

# Queue Layouts of Graph Products and Powers<sup>†</sup>

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A *k*-queue layout of a graph  $G$  consists of a linear order  $\sigma$  of  $V(G)$ , and a partition of  $E(G)$  into  $k$  sets, each of which contains no two edges that are nested in  $\sigma$ . This paper studies queue layouts of graph products and powers.

**Keywords:** graph, queue layout, cartesian product,  $d$ -dimensional grid graph,  $d$ -dimensional toroidal grid graph, Hamming graph.

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## 1 Introduction

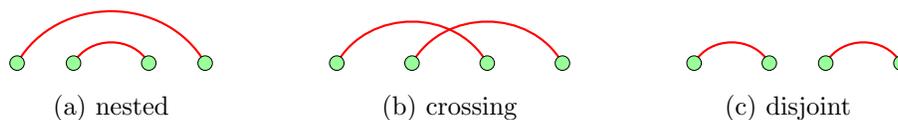
Let  $G$  be a graph. (All graphs considered are finite, simple and undirected.) The vertex and edge sets of  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. The minimum and maximum degree of  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. The *density* of  $G$  is  $\eta(G) := |E(G)|/|V(G)|$ .

A *vertex ordering* of  $G$  is a bijection  $\sigma : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$ . In a vertex ordering  $\sigma$  of  $G$ , let  $L_\sigma(e)$  and  $R_\sigma(e)$  denote the endpoints of each edge  $e \in E(G)$  such that  $\sigma(L_\sigma(e)) < \sigma(R_\sigma(e))$ . Where the vertex ordering  $\sigma$  is clear from the context, we will abbreviate  $L_\sigma(e)$  and  $R_\sigma(e)$  by  $L_e$  and  $R_e$ , respectively. For edges  $e$  and  $f$  of  $G$  with no endpoint in common, there are the following three possible relations with respect to  $\sigma$ , as illustrated in Figure 1:

- (a)  $e$  and  $f$  *nest* if  $\sigma(L_e) < \sigma(L_f) < \sigma(R_f) < \sigma(R_e)$ ,
- (b)  $e$  and  $f$  *cross* if  $\sigma(L_e) < \sigma(L_f) < \sigma(R_e) < \sigma(R_f)$ ,
- (c)  $e$  and  $f$  are *disjoint* if  $\sigma(L_e) < \sigma(R_e) < \sigma(L_f) < \sigma(R_f)$ .

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**Fig. 1:** Relationships between pairs of edges with no common endpoint in a vertex ordering.

A *queue* in  $\sigma$  is a set of edges  $Q \subseteq E(G)$  such that no two edges in  $Q$  are nested. Observe that when traversing  $\sigma$  from left to right, the left and right endpoints of the edges in a queue are reached in first-in-first-out order—hence the name ‘queue’. Observe that  $Q \subseteq E(G)$  is a queue if and only if for all edges  $e, f \in Q$ ,

$$\begin{aligned} \sigma(L_e) \leq \sigma(L_f) \text{ and } \sigma(R_e) \leq \sigma(R_f) , \\ \text{or } \sigma(L_f) \leq \sigma(L_e) \text{ and } \sigma(R_f) \leq \sigma(R_e) . \end{aligned} \quad (1)$$

A *k*-queue layout of  $G$  is a pair

$$(\sigma, \{Q_1, Q_2, \dots, Q_k\})$$

where  $\sigma$  is a vertex ordering of  $G$ , and  $\{Q_1, Q_2, \dots, Q_k\}$  is a partition of  $E(G)$ , such that each  $Q_i$  is a queue in  $\sigma$ . The *queue-number* of a graph  $G$ , denoted by  $\text{qn}(G)$ , is the minimum  $k$  such that there is a  $k$ -queue layout of  $G$ .

Queue layouts were introduced by Heath et al. [15, 19]. Applications of queue layouts include sorting permutations [12, 20, 22, 24, 27], parallel process scheduling [3], matrix computations [23], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in the literature [7, 9, 10, 13, 25, 26]. Queue layouts of directed graphs [5, 11, 17, 18] and posets [16] have also been investigated.

Table 1 describes the best known upper bounds on the queue-number of various classes of graphs. Planar graphs are an interesting class of graphs for which it is not known whether the queue-number is bounded (see [6, 23]).

This paper studies queue layouts of graph products and graph powers. To prove optimality we use the following lower bound by Heath and Rosenberg [19]. See Pemmaraju [23] and Dujmović and Wood [9] for slightly more exact lower bounds.

**Lemma 1 ([19])** *Every graph  $G$  has queue-number  $\text{qn}(G) > \eta(G)/2$ .*

This paper is organised as follows. In Section 2 we introduce the concepts of strict queue layout and strict queue-number. Many of the upper bounds on the queue-number that are presented in later sections will be expressed as functions of the strict queue-number. In Section 3 we prove bounds on the queue-number of the power of a graph in terms of the queue-number of the underlying graph. In Section 4 we define the graph products that will be studied in later sections. In Section 5 we study the queue-number of the cartesian product of graphs. Finally in Section 6 we study the queue-number of the direct and strong products of graphs.

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<sup>‡</sup> Dujmović and Wood [8] gave a simple proof of this result.

**Tab. 1:** Upper bounds on the queue-number.

graph family	queue-number	reference
$n$ vertices	$\lfloor \frac{n}{2} \rfloor$	Heath and Rosenberg [19]
$m$ edges	$e\sqrt{m}$	Dujmović and Wood [9]
tree-width $w$	$3^w \cdot 6^{(4^w - 3w - 1)/9} - 1$	Dujmović <i>et al.</i> [6]
tree-width $w$ , max. degree $\Delta$	$36\Delta w$	Wood [29]
path-width $p$	$p$	Dujmović <i>et al.</i> [6]
band-width $b$	$\lfloor \frac{b}{2} \rfloor$	Heath and Rosenberg [19]
track-number $t$	$t - 1$	Dujmović <i>et al.</i> [6]
2-trees	3	Rengarajan and Veni Madhavan [25] <sup>‡</sup>
$k$ -ary butterfly	$\lfloor \frac{k}{2} \rfloor + 1$	Hasunuma [14]
$d$ -ary de Bruijn	$d$	Hasunuma [14]
Halin	3	Ganley [13]
X-trees	2	Heath and Rosenberg [19]
outerplanar	2	Heath <i>et al.</i> [15]
arched levelled planar	1	Heath <i>et al.</i> [15]
trees	1	Heath and Rosenberg [19]

## 2 Strict Queue Layouts

Let  $\sigma$  be a vertex ordering of a graph  $G$ . We say an edge  $e$  is *inside* a distinct edge  $f$ , and  $e$  and  $f$  *overlap*, if

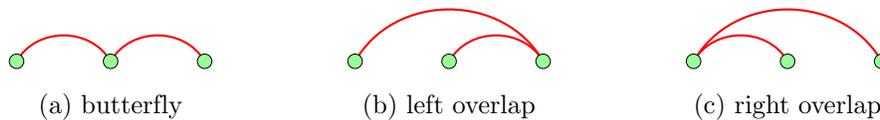
$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f) .$$

A set of edges  $Q \subseteq E(G)$  is a *strict queue* in  $\sigma$  if no edge in  $Q$  is inside another edge in  $Q$ . Alternatively,  $Q$  is a *strict queue* in  $\sigma$  if

$$\begin{aligned} &\sigma(L_e) < \sigma(L_f) \text{ and } \sigma(R_e) < \sigma(R_f) , \\ \text{or } &\sigma(L_f) < \sigma(L_e) \text{ and } \sigma(R_f) < \sigma(R_e) . \end{aligned} \tag{2}$$

Note that Equation (2) is obtained from Equation (1) by replacing “ $\leq$ ” by “ $<$ ”.

Hence a strict queue is a set of edges, no two of which are nested or overlapping, as illustrated in Figure 2. Note that edges forming a ‘butterfly’ can be in a single strict queue.

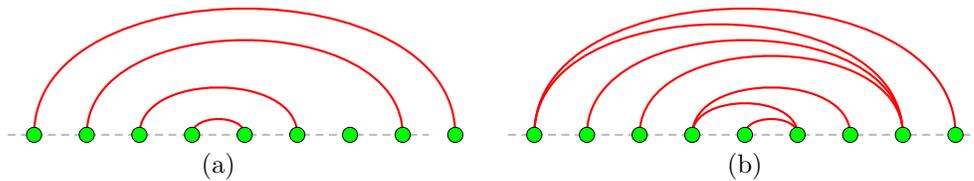


**Fig. 2:** Relationships between pairs of edges with a common endpoint in a vertex ordering.

A *strict  $k$ -queue layout* of  $G$  is a pair  $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$  where  $\sigma$  is a vertex ordering of  $G$ , and  $\{Q_1, Q_2, \dots, Q_k\}$  is a partition of  $E(G)$ , such that each  $Q_i$  is a strict queue in  $\sigma$ . We sometimes write

$\text{queue}(e) = i$  for each edge  $e \in Q_i$ . The *strict-queue-number* of a graph  $G$ , denoted by  $\text{sqn}(G)$ , is the minimum  $k$  such that there is a strict  $k$ -queue layout of  $G$ .

Heath and Rosenberg [19] proved that a fixed vertex ordering of a graph  $G$  admits a  $k$ -queue layout of  $G$  if and only if it has no  $(k + 1)$ -edge rainbow, where a *rainbow* is a set of pairwise nested edges, as illustrated in Figure 3(a). Consider the analogous problem for strict queues: assign the edges of a graph  $G$  to the minimum number of strict queues given a fixed vertex ordering  $\sigma$  of  $G$ . As illustrated in Figure 3(b), a *weak rainbow* in  $\sigma$  is a set of edges  $R$  such that for every pair of edges  $e, f \in R$ ,  $e$  is inside  $f$  or  $f$  is inside  $e$ .



**Fig. 3:** (a) rainbow, (b) weak rainbow

**Lemma 2** A vertex ordering of a graph  $G$  admits a strict  $k$ -queue layout of  $G$  if and only if it has no  $(k + 1)$ -edge weak rainbow.

**Proof:** A strict  $k$ -queue layout has no  $(k + 1)$ -edge weak rainbow since each edge of a weak rainbow must be in a distinct strict queue. Conversely, suppose we have a vertex ordering with no  $(k + 1)$ -edge weak rainbow. For every edge  $e \in E(G)$ , let  $\text{queue}(e)$  be one plus the maximum number of edges in a weak rainbow consisting of edges that are inside  $e$ . If  $e$  is inside  $f$  then  $\text{queue}(e) < \text{queue}(f)$ . Hence we have a valid strict queue assignment. The number of strict queues is at most  $k$ .  $\square$

A *linear forest* is a graph in which every component is a path. The *linear arboricity* of a graph  $G$ , denoted by  $\text{la}(G)$ , is the minimum integer  $k$  such that  $E(G)$  can be partitioned in  $k$  linear forests; see [1, 2, 30, 31]. We have the following lower bounds on  $\text{sqn}(G)$ .

**Lemma 3** The strict queue-number of every graph  $G$  satisfies:

- (a)  $\text{sqn}(G) \geq \text{la}(G) > \eta(G)$ ,
- (b)  $\text{sqn}(G) \geq \text{la}(G) \geq \Delta(G)/2$ , and
- (c)  $\text{sqn}(G) \geq \delta(G)$ .

**Proof:** Say  $Q$  is a strict queue in a vertex ordering  $\sigma$  of  $G$ . Every 2-edge path  $(u, v, w)$  in  $Q$  has  $\sigma(u) < \sigma(v) < \sigma(w)$  (or  $\sigma(w) < \sigma(v) < \sigma(u)$ ). Thus no vertex is incident to three edges in  $Q$ , and  $Q$  induces a linear forest. Hence  $\text{la}(G) \leq \text{sqn}(G)$ .

Since a linear forest in  $G$  has at most  $|V(G)| - 1$  edges,  $\text{la}(G) \geq |E(G)| / (|V(G)| - 1) > \eta(G)$ . This proves (a). At most two edges incident to each vertex are a linear forest. Thus  $\text{la}(G) \geq \Delta(G)/2$ . This proves (b).

In every vertex ordering of  $G$ , every edge incident to the first vertex is in a distinct strict queue. Hence  $\text{sqn}(G) \geq \delta(G)$ . This proves (c).  $\square$

Obviously a proper edge  $(\Delta(G) + 1)$ -colouring [28] can be combined with a  $\text{qn}(G)$ -queue layout to obtain a strict queue layout.

**Lemma 4** Every graph  $G$  has strict queue-number  $\text{sqn}(G) \leq (\Delta(G) + 1) \cdot \text{qn}(G)$ .  $\square$

### 3 Graph Powers

Let  $G$  be a graph, and let  $d \in \mathbb{Z}^+$ . The  $d$ -th power of  $G$ , denoted by  $G^d$ , is the graph with vertex set  $V(G^d) = V(G)$ , where  $vw \in E(G^d)$  if and only if the distance between  $v$  and  $w$  in  $G$  is at most  $d$ . The following general result is similar to a theorem of Dujmović and Wood [10].

**Theorem 1** For every graph  $G$  and  $d \in \mathbb{Z}^+$ ,

$$\text{qn}(G^d) \leq \frac{(2 \text{sqn}(G))^{d+1} - 1}{2 \text{sqn}(G) - 1} - \text{sqn}(G) - 1 .$$

**Proof:** Let  $\sigma$  be the vertex ordering in a strict  $\text{sqn}(G)$ -queue layout of  $G$ . Consider  $\sigma$  to be a vertex ordering of  $G^d$ . For every pair of vertices  $v, w \in V(G)$  with  $\sigma(v) < \sigma(w)$  and at distance  $\ell \leq d$ , fix a path  $P(vw)$  from  $v$  to  $w$  in  $G$  with exactly  $\ell$  edges. Suppose  $P(vw) = (x_0, x_1, \dots, x_\ell)$ , where  $v = x_0$  and  $w = x_\ell$ . For each  $1 \leq i \leq \ell$ , let  $\text{dir}(x_{i-1}x_i)$  be ‘+’ if  $\sigma(x_{i-1}) < \sigma(x_i)$ , and ‘-’ otherwise. Let  $f(vw)$  be the vector

$$f(vw) = \left[ (\text{queue}(x_{i-1}x_i), \text{dir}(x_{i-1}x_i)) : 1 \leq i \leq \ell \right] .$$

Consider two edges  $vw, pq \in E(G^d)$  with  $f(vw) = f(pq)$ . Then  $|P(vw)| = |P(pq)|$ . Let  $P(vw) = (x_0, x_1, \dots, x_\ell)$  and  $P(pq) = (y_0, y_1, \dots, y_\ell)$ . We have  $\text{dir}(x_0x_1) = \text{dir}(y_0y_1)$  and  $\text{queue}(x_0x_1) = \text{queue}(y_0y_1)$ . Thus  $x_0 \neq y_0$ . Without loss of generality  $\sigma(x_0) < \sigma(y_0)$ . By Equation (2),  $\sigma(x_1) < \sigma(y_1)$ . In general,  $\sigma(x_{i-1}) < \sigma(y_{i-1})$  implies  $\sigma(x_i) < \sigma(y_i)$ , since  $\text{queue}(x_{i-1}x_i) = \text{queue}(y_{i-1}y_i)$  and  $\text{dir}(x_{i-1}x_i) = \text{dir}(y_{i-1}y_i)$ . By induction,  $\sigma(x_i) < \sigma(y_i)$  for all  $0 \leq i \leq \ell$ . In particular,  $\sigma(w) < \sigma(q)$ . Thus  $vw$  and  $pq$  can be in the same strict queue. If we partition the edges of  $G^d$  by the value of  $f$  we obtain a strict queue layout of  $G^d$ . The number of queues is

$$\sum_{\ell=1}^d (2 \text{sqn}(G))^\ell = \frac{(2 \text{sqn}(G))^{d+1} - 1}{2 \text{sqn}(G) - 1} - 1 .$$

Observe that for the edges of  $G$  we have counted  $2 \text{sqn}(G)$  queues. Of course we need only  $\text{sqn}(G)$  queues. Thus the total number of queues is as claimed.  $\square$

#### 3.1 Powers of Paths and Cycles

In a vertex ordering  $\sigma$  of a graph  $G$ , the *width* of an edge  $e$  is  $\sigma(R_e) - \sigma(L_e)$ . The *bandwidth* of  $\sigma$  is the maximum width of an edge of  $G$ . The *bandwidth* of  $G$ , denoted by  $\text{bw}(G)$ , is the minimum bandwidth of a vertex ordering of  $G$ . Alternatively,  $\text{bw}(G) = \min\{k : G \subseteq P_n^k\}$  for every  $n$ -vertex graph  $G$ .

Heath and Rosenberg [19] observed that edges whose widths differ by at most one are not nested. Thus  $\text{qn}(G) \leq \lceil \text{bw}(G)/2 \rceil$ , as mentioned in Table 1. In a vertex ordering, edges with the same width are not nested or overlapping, and thus form a strict queue. The next lemma follows.

**Lemma 5** *Every graph  $G$  has strict queue-number  $\text{sqn}(G) \leq \text{bw}(G)$ .*  $\square$

We have the following results that give more precise bounds on the queue-number and strict-queue-number of powers of paths and cycles than Theorem 1.

**Lemma 6** *The  $k$ -th power of a path  $P_n$  ( $n \geq k + 1$ ) has queue-number  $\text{qn}(P_n^k) = \lceil k/2 \rceil$  and strict queue-number  $\text{sqn}(P_n^k) = k$*

**Proof:** The bandwidth of a graph  $G$  can be thought of as the minimum integer  $k$  such that  $G \subseteq P_n^k$ . Thus the upper bound is nothing more than the result  $\text{qn}(G) \leq \lceil \text{bw}(G)/2 \rceil$  of Heath and Rosenberg [19]. The lower bound follows since  $P_n^k$  contains a  $(k + 1)$ -clique, which contains  $\lceil k/2 \rceil$  pairwise nested edges in any vertex ordering, all of which must be assigned to distinct queues.

The natural vertex-ordering of  $P_n^k$  has no  $(k + 1)$ -edge weak rainbow. Thus  $\text{sqn}(P_n^k) \leq k$  by Lemma 2. The lower bound follows since  $P_n^k$  contains a  $(k + 1)$ -clique, which contains a  $k$ -edge weak rainbow in any vertex ordering.  $\square$

A graph is *unicyclic* if every connected component has at most one cycle. Heath and Rosenberg [19] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle has a 1-queue layout. More generally,

**Lemma 7** *The  $k$ -th power of a cycle  $C_n$  ( $n \geq 2k$ ) has queue-number  $\frac{k}{2} < \text{qn}(C_n^k) \leq k$ , and strict queue-number  $\text{sqn}(C_n^k) = 2k$ .*

**Proof:** Observe that  $\delta(C_n^k) = \Delta(C_n^k) = 2k$  and  $\eta(C_n^k) = k$ . Thus the claimed lower bounds follow from Lemmata 1 and 3. For the upper bounds, say  $C_n = (v_1, v_2, \dots, v_n)$ . By considering the vertex ordering

$$(v_1, v_n; v_2, v_{n-1}; \dots; v_i, v_{n-i+1}; \dots; v_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}) , \quad (3)$$

we see that  $C_n^k \subseteq P_n^{2k}$ . The result follows from Lemma 6.  $\square$

## 4 Graph Products

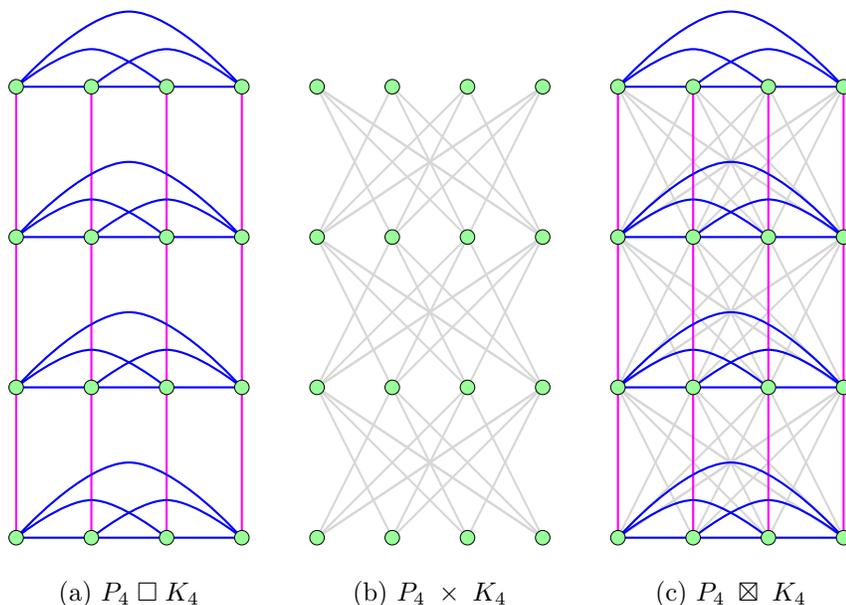
Let  $G_1$  and  $G_2$  be graphs. Below we define a number of graph products whose vertex set is

$$V(G_1) \times V(G_2) = \{(a, v) : a \in V(G_1), v \in V(G_2)\} .$$

We classify a potential edge  $(a, v)(b, w)$  as follows:

- $G_1$ -edge:  $ab \in E(G_1)$  and  $v = w$ .
- $G_2$ -edge:  $a = b$  and  $vw \in E(G_2)$ .
- direct edge:  $ab \in E(G_1)$  and  $vw \in E(G_2)$ .

The *cartesian product*  $G_1 \square G_2$  consists of the  $G_1$ -edges and the  $G_2$ -edges. The *direct product*  $G_1 \times G_2$  consists of the direct edges. The *strong product*  $G_1 \boxtimes G_2$  consists of the  $G_1$ -edges, the  $G_2$ -edges, and the direct edges. That is,  $G_1 \boxtimes G_2 = (G_1 \square G_2) \cup (G_1 \times G_2)$ . Note that other names abound for these graph products. Our notation is taken from the survey by Klavžar [21]. Assuming isomorphic graphs are equal, each of the above three products are associative, and for instance,  $G_1 \square G_2 \square \dots \square G_d$  is well-defined. Figure 4 illustrates these three types of graph products.



**Fig. 4:** Examples of graph products: (a) cartesian, (b) direct, (c) strong.

The following lemma is well-known and easily proved.

**Lemma 8** For all graphs  $G_1$  and  $G_2$ , the density satisfies

(a)  $\eta(G_1 \square G_2) = \eta(G_1) + \eta(G_2)$ ,

(b)  $\eta(G_1 \times G_2) = 2\eta(G_1) \cdot \eta(G_2)$ ,

(c)  $\eta(G_1 \boxtimes G_2) = 2\eta(G_1) \cdot \eta(G_2) + \eta(G_1) + \eta(G_2)$ .

## 5 The Cartesian Product

We have the following bounds on the queue-number of a cartesian product. In a vertex ordering  $\sigma$  of a graph product, we abbreviate  $\sigma((v, a))$  by  $\sigma(v, a)$ .

**Theorem 2** For all graphs  $G$  and  $H$ ,

$$\text{qn}(G \square H) \leq \text{sqn}(G) + \text{qn}(H) .$$

Furthermore, if for some constant  $c$  we have  $\text{sqn}(G) \leq c \cdot \eta(G)$  and  $\text{qn}(H) \leq c \cdot \eta(H)$ , then

$$\text{qn}(G \square H) \geq \frac{1}{2c} (\text{sqn}(G) + \text{qn}(H)) .$$

**Proof:** First we prove the upper bound. Let  $\sigma$  be the vertex ordering in a strict  $\text{sqn}(G)$ -queue layout of  $G$ . Let  $\pi$  be the vertex ordering in a  $\text{qn}(H)$ -queue layout of  $H$ . Let  $\phi$  be the vertex ordering of  $G \square H$  in which  $\phi(v, a) < \phi(w, b)$  if and only if  $\sigma(v) < \sigma(w)$ , or  $v = w$  and  $\pi(a) < \pi(b)$ .

For all edges  $e$  of  $G$  and for all vertices  $a$  of  $H$ , we have  $\phi(L_e, a) < \phi(R_e, a)$ . Similarly, for all edges  $e$  of  $H$  and for all vertices  $v$  of  $G$ , we have  $\phi(v, L_e) < \phi(v, R_e)$ .

Consider two  $G$ -edges  $(L_e, a)(R_e, a)$  and  $(L_f, b)(R_f, b)$  of  $G \square H$ , for which  $e$  and  $f$  are in the same strict queue of  $G$ . By Equation (2), without loss of generality,  $\sigma(L_e) < \sigma(L_f)$  and  $\sigma(R_e) < \sigma(R_f)$ . Thus  $\phi(L_e, a) < \phi(L_f, b)$  and  $\phi(R_e, a) < \phi(R_f, b)$ . Hence for each strict queue in  $G$ , the corresponding  $G$ -edges of  $G \square H$  form a strict queue in  $\phi$ .

Consider two  $H$ -edges  $(v, L_e)(v, R_e)$  and  $(w, L_f)(w, R_f)$  of  $G \square H$ , for which  $e$  and  $f$  are in the same queue of  $H$ . By Equation (1), without loss of generality,  $\pi(L_e) \leq \pi(L_f)$  and  $\pi(R_e) \leq \pi(R_f)$ . First suppose that  $\sigma(v) \leq \sigma(w)$ . Then  $\phi(v, L_e) \leq \phi(w, L_f)$  and  $\phi(v, R_e) \leq \phi(w, R_f)$ . Thus  $(v, L_e)(v, R_e)$  and  $(w, L_f)(w, R_f)$  are not nested in  $\phi$ . Now suppose that  $\sigma(w) < \sigma(v)$ . Then  $\phi(w, L_f) < \phi(w, R_f) < \phi(v, L_e) < \phi(v, R_e)$ . Thus  $(v, L_e)(v, R_e)$  and  $(w, L_f)(w, R_f)$  are disjoint. Thus for each queue in  $H$ , the corresponding  $H$ -edges of  $G \square H$  form a queue in  $\phi$ . Therefore  $\phi$  admits a  $(\text{sqn}(G) + \text{qn}(H))$ -queue layout of  $G \square H$ .

Now we prove the lower bound. By Lemmata 1 and 8(a),  $\text{qn}(G \square H) > \eta(G \square H)/2 = (\eta(G) + \eta(H))/2$ . The result follows since  $\eta(G) \geq \frac{1}{c} \text{sqn}(G)$  and  $\eta(H) \geq \frac{1}{c} \text{qn}(H)$ .  $\square$

Theorem 2 has the following immediate corollary.

**Corollary 1** For all graphs  $G_1, G_2, \dots, G_d$ ,

$$\text{qn}(G_1 \square G_2 \square \dots \square G_d) \leq \text{qn}(G_1) + \sum_{i=2}^d \text{sqn}(G_i) .$$

$\square$

## 5.1 Grids

A  $d$ -dimensional grid is a graph  $P_{n_1} \square P_{n_2} \square \dots \square P_{n_d}$ , for all  $n_i \geq 1$ . Heath and Rosenberg [19] determined the queue-number of every 2-dimensional grid.

**Lemma 9 ([19])** Every 2-dimensional grid has queue-number one.  $\square$

A generalised  $d$ -dimensional grid is a graph  $G = P_{n_1}^k \square P_{n_2}^k \square \dots \square P_{n_d}^k$ , for all  $k \geq 1$  and  $n_i \geq k + 1$ . Now  $P_n^k$  has  $kn - k(k + 1)/2$  edges. Thus  $\eta(P_n^k) = k - \frac{k(k+1)}{2n}$ . By Lemma 8(a),

$$\eta(G) = \sum_{i=1}^d \left( k - \frac{k(k+1)}{2n_i} \right) = dk - \frac{1}{2} k(k+1) \sum_{i=1}^d \frac{1}{n_i} . \quad (4)$$

Lemma 9 generalises as follows.

**Theorem 3** For all  $d \geq 2$ , the queue-number of a  $d$ -dimensional grid  $G = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d}$  satisfies:

$$\frac{d}{4} \leq \frac{1}{2} \left( d - \sum_{i=1}^d \frac{1}{n_i} \right) < \text{qn}(G) \leq d - 1 .$$

**Proof:** The lower bound follows from Lemma 1 and Equation (4) with  $k = 1$ .

For the upper bound, we have  $\text{qn}(P_{n_1} \square P_{n_2}) = 1$  by Lemma 9. Obviously  $\text{sqn}(P_{n_i}) = 1$  for all  $i \geq 3$ . Thus  $\text{qn}(G) \leq d - 1$  by Corollary 1.

We now give an alternative proof of the upper bound using a different construction. The graph  $G$  can be thought of as having vertex set  $\{(x_1, x_2, \dots, x_d) : 1 \leq x_i \leq n_i, 1 \leq i \leq d\}$ , where two vertices  $(x_1, x_2, \dots, x_d)$  and  $(y_1, y_2, \dots, y_d)$  are adjacent if and only if  $|x_i - y_i| = 1$  for some  $i$ , and  $x_j = y_j$  for all  $j \neq i$ . We say this edge is in the  $i$ -th dimension. For all  $s \geq 0$ , let  $V_s$  be the set of vertices

$$V_s = \{(x_1, x_2, \dots, x_d) : \sum_{i=1}^d x_i = s\} .$$

Order the vertices  $(V_0, V_1, \dots)$ , where each  $V_s$  is ordered lexicographically. If  $vw$  is an edge then  $v$  and  $w$  differ in exactly one coordinate, and  $v \in V_s$  and  $w \in V_{s+1}$  for some  $s$ . Thus if two edges  $vw$  and  $pq$  are nested then  $v, p \in V_s$  and  $w, q \in V_{s+1}$  for some  $s$ . Let  $Q_i$  be the set of edges in the  $i$ -th dimension. Consider two edges  $e$  and  $f$  in  $Q_i$ . Say

$$e = (x_1, x_2, \dots, x_d)(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_d) ,$$

and

$$f = (y_1, y_2, \dots, y_d)(y_1, \dots, y_{i-1}, y_i + 1, y_{i+1}, \dots, y_d) .$$

Without loss of generality  $(x_1, x_2, \dots, x_d) \prec (y_1, y_2, \dots, y_d)$ , which implies that

$$(x_1, \dots, x_{i-1}, x_i + j, x_{i+1}, \dots, x_d) \prec (y_1, \dots, y_{i-1}, y_i + j, y_{i+1}, \dots, y_d) .$$

Thus  $e$  and  $f$  are not nested, and  $Q_i$  is a queue. Hence we have a  $d$ -queue layout. (At this point we have in fact proved that the lexicographical order admits a  $d$ -queue layout.)

We now prove that  $Q_{d-1} \cup Q_d$  is a queue, and thus we obtain the claimed  $(d-1)$ -queue layout. Suppose two edges  $e \in Q_{d-1}$  and  $f \in Q_d$  are nested. Say

$$e = (x_1, x_2, \dots, x_d)(x_1, x_2, \dots, x_{d-1} + 1, x_d) ,$$

and

$$f = (y_1, y_2, \dots, y_d)(y_1, y_2, \dots, y_{d-1}, y_d + 1) .$$

Then for some  $s$ , both  $(x_1, x_2, \dots, x_d)$  and  $(y_1, y_2, \dots, y_d)$  are in  $V_s$ , and both  $(x_1, x_2, \dots, x_{d-1} + 1, x_d)$  and  $(y_1, y_2, \dots, y_{d-1}, y_d + 1)$  are in  $V_{s+1}$ .

**Case 1.**  $(x_1, x_2, \dots, x_d) \prec (y_1, y_2, \dots, y_d)$ : Let  $j$  be the first dimension for which  $x_j < y_j$ . If  $j \leq d - 2$  then

$$(x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \dots, y_{d-1}, y_d + 1) ,$$

which implies that  $e$  and  $f$  are not nested. Observe that  $j \neq d$  as  $(x_1, x_2, \dots, x_d)$  and  $(y_1, y_2, \dots, y_d)$  differ in at least two coordinates, since  $\sum_i x_i = \sum_i y_i$ . Thus  $j = d - 1$ . That is,

$$x_{d-1} \leq y_{d-1} - 1 . \quad (5)$$

Since  $e$  and  $f$  are nested, we have  $(y_1, y_2, \dots, y_{d-1}, y_d + 1) \prec (x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d)$ , which implies that  $y_{d-1} \leq x_{d-1} + 1$ . By Equation (5),  $x_{d-1} = y_{d-1} - 1$ . Since  $x_{d-1} + x_d = y_{d-1} + y_d$ , we have  $x_d = y_d + 1$ , which implies that

$$(y_1, y_2, \dots, y_{d-1}, y_d + 1) = (x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) .$$

That is, the right-hand endpoints of  $e$  and  $f$  are the same vertex. Hence  $e$  and  $f$  are not nested.

**Case 2.**  $(y_1, y_2, \dots, y_d) \prec (x_1, x_2, \dots, x_d)$ : By the same argument employed above, the first coordinate for which  $(y_1, y_2, \dots, y_d)$  and  $(x_1, x_2, \dots, x_d)$  differ is  $d - 1$ . That is,

$$y_{d-1} < x_{d-1} . \quad (6)$$

Since  $e$  and  $f$  are nested, we have  $(x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \dots, y_{d-1}, y_d + 1)$ . Thus  $x_{d-1} + 1 < y_{d-1}$ , which contradicts Equation (6). Hence  $e$  and  $f$  are not nested.

Therefore  $Q_1, Q_2, \dots, Q_{d-2}, Q_{d-1} \cup Q_d$  is the desired  $(d - 1)$ -queue layout.  $\square$

More generally we have the following.

**Theorem 4** *The queue-number of a generalised  $d$ -dimensional grid  $G = P_{n_1}^k \square P_{n_2}^k \square \dots \square P_{n_d}^k$  (where  $n_i \geq k + 1$ ) satisfies:*

$$\frac{dk}{4} \leq \frac{dk}{2} - \frac{k(k+1)}{4} \sum_{i=1}^d \frac{1}{n_i} < \text{qn}(G) \leq \lceil (d - \frac{1}{2})k \rceil .$$

**Proof:** By Lemma 6,  $\text{qn}(P_n^k) = \lceil \frac{k}{2} \rceil$  and  $\text{sqn}(P_n^k) \leq k$ . Thus, the upper bound follows from Corollary 1. Thus the lower bound follows from Lemma 1 and Equation (4).  $\square$

By Theorem 4 with  $k = n - 1$  we have the following.

**Corollary 2** *The queue-number of the  $d$ -dimensional Hamming graph  $G = K_n \square K_n \square \dots \square K_n$  satisfies:*

$$\frac{d(n-1)}{4} < \text{qn}(G) \leq \lceil (d - \frac{1}{2})(n-1) \rceil .$$

A generalised  $d$ -dimensional toroidal grid is a graph  $C_{n_1}^k \square C_{n_2}^k \square \dots \square C_{n_d}^k$  for all  $k \geq 1$  and  $n_i \geq 2k + 1$ .

**Theorem 5** *The queue-number of a generalised toroidal grid  $G = C_{n_1}^k \square C_{n_2}^k \square \dots \square C_{n_d}^k$  (where  $n_i \geq 2k + 1$ ) satisfies:*

$$\frac{kd}{2} < \text{qn}(G) \leq (2d - 1)k .$$

**Proof:** Since  $\eta(G) = kd$ , we have that  $\text{qn}(G) > \frac{kd}{2}$  by Lemma 1. Thus  $\text{qn}(G) \geq \lfloor \frac{d}{2} \rfloor + 1$ . By Lemma 7,  $\text{qn}(C_{n_1}^k) \leq k$  and  $\text{sqn}(C_{n_1}^k) \leq 2k$ . By Corollary 1,  $\text{qn}(G) \leq 2k(d - 1) + k = (2d - 1)k$   $\square$

## 6 Direct and Strong Products

We have the following bounds on the queue-number of direct and strong products.

**Theorem 6** For all graphs  $G$  and  $H$ ,

$$\text{qn}(G \times H) \leq 2 \text{sqn}(G) \cdot \text{qn}(H) .$$

Furthermore, if  $\text{sqn}(G) \leq c \cdot \eta(G)$  and  $\text{qn}(H) \leq c \cdot \eta(H)$ , then

$$\text{qn}(G \times H) > \frac{1}{c^2} \text{sqn}(G) \cdot \text{qn}(H) .$$

**Proof:** First we prove the upper bound. Let  $k := \text{sqn}(G)$ , and let  $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$  be a strict  $k$ -queue layout of  $G$ . Let  $\ell := \text{qn}(H)$ , and let  $(\pi, \{P_1, P_2, \dots, P_\ell\})$  be an  $\ell$ -queue layout of  $H$ . For  $1 \leq i \leq k$  and  $1 \leq j \leq \ell$ , let

$$\begin{aligned} E'_{i,j} &:= \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(a) < \pi(b)\} \\ E''_{i,j} &:= \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(b) < \pi(a)\} \end{aligned}$$

Then  $\{E'_{i,j}, E''_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$  is a partition of  $E(G \times H)$  into  $2k\ell$  sets. Let  $\phi$  be the vertex ordering of  $G \times H$  in which  $\phi(v, a) < \phi(w, b)$  if and only if  $\sigma(v) < \sigma(w)$ , or  $v = w$  and  $\pi(a) < \pi(b)$ .

We claim that each set  $E'_{i,j}$  and  $E''_{i,j}$  is a queue in  $\phi$ .

Suppose that two edges  $(v, a)(w, b), (x, c)(y, d) \in E'_{i,j}$  are nested. Without loss of generality,  $\phi(v, a) < \phi(x, c) < \phi(y, d) < \phi(w, b)$ . If  $v \neq x$  and  $y \neq w$ , then  $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$ , and the edges  $vw, xy \in Q_i$  are nested in  $\sigma$ . If  $v \neq x$  and  $y = w$ , then  $\sigma(v) < \sigma(x) < \sigma(y) = \sigma(w)$ , and the edges  $vw, xy \in Q_i$  overlap in  $\sigma$ . If  $v = x$  and  $y \neq w$ , then  $\sigma(v) = \sigma(x) < \sigma(y) < \sigma(w)$ , and the edges  $vw, xy \in Q_i$  overlap in  $\sigma$ . Each of these outcomes contradict the assumption that  $Q_i$  is a strict queue in  $\sigma$ . Otherwise  $v = x$  and  $y = w$ , in which case  $\pi(a) < \pi(c) < \pi(d) < \pi(b)$ , and  $ab$  and  $cd$  are nested in  $\pi$ . This contradicts the assumption that  $P_j$  is a queue in  $\pi$ . Thus each  $E'_{i,j}$  is queue in  $\phi$ . By symmetry, each  $E''_{i,j}$  is also a queue in  $\phi$ .

Now we prove the lower bound. Lemmata 1 and 8(b) imply that

$$\text{qn}(G \times H) > \eta(G \times H)/2 = \eta(G) \cdot \eta(H) \geq \frac{1}{c} \text{sqn}(G) \cdot \frac{1}{c} \text{qn}(H) .$$

□

**Theorem 7** For all graphs  $G$  and  $H$ ,

$$\text{qn}(G \boxtimes H) \leq 2 \text{sqn}(G) \cdot \text{qn}(H) + \text{sqn}(G) + \text{qn}(H) .$$

Furthermore, if  $\text{sqn}(G) \leq c \cdot \eta(G)$  and  $\text{qn}(H) \leq c \cdot \eta(H)$ , then

$$\text{qn}(G \boxtimes H) > \frac{1}{c^2} \text{sqn}(G) \cdot \text{qn}(H) + \frac{1}{2c} (\text{sqn}(G) + \text{qn}(H)) .$$

**Proof:** To prove the upper bound, observe that the vertex ordering  $\phi$  defined in Theorems 2 and 6 is the same. By Theorem 2,  $\phi$  admits a  $\text{sqn}(G) + \text{qn}(H)$ -queue layout of  $G \square H$ . By Theorem 6,  $\phi$  admits a

$2 \text{sqn}(G) \cdot \text{qn}(H)$ -queue layout of  $G \times H$ . Since  $G \boxtimes H = (G \square H) \cup (G \times H)$ ,  $\phi$  admits the claimed queue layout of  $G \boxtimes H$ .

For the lower bound, Lemmata 1 and 8(c) imply that

$$\text{qn}(G \boxtimes H) > \frac{1}{2}\eta(G \boxtimes H) = \eta(G) \cdot \eta(H) + \frac{1}{2}(\eta(G) + \eta(H)) \geq \frac{1}{c} \text{sqn}(G) \cdot \frac{1}{c} \text{qn}(H) .$$

□

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