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A sufficient condition for bicolorable hypergraphs

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In this note we prove Sterboul's conjecture, that provides a sufficient condition for the bicolorability of hypergraphs.

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In [2], Fournier and Las Vergnas gave a sufficient condition for the bicolorability of hypergraphs. Their theorem was a weaker form of a conjecture due to Sterboul, that we prove here. These facts are reproduced in [1] and [3].

A hypergraph is a pair $H = (V, \mathcal{E})$, where the elements of V are the *vertices*, and the elements of $\mathcal{E}$ are subsets of V and are called the *edges*. A function $c : V \to \{1, 2\}$ is called a *bipartition* of V, and for $x \in V$, we call $c(x)$ the *color* of x. If c is such that for any $e \in \mathcal{E}$ with $|e| \geq 2$ both colors occur, then c is called a *bicoloration* of H . If only one color occurs, the edge is said to be *monochromatic*. If a hypergraph admits a bicoloration we say that it is *bicolorable*.

A sequence $(x_1, e_1, x_2, ..., e_k, x_1)$, where the e_i's are distinct edges, the x_i's are distinct vertices, and $k \geq 3$, is said to be a *cycle* if $x_i \in e_{i-1} \cap e_i$ for $i = 2, ..., k$ and $x_1 \in e_1 \cap e_k$. A cycle is said to be *odd* if it has an odd number of edges.

An odd cycle $(x_1, e_1, x_2, ..., e_k, x_1)$ such that two non-consecutive edges are disjoint and $|e_i \cap e_{i+1}| = 1$ for $i = 1, 2, ..., k-1$, is called a *Sterboul cycle*. If a hypergraph H has no Sterboul cycle, it is said to be a *Sterboul hypergraph*.

Then we can word Sterboul's conjecture as follows:

Theorem 1 *If H is a Sterboul hypergraph, then H is bicolorable.*

Proof: The proof works by induction on the number of edges.

When the hypergraph has no edge, the theorem clearly holds.

The general step assumes that we have a hypergraph $H = (V, \mathcal{E})$ and $e_0 \in \mathcal{E}$ such that $H \backslash e_0 = (V, \mathcal{E} \backslash e_0)$ has a bicoloration $c : V \to \{1, 2\}$.

We can assume that e_0 has size at least 2, and that c leaves e_0 monochromatic, or else we have nothing to do.

Now we use the following algorithm to transform the bipartition c into a bicoloration of H. The algorithm switches successively the colors of some vertices of H that are contained in a monochromatic edge in the current bipartition. It constructs an arborescence $G_0 = (V_0, E_0)$ and a mapping $g: V_0 \to \mathcal{E}$ that keep track of the running of the algorithm: the vertices of G_0 are those whose colors were switched, and g associates a vertex with the monochromatic edge that caused its color switch.

The vertices are chosen with a DFS (Depth-First Search) method, and to do so the algorithm uses a LIFO (Last In First Out) stack $\mathcal{P}$ that contains the set of vertices whose colors have been switched, and so that might be in a monochromatic edge. top($\mathcal{P}$) returns the last vertex entered in $\mathcal{P}$; drop($\mathcal{P}$) removes top($\mathcal{P}$) out of $\mathcal{P}$; and put(x,$\mathcal{P}$) enters a new vertex x in $\mathcal{P}$.

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INPUT: A hypergraph H = (V,E) and a bipartition c such that e0 ∈ E is the
only monochromatic edge.
OUTPUT: A bicoloration of H or an Error message only if H is not a Sterboul
hypergraph.

let x0 ∈ e0
G0 := ({x0}, ∅)
g(x0) := e0
switch c(x0)
put(x0, P)
While P ≠ ∅ do
     let v = top(P)
     If there exists e ∈ E, |e| ≥ 2, monochromatic such that v ∈ e then
          If e\V0 = ∅ then
               return Error
          else
               let w ∈ e\V0
               V0 := V0 ∪ w ; E0 := E0 ∪ (vw)
               g(w) := e
               switch c(w)
               put(w, P)
          end If
     else
          drop(P)
     end If
end While
```

First we remark that G_0 is indeed an arborescence since the end point of each new arc of G_0 is a new vertex. Then for a given $x \in V_0$ there is a unique path in G_0 from x_0 to x. Moreover when x is at the top of $\mathcal{P}$, then $\mathcal{P}$ contains exactly the vertices of that path (because $\mathcal{P}$ is a LIFO stack).

We can also remark that if the algorithm does not return Error, then at each iteration either a new vertex is put into $\mathcal{P}$, or a vertex is dropped out of $\mathcal{P}$. Since a vertex appears at most once in G_0 and thus can be put at most once in $\mathcal{P}$, we have at most $2|V|$ iterations, and the algorithm ends.

We note $\mathcal{P}^{(i)}$, $G_0^{(i)} = (V_0^{(i)}, E_0^{(i)})$, $c^{(i)}$, $g^{(i)}$ the values of $\mathcal{P}$, $G_0 = (V_0, E_0)$, c, g (respectively) at the beginning of the i-th iteration. We also note $c^{(0)}$ the original bipartition (which is different from $c^{(1)}$ because of the switch of $c(x_0)$).

To prove the validity of the algorithm, we have to prove that:
- `Error` cannot be returned if H is a Sterboul hypergraph.
- The output of the algorithm if no `Error` occurs is a bicoloration.

Before proving those points, we claim the following:

Claim 1 *Suppose that H is a Sterboul hypergraph. Consider the beginning of the i-th iteration. Let $\mathcal{P}^{(i)} = (x_k...x_0)$, and $e_j = g^{(i)}(x_j)$ for $j = 0, ..., k$. Then we have:*
(a) *For each $j = 0, ..., k$, x_j is the only vertex of its color in e_j.*
(b) *For each $j = 0, ..., k-1$ we have $e_j \cap e_{j+1} = \{x_j\}$.*
(c) *Two non-consecutive edges are disjoint.*

Proof: The proof works by induction on i.
For $i = 1$ the claim clearly holds since $\mathcal{P}^{(1)} = (x_0)$.
We now consider $i \geq 1$ and we suppose the claim holds at iteration i. We are going to prove that it also holds at iteration $i+1$. Let $\mathcal{P}^{(i)} = (x_k...x_0)$, and $e_j = g^{(i)}(x_j)$ for $j = 0, ..., k$.
If during the i-th iteration the algorithm dropped x_k out of $\mathcal{P}$ (that is $\mathcal{P}^{(i+1)} = (x_{k-1}...x_0)$), the claim clearly holds at iteration $i+1$. Thus we assume that the algorithm found $e_{k+1} \in \mathcal{E}$ with $x_k \in e_{k+1}$ that is monochromatic for $c^{(i)}$, and $x_{k+1} \in e_{k+1} \backslash V_0^{(i)}$ (because we assume that there is a $(i+1)$-th iteration or else the claim is true) so that $\mathcal{P}^{(i+1)} = (x_{k+1} x_k ... x_0)$.
Since (a) holds at iteration i, we know that if $w \in e_k \backslash x_k$ then $c^{(i)}(w) \neq c^{(i)}(x_k)$. As $x_k \in e_{k+1}$ and e_{k+1} is monochromatic for $c^{(i)}$, then $e_k \cap e_{k+1} = \{x_k\}$ and (b) holds at iteration $i+1$.
Suppose $j_0 = max\{0 \leq j \leq k-1 | e_j \cap e_{k+1} \neq \emptyset\}$ exists, and let $y \in e_{j_0} \cap e_{k+1}$. If $k - j_0$ is odd, then $(y, e_{j_0}, x_{j_0}, ..., e_{k+1}, y)$ is a Sterboul cycle (because (c) holds at iteration i and (b) holds at iteration $i+1$), so $k - j_0$ is even. But then since (a) holds at iteration i, we have $c^{(i)}(y) \neq c^{(i)}(x_{j_0})$ ($y \neq x_{j_0}$ by definition of j_0), $c^{(i)}(x_{j_0}) \neq c^{(i)}(x_{j_0+1})$, ..., $c^{(i)}(x_{k-1}) \neq c^{(i)}(x_k)$ and then $c^{(i)}(y) \neq c^{(i)}(x_k)$, which is impossible because e_{k+1} is monochromatic for $c^{(i)}$. Hence j_0 does not exist, and (c) holds at iteration $i+1$.
Thus $x_{k+1} \notin e_j$ for all $j = 0, ..., k$. Since the only color switch done during the i-th iteration concerns x_{k+1}, then (a) holds at iteration $i+1$.
This achieves to prove the claim. □

Now we are able to prove the validity of the algorithm.

- Suppose that H is a Sterboul hypergraph. Consider an iteration i, and let $\mathcal{P}^{(i)} = (x_k...x_0)$. If $k = 1$ then from (b) of the claim we have $x_1 \notin e_0$. If $k \geq 2$ then from (c) of the claim we also have $x_k \notin e_0$. This proves that we always have $e_0 \cap V_0 = \{x_0\}$.

If `Error` is returned, it means that at a given iteration i_0, the algorithm found an edge e monochromatic for $c^{(i_0)}$ such that $e \backslash V_0^{(i_0)} = \emptyset$. Then e was also monochromatic for $c^{(0)}$, but e_0 was the only such edge, so we have a contradiction because we have just seen that we must have $e_0 \cap V_0^{(i_0)} = \{x_0\}$. Thus if H is a Sterboul hypergraph, `Error` cannot be returned.

- Finally, if the bipartition obtained by the algorithm is not a bicoloration then we have some $e \in \mathcal{E}$ that is monochromatic. Consider i_0 the last iteration during which the algorithm dropped a vertex of e out of $\mathcal{P}$ (i_0 exists or else e was monochromatic with $c^{(0)}$, but e_0 was the only such edge and e_0 is not monochromatic at the end of the algorithm), and let y_0 be that vertex. Then e was not monochromatic with $c^{(i_0)}$ or else the algorithm would have considered e during the iteration i_0 instead of dropping y_0. But since no color switch concerning a vertex of e occurs afterwards (by choice of i_0), we have a contradiction. Thus the final bipartition is a bicoloration.

So the algorithm is correct and the theorem is proved. □

We can slightly modify the algorithm so that it gives a Sterboul cycle instead of just returning `Error` when the hypergraph is not Sterboul (in order to have a certificate that the hypergraph is not Sterboul).

To do so we just have to check at each iteration that the properties of Claim 1 still hold. If not it means that the monochromatic edge considered intersects the path induced by the stack, and a Sterboul cycle can be easily found.

Our algorithm finds a bicoloration for Sterboul hypergraphs in polynomial time. However it cannot be used to recognize bicolorable hypergraphs (since a Sterboul hypergraph may be bicolorable) and neither to recognize Sterboul hypergraphs (since it may happen that it gives a bicoloration for a hypergraph that is not Sterboul).

The problem of recognizing bicolorable hypergraphs is well-known to be NP-complete [4]. But we leave the following question open: what is the complexity of recognizing Sterboul hypergraphs?

References

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