. Wedeniwski (1999) to obtain up to 128 million correct decimal places. The terms of the series in (2) are $O((110592)^{-j}) \approx O(10^{-5j})$, while the terms of the second series are $O((\frac{64}{531441})^j) \approx O(10^{-4j})$.

Instead, if we take $H(1, 5)^3$, we get

$$P(x, z) = \frac{2\sqrt{3}}{3\sqrt{\pi}} \frac{(2x-1/2)^3 (2x)!^5 (4096)^x}{(729)^x \Gamma(2x+2/3) \Gamma(2x+1/3)} ,$$

and a corresponding $G(x, z)$. Let

$$F(x, z) = H(1, 5)^3 P(x, z) ,$$

and let $M(n)$ be as above.

Then theorem 2 gives $\zeta(3) = M(n)$, $\forall n = 0, 1, 2, 3, 4, \dots$ and in particular

$$\zeta(3) = M(0) = \frac{16}{81} \sum_{x=0}^{\infty} \frac{P(x)(4096)^x ((4x)!)^3 ((2x)!)^2 ((2x+1)!)^4 (-1)^x}{((6x+5)!)^4} , \quad (3)$$

where

$$\begin{aligned} P(x) := & 5561689253120x^{13} + 41827852352256x^{12} + 143295193251200x^{11} + 295842983236608x^{10} \\ & + 410324548816928x^9 + 403368918753744x^8 + 288879369092920x^7 + 152460289970616x^6 \\ & + 59240414929957x^5 + 16722886152858x^4 + 3330604771504x^3 \\ & + 442815051024x^2 + 35195802021x + 1261871244 . \end{aligned}$$

The terms of this series are $O((\frac{4096}{282429536481})^j) \approx O(10^{-8j})$.

This improves the previous record (2).

Example 10. If we start with

$$H(x, z) = \binom{x+a}{a+z} \binom{x+b}{b+z} ,$$

we get

$$P(x, z) = \frac{(a+b)!(x+a)!(x+b)!}{(a+2x+b)!a!b!} ,$$

and

$$G(x, z) = \frac{(3x^2 + 2xa + 2xb + 6x - 2xz + 2b + 2a - 2z + 3 - za + ab - zb)(a+z)(b+z)}{(a+2x+1+b)(2x+b+2+a)(x+1-z)^2} H(x, z) P(x, z) .$$

One can easily check that $G(x, \pm\infty) = 0$.

Hence, we get

$$\sum_{z=-\infty}^{\infty} \binom{x+a}{a+z} \binom{x+b}{b+z} = \frac{(a+2x+b)!a!b!}{(a+b)!(x+a)!(x+b)!} .$$

This is a derivation of the classical Chu-Vandermonde summation formula, in the framework of the MWZ-method. The Markov-WZ method can sometimes lead to a discovery of new identities with appropriate $H(x, z)$.

Example 11. Let

$$H_s(x, z) := \left(\frac{(-1)^z (m)_z}{(m+\delta)_{x+z}} \right)^s .$$

In this example we will show how to use implementations of some numerical methods together with the Markov-WZ Method to give new WZ-pairs. The steps are:

- Take the output from Markov in MarkovWZ (see [MZ]), which is a system of first order linear recurrence relation(s) for the unknown coefficient functions $a_i(x)$'s.
- Crank out some terms for the unknown coefficients, i.e. use the recurrence equation outputted by the program and find the first few terms.

- (c) Use the Salvy-Zimmermann `gfund` program in the Algolib library available from `algo.inria.fr`, or **findrec** in **EKHAD**[†], to find a recurrence equation satisfied by the coefficient functions.
- (d) Finally, solve the recurrence relations to find a closed form for the coefficients (if there exists one) (for example, in Maple, use `rsolve`).

11.1 Starting with $H_2(x, z)$, we find that $L = 0$ and

$$P(x, z) := \frac{\Gamma(\delta + x)^3 \Gamma(\delta - 1/2)}{4^x \Gamma(\delta + x - 1/2) \Gamma(\delta)^3}.$$

Therefore we get a WZ-pair (F, G) (*not MWZ!*), where $F(x, z) := H_2(x, z)P(x, z)$, and

$$G(x, z) := F(x, z) \frac{(3x + 2z + 2m - 2 + 3\delta)}{2(2x + 2\delta - 1)},$$

and by applying corollary 1, we get the identity

$$\sum_{z=0}^{\infty} \frac{\Gamma(z+m)^2 \Gamma(m+\delta)^2}{\Gamma(m)^2 \Gamma(m+\delta+z)^2} = \frac{1}{2} \sum_{x=0}^{\infty} \frac{(3x + 3\delta + 2m - 2) \Gamma(\delta + x)^3 \Gamma(\delta - 1/2) \Gamma(m+\delta)^2}{\Gamma(\delta + x - 1/2) \Gamma(\delta)^3 \Gamma(m+x+\delta)^2 (2x + 2\delta - 1)} \left(\frac{1}{4}\right)^x,$$

for $\delta = 0, 1, 2, 3, \dots, m = 0, 1, 2, 3, \dots$. If we specialize to $m = 1$ and $\delta = 1$, we get the formula for $\zeta(2)$, which is

$$\zeta(2) = \frac{3\sqrt{\pi}}{4} \sum_{x=0}^{\infty} \frac{\Gamma(x+1)}{(x+1)\Gamma(3/2+x)} \left(\frac{1}{4}\right)^x = \frac{3}{2} {}_3F_2\left(\begin{matrix} 1, 1, 1 \\ 2, \frac{3}{2} \end{matrix}; \frac{1}{4}\right)$$

11.2 Starting with $H_3(x, z)$ we find that $L = 1$ and there is a vector first order recurrence relations for the polynomials $a_0(x), a_1(x)$. That means if we set

$$a(x) := [a_0(x), a_1(x)]^T,$$

then there is a 2 by 2 matrix $\mathbf{A}(x)$ such that $a(x+1) = \mathbf{A}(x)a(x)$, and by using **findrec** in **EKHAD** we get

$$a_0(x) := \frac{(-1)^x \Gamma(x+\delta)^3 (x+\delta-1) \Gamma(\delta-1/2)}{\Gamma(\delta)^3}, \text{ and } a_1(x) := \frac{2(-1)^x \Gamma(\delta+x)^3 \Gamma(\delta-1/2)}{\Gamma(\delta)^3}.$$

Hence our polynomial is $P(x, z) = a_0(x) + a_1(x)(z+m)$, and the corresponding WZ-pair is $(H_3(x, z)P(x, z), G(x, z))$, where

$$G(x, z) := \frac{2x + 2\delta + z + m - 1}{2z + 2m + \delta + x - 1} P(x, z) H_3(x, z),$$

as outputted by **zeil** in **EKHAD**. Applying corollary 1, we get the identity

$$\sum_{z=0}^{\infty} \frac{(-1)^z (2z + 2m + \delta - 1) \Gamma(m+z)^3}{\Gamma(m)^3 \Gamma(m+\delta+z)^3} = \sum_{x=0}^{\infty} \frac{(-1)^x (2x + 2\delta + m - 1) \Gamma(x+\delta)^3}{\Gamma(\delta)^3 \Gamma(m+\delta+x)^3},$$

for $\delta = 0, 1, 2, 3, \dots$, and $m = 0, 1, 2, 3, \dots$.

[†] download-able free from: <http://www.math.rutgers.edu/~zeilberg/>

11.3 Starting with $H_4(x, z)$ we find that $L = 1$ and there is a first order vector recurrence relations for the polynomials $a_0(x), a_1(x)$. Using findrec in EKHAD we get

$$a_0(x) := \frac{(-1)^x \Gamma(\delta + x)^5 (\delta + x - 1) \Gamma(\delta - 1/2)}{\Gamma(\delta + x - 1/2) \Gamma(\delta)^5 4^x},$$

and

$$a_1(x) := \frac{2(-1)^x \Gamma(\delta + x)^5 \Gamma(\delta - 1/2)}{4^x \Gamma(\delta + x - 1/2) \Gamma(\delta)^5}.$$

This leads to the WZ-pair $(F(x, z)(a_0(x) + a_1(x)(m + z)), G)$, where G is

$$G := \frac{5x^2 + 6mx + 10\delta x + 6m\delta + 5\delta^2 + 2m^2 + 6xz - 6x + 6\delta z - 6\delta + 4mz - 4m + 2z^2 - 4z + 2}{2(2x + 2\delta - 1)(2m + 2z + x + \delta - 1)}.$$

Application of corollary 1 yields the identity

$$\sum_{z=0}^{\infty} \frac{\Gamma(m+z)^4 (2m+2z+\delta-1)}{\Gamma(m+\delta+z)^4} = \frac{1}{4} \sum_{x=0}^{\infty} \frac{\Gamma(m)^4 \Gamma(x+\delta)^5 \Gamma(\delta-1/2) P(x)}{\Gamma(x+1/2+\delta) \Gamma(m+\delta+x)^4 \Gamma(\delta)^5} \left(\frac{-1}{4}\right)^x,$$

that holds for $\delta = 0, 1, 2, 3, \dots$, and $m = 0, 1, 2, 3, 4, \dots$, where

$$P(x) := 5x^2 + 10x\delta + 6xm + 2m^2 + 5\delta^2 + 6\delta m + 2 - 6x - 4m - 6\delta.$$

If we specialize to $m = 1$ and $\delta = 1$, we find the motivation for Andrei Markov's beautiful work, namely

$$\zeta(3) = \frac{5\sqrt{\pi}}{4} \sum_{x=0}^{\infty} \frac{\Gamma(x+1)}{(x+1)^2 \Gamma(x+3/2)} \left(\frac{-1}{4}\right)^x = \frac{5}{4} {}_4F_3\left(\begin{matrix} 1, 1, 1, 1 \\ 2, 2, \frac{3}{2} \end{matrix}; \frac{-1}{4}\right).$$

11.4 Starting with $H_5(x, z)$ we found that $L = 3$. The corresponding polynomial satisfies a recurrence relation of order ≥ 2 , for which we couldn't find an explicit closed form solution for the polynomial. Nonetheless, as described in [MZ], we have an accelerating formula for $\zeta(5)$ (see [MZ] for $5 \leq n \leq 9$).

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