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# Asymptotic properties of the magnetic integrated density of states *

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**Abstract**

This article could be regarded as a supplement to [11] where we considered the Schrödinger operator with constant magnetic field and decaying electric potential, and studied the asymptotic behaviour of the discrete spectrum as the coupling constant of the magnetic field tends to infinity. To describe this behaviour when the kernel of the magnetic field is not trivial, we introduced a measure $\mathcal{D}(\lambda)$ defined on $(-\infty, 0)$ called the "magnetic integrated density of states". In this article, we study the asymptotic behaviour of this measure as $\lambda \uparrow 0$ and as $\lambda \downarrow \lambda_0$, $\lambda_0$ being the lower bound of the support of $\mathcal{D}$.

## 1 Introduction

In [11], we considered the Schrödinger operator

$$H(\mu) = H_0(\mu) + V, \quad \text{with} \quad H_0(\mu) = (i\nabla + \mu A)^2 ,$$

where $A : \mathbb{R}^m \to \mathbb{R}^m$, $m \geq 2$, is the magnetic potential, $V : \mathbb{R}^m \to \mathbb{R}$ is the electric potential, and $\mu \geq 0$ is the magnetic-field coupling constant. Under the assumptions that the magnetic field

$$B := \{B_{j,l}\}_{j,l=1}^m , \quad B_{j,l} := \frac{\partial A_l}{\partial X_j} - \frac{\partial A_j}{\partial X_l}, \quad j, l = 1, \dots, m ,$$

is constant with respect to $X \in \mathbb{R}^m$, $B \neq 0$, and $V$ is $-\Delta$-form-compact, the asymptotic behaviour as $\mu \to \infty$ of the discrete spectrum of $H(\mu)$ was investigated. The main result in [11] concerning the case

$$k := \dim \operatorname{Ker} B \geq 1 , \tag{1.1}$$

is the source of motivation for the present paper. That is why, this result is reproduced here in detail.

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In the sequel, we shall always assume that (1.1) is satisfied. Hence, in particular, $m \geq 3$. Moreover, (1.1) implies

$$\sigma(H_0(\mu)) = \sigma_{\rm ess}(H_0(\mu)) = [\mu\Lambda, \infty)\,,$$

where $\Lambda := \frac{1}{2}{\rm Tr}\sqrt{B^*B}$ is the first Landau level. On the other hand, since $V$ is assumed to be $-\Delta$-form-compact, the Kato-Simon inequality implies that $V$ is also $H_0(\mu)$-form-compact, $\mu > 0$, and hence

$$\sigma_{\rm ess}(H(\mu)) = \sigma_{\rm ess}(H_0(\mu)),\ \mu \geq 0\,.$$

Set $2d := m - k = {\rm rank}\, B$. On $\mathbb{R}^m \equiv \mathbb{R}^{2d+k}$, introduce the Cartesian coordinates $X = (x, y, z)$ with $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, and $z \in \mathbb{R}^k$, such that $(x, y) \in {\rm Ran}\, B$, $z \in {\rm Ker}\, B$ and $\mathcal{B} := d\mathcal{A} = \sum_{j=1}^d b_j dy^j \wedge dx^j$ where $\mathcal{A} := \sum_{l=1}^m A_l dX^l$ (see [8, Subsection 2.3]). Denote by $B_+$ the restriction of the matrix $\sqrt{B^*B}$ to ${\rm Ran}\, B$. If we consider ${\rm Ran}\, B = \mathbb{R}^{2d}$ as a symplectic vector space with symplectic form $\mathcal{B}$, then

$$\frac{\mathcal{B}^d}{d!} = b_1 \dots b_d\, dx^1 \wedge dy^1 \dots dx^d \wedge dy^d = \sqrt{\det B_+}\, dx^1 \wedge dy^1 \dots dx^d \wedge dy^d$$

is a volume form (see [5, p. 274]). In what follows we shall use the short-hand notation $X_\perp = (x, y)$, $X_\perp \in {\rm Ran}\, B = \mathbb{R}^{2d}$.

Our next goal is to define an auxiliary operator $\chi(X_\perp)$ which acts in $L^2(\mathbb{R}^k)$, with $\mathbb{R}^k = {\rm Ker}\, B$, and depends on the parameter $X_\perp \in \mathbb{R}^{2d}$, $\mathbb{R}^{2d} = {\rm Ran}\, B$.

We shall write $V \in \mathcal{L}_r$, $r \geq 1$, if for each $\varepsilon > 0$,

$$V = V_1 + V_2, \tag{1.2}$$

where $V_1 \in L^r(\mathbb{R}^m)$ and $\sup_{X\in\mathbb{R}^m} |V_2(X)| \leq \varepsilon$.

Assume $V \in \mathcal{L}_{m/2}$. Hence, in particular, $V$ is $-\Delta$-form-compact.

Fix $\varepsilon > 0$ and write $V$ as in (1.2). We shall say that $X_\perp \in \mathbb{R}^{2d}$ is in the regularity set of $V_1$ if the integral $\int_{\mathbf{R}^k} |V_1(X_\perp, z)|^{m/2}\, dz$ is defined and finite. Obviously, the complement of the regularity set of $V_1$ is a null set. Moreover, $V(X_\perp, .)$ is $-\Delta$-form-bounded with zero bound for every $X_\perp$ in the regularity set of $V_1$. Fix $X_\perp$ in the regularity set of $V_1$, and set

$$\chi(X_\perp) = \chi(V(X_\perp)) := -\Delta_z + V(X_\perp, z)$$

where the sum should be understood in the sense of quadratic forms.

Obviously, for almost every $X_\perp$ in the regularity set of $V_1$, the operator $V(X_\perp, .)$ is $-\Delta$-form-compact, and we have $\sigma_{\rm ess}(\chi(X_\perp)) = [0, \infty)$.

Let $T$ be a linear selfadjoint operator in a Hilbert space. Denote by $P_I(T)$ the spectral projection of $T$ corresponding to an interval $I \subset \mathbb{R}$. Set

$$N(\lambda; T) := {\rm rank}\, P_{(-\infty,\lambda)}(T), \quad \lambda \in \mathbb{R},$$

$$n_\pm(\lambda; T) := {\rm rank}\, P_{(\lambda,\infty)}(\pm T), \quad \lambda > 0.$$

For $\lambda < 0$ introduce the magnetic integrated density of states

$$\mathcal{D}(\lambda) := \int_{\mathbb{R}^{2d}} N(\lambda; \chi(X_\perp))\, dX_\perp. \tag{1.3}$$

In [11] it is shown that under the assumptions $V \in \mathcal{L}_{m/2}$ and $\lambda < 0$, the right-hand side of (1.3) is well-defined. In particular, $\mathcal{D}(\lambda)$ is independent of the particular choice of $\varepsilon > 0$ and the related decomposition (1.2) used for the definition of $\chi(X_\perp)$.

The main result of [11] concerning the case $k \geq 1$ (see (1.1)) is reproduced as follows.

**Theorem 1.1** [11, Theorem 2.2] *Suppose that $B$ is constant, $B \neq 0$, and $k \geq 1$. Assume that $V \in \mathcal{L}_{m/2}$, $V \leq 0$, and $V \not\equiv 0$. Let $\lambda < 0$ be a continuity point of the function $\mathcal{D}$. Then we have*

$$\lim_{\mu \to \infty} \mu^{-d} N(\mu\Lambda + \lambda; H(\mu)) = (2\pi)^{-d} \sqrt{\det B_+}\, \mathcal{D}(\lambda). \tag{1.4}$$

In order to explain why $\mathcal{D}(\lambda)$ was named the "magnetic integrated density states" in [11], we shall recall briefly the definition and the basic property of the usual (spatial) integrated density of states for the operator $\tilde{H} := -\Delta + Q$ where $Q$ is a periodic function on $\mathbb{R}^m$. Denote by $\Gamma$ (respectively, by $\Gamma^*$) the lattice (respectively, the dual lattice) of the periods of $Q$, and set $\mathbf{T} := \mathbb{R}^m/\Gamma$, $\mathbf{T}^* := \mathbb{R}^m/\Gamma^*$. Define the auxiliary operator $\tilde{\chi}(\mathbf{k}) := (i\nabla - \mathbf{k})^2 + Q$, $\mathbf{k} \in \mathbf{T}^*$, on the Sobolev space $\mathrm{H}^2(\mathbf{T})$. Introduce the (spatial) integrated density of states

$$\tilde{\mathcal{D}}(\lambda) := \int_{\mathbf{T}^*} N(\lambda; \tilde{\chi}(\mathbf{k}) d\mathbf{k},\ \lambda \in \mathbb{R},$$

(note that our definition differs slightly from the standard one: usually, the integrated density of states is defined as $(2\pi)^{-m}\tilde{\mathcal{D}}(\lambda)$). Further, set $\mathbf{T}_r := \mathbb{R}^m/r\Gamma$ with an integer $r \geq 1$. Evidently, $\mathrm{vol}\,\mathbf{T}_r = r^m \mathrm{vol}\,\mathbf{T}$. Define the operator $\tilde{H}_r := -\Delta + Q$ on $\mathrm{H}^2(\mathbf{T}_r)$. Then we have

$$\lim_{r \to \infty} r^{-m} N(\lambda; \tilde{H}_r) = (2\pi)^{-m}\, \mathrm{vol}\,\mathbf{T}\, \tilde{\mathcal{D}}(\lambda),\ \lambda \in \mathbb{R}, \tag{1.5}$$

(see e.g. [13, Theorem XIII.101] or [14, Subsection 4.3])).

The formal resemblance of (1.4) and (1.5) is the motivation for the choice of the name "magnetic integrated density of states" for $\mathcal{D}(\lambda)$.

The function $\mathcal{D}(\lambda)$, $\lambda < 0$, is non-negative, and non-decreasing. Set

$$\lambda_0 := \begin{cases} -\infty \text{ if } \quad \mathcal{D}(\lambda) > 0 \quad \text{for all} \quad \lambda < 0, \\ \sup\{\lambda \in (-\infty, 0) | \mathcal{D}(\lambda) = 0\} \quad \text{otherwise.} \end{cases} \tag{1.6}$$

Throughout the paper we suppose $\lambda_0 < 0$.

The aim of the paper is to study the asymptotic behaviour of $\mathcal{D}(\lambda)$ as $\lambda \uparrow 0$, and as $\lambda \downarrow \lambda_0$.

In Section 2 we investigate the asymptotic behaviour of $\mathcal{D}(\lambda)$ as $\lambda \uparrow 0$, imposing some supplementary regularity assumptions on $V$; in particular, we assume that $V$ admits a power-like decay, i.e.

$$-V(X) \asymp \langle X \rangle^{-\alpha},\ \alpha > 0,\ \langle X \rangle := (1+|X|^2)^{1/2},\ X \in \mathbb{R}^m, \tag{1.7}$$

where $-V(X) \asymp \langle X \rangle^{-\alpha}$, $X \in \mathbb{R}^m$, means that there exist positive constants $c_1$ and $c_2$ such that the inequalities $c_1\langle X \rangle^{-\alpha} \leq -V(X) \leq c_2\langle X \rangle^{-\alpha}$ hold for each $X \in \mathbb{R}^m$; analogous short-hand notations are systematically used in the sequel.

The asymptotic behaviour of $\mathcal{D}(\lambda)$ is essentially different in the case of rapid decay (i.e. $\alpha > 2$), or slow decay (i.e. $\alpha \in (0,2]$). If $V$ decays rapidly, the type of behaviour of $\mathcal{D}(\lambda)$ depends on $k$: if $k = 1$, the unbounded growth of $\mathcal{D}(\lambda)$ as $\lambda \uparrow 0$ is described by a power-like function, if $k = 2$, this growth is described by a logarithmic function, and if $k \geq 3$, $\mathcal{D}(\lambda)$ remains bounded as $\lambda \uparrow 0$ (see Theorems 2.1-2.3 below). If $V$ decays slowly, the asymptotic behaviour of $\mathcal{D}(\lambda)$ as $\lambda \uparrow 0$ is of the same type for all $k \geq 1$ (see Theorem 2.5 below). Moreover, in Theorem 2.4 below we treat a particular example illustrating the asymptotic behaviour of $\mathcal{D}(\lambda)$ as $\lambda \uparrow 0$ in the border-line case $\alpha = 2$.

It should be noted that in the 1980s and the early 1990s the asymptotic behaviour of the quantity $N(\Lambda + \lambda; H(1))$ as $\lambda \uparrow 0$ was investigated by several authors (see [15], [17], [9], [10], [6]). The variety of apparently non-related asymptotic formulas concerning different values of the decay rate $\alpha$ and the deficiency index $k$, was somewhat unsatisfactory, and the problem to derive a uniform formula describing the asymptotic behaviour of $N(\Lambda+\lambda; H(1))$ as $\lambda \uparrow 0$ remained open. Comparing the earlier results on the asymptotics as $\lambda \uparrow 0$ of $N(\Lambda + \lambda; H(1))$ and the present results on the behaviour of $\mathcal{D}(\lambda)$, we find that generically the asymptotic equivalence

$$N(\Lambda + \lambda; H(1)) \sim (2\pi)^{-d}\sqrt{\det\, B_+}\,\mathcal{D}(\lambda) \tag{1.8}$$

holds as $\lambda \uparrow 0$.

In Section 3 we investigate the asymptotic behaviour as $\lambda \downarrow \lambda_0$ of $\mathcal{D}(\lambda)$ in the case $\lambda_0 > -\infty$. More precisely, we impose some regularity assumptions on $V$ which, in particular, imply that

$$\lambda_0 = \mathcal{E}_0 > -\infty$$

where

$$\mathcal{E}_0 := \inf_{X_\perp \in \mathbb{R}^{2d}} \mathcal{E}(X_\perp), \tag{1.9}$$

and

$$\mathcal{E}(X_\perp) := \inf \sigma(\chi(X_\perp)),\ X_\perp \in \mathbb{R}^{2d}, \tag{1.10}$$

and study the asymptotics as $\lambda \downarrow \mathcal{E}_0$ of $\mathcal{D}(\lambda)$.

An important special case of the potentials considered in Section 3 are the homogeneous ones $V(X) = -g|X|^{-\alpha}$ with $g > 0$ and $\alpha \in (0,1)$, in the case $m = 3$ (i.e. $k = 1$ and $d = 1$). The first asymptotic term of $\mathcal{D}(\lambda)$ as $\lambda \downarrow \mathcal{E}_0$ is calculated explicitly at the end of Section 3.

Finally, in Section 4 we assume $m = 3$, and introduce a class of asymptotically homogeneous potentials of order $\alpha \in [1, 2)$; in this case we have $\lambda_0 = -\infty$. If $\alpha = 1$, we find that $\mathcal{D}(\lambda)$ decays exponentially as $\lambda \to -\infty$, while in the case $\alpha \in (1, 2)$ its decay is power-like.

The main technical tools utilized in the proofs of these results consist of variational methods and continuous perturbation theory. Moreover, in the cases $k = 1, 2$, we decompose of the Birman-Schwinger operator $|V(X_\perp, .)|^{1/2}(-\Delta_z - \lambda)^{-1}|V(X_\perp, .)|^{1/2}$, $\lambda < 0$, or some related operators, into a sum of a rank-one operator divergent as $\lambda \uparrow 0$, and a compact operator of lower-order growth as $\lambda \uparrow 0$. Similar decompositions have been used in [16], [4], [3], and later in [9] (see also [8, Section 4.2]).

## 2 Asymptotic behaviour of $\mathcal{D}(\lambda)$ near the origin

**2.1.** In this subsection we treat the case of rapidly decaying potentials, i.e. we suppose that $V$ satisfies estimates (1.7) with $\alpha > 2$. In the first two theorems we deal with dimensions $k = 1$ and $k = 2$.

For $s > 0$, $\alpha > 2$, and $k = 1, 2$, set

$$\nu_1(s) := \mathrm{vol}\left\{X_\perp \in \mathbb{R}^{2d} | -\int_{\mathbb{R}^k} V(X_\perp, z)\, dz > s\right\}.$$

Note that

$$\nu_1(s) \asymp s^{-2d/(\alpha-k)},\ s \downarrow 0.$$

**Theorem 2.1** *Let $k = 1$. Suppose that $V$ satisfies* (1.7) *with $\alpha > 2$. In addition, assume*

$$\lim_{\delta\downarrow 0} \limsup_{s\downarrow 0} \frac{\nu_1((1-\delta)s)}{\nu_1(s)} = 1. \tag{2.1}$$

*Then we have*

$$\mathcal{D}(\lambda) \sim \nu_1(2|\lambda|^{1/2}),\ \lambda \uparrow 0. \tag{2.2}$$

**Remark.** The assumptions of Theorem 2.1 entail

$$\nu_1(2|\lambda|^{1/2}) \asymp |\lambda|^{-d/(\alpha-1)},\ \lambda \uparrow 0. \tag{2.3}$$

**Proof of Theorem** 2.1. The Birman-Schwinger principle implies

$$\mathcal{D}(\lambda) = \int_{\mathbb{R}^{2d}} n_+(1; G^{(1)}(\lambda, X_\perp))\, dX_\perp \tag{2.4}$$

where $G^{(1)}(\lambda, X_\perp)$ is an integral operator with kernel

$$\mathcal{G}^{(1)}(z_1, z_2; \lambda, X_\perp) := |V(X_\perp, z_1)|^{1/2}\mathcal{R}^{(1)}(z_1 - z_2; \lambda)|V(X_\perp, z_2)|^{1/2},$$

$$\mathcal{R}^{(1)}(z; \lambda) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{iz\zeta}}{\zeta^2 + |\lambda|}\, d\zeta = \frac{1}{2|\lambda|^{1/2}} e^{-|\lambda|^{1/2}|z|}.$$

Set

$$\mathcal{R}_1^{(1)}(\lambda) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{d\zeta}{\zeta^2+|\lambda|} = \frac{1}{2|\lambda|^{1/2}},$$

$$\mathcal{R}_2^{(1)}(z;\lambda) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{iz\zeta}-1}{\zeta^2+|\lambda|}\,d\zeta = \frac{1}{2|\lambda|^{1/2}}\left(e^{-|\lambda|^{1/2}|z|}-1\right),$$

$$\mathcal{G}_j^{(1)}(z_1,z_2;\lambda,X_\perp) := |V(X_\perp,z_1)|^{1/2}\mathcal{R}_j^{(1)}(z_1-z_2;\lambda)|V(X_\perp,z_2)|^{1/2},\ j=1,2,$$

and denote by $G_j^{(1)}(\lambda,X_\perp)$ the integral operator with kernel $\mathcal{G}_j^{(1)}(z_1,z_2;\lambda,X_\perp)$, $j=1,2$. Obviously,

$$G^{(1)}(\lambda,X_\perp) = G_1^{(1)}(\lambda,X_\perp)+G_2^{(1)}(\lambda,X_\perp)$$

and, therefore, the inequalities

$$n_+(1+\delta;G_1^{(1)}(\lambda,X_\perp)) - n_-(\delta;G_2^{(1)}(\lambda,X_\perp)) \le n_+(1;G^{(1)}(\lambda,X_\perp)) \le$$
$$n_+(1-\delta;G_1^{(1)}(\lambda,X_\perp)) + n_+(\delta;G_2^{(1)}(\lambda,X_\perp)) \tag{2.5}$$

hold for each $\delta\in(0,1)$. Next, we make use of the estimate

$$n_\pm(\delta;G_2^{(1)}(\lambda,X_\perp)) \le \operatorname{ent}\left\{\delta^{-2}\|G_2^{(1)}(\lambda,X_\perp)\|_2^2\right\},\ \delta>0, \tag{2.6}$$

where $\operatorname{ent} t$ denotes the integral part of the real number $t$, and $\|.\|_2$ denotes the Hilbert-Schmidt norm. Employing (1.7) and the elementary estimate

$$|\mathcal{R}_2^{(1)}(z;\lambda)| \le |\lambda|^{-(1-\varepsilon)/2}|z|^\varepsilon,\ z\in\mathbb{R},\ \lambda\ne 0,\ \varepsilon\in(0,1],$$

we get

$$\|G_2^{(1)}(\lambda,X_\perp)\|_2^2 \le$$
$$c_1'|\lambda|^{-(1-\varepsilon)}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|X_\perp|^2+z_1^2)^{-\alpha/2}(1+|X_\perp|^2+z_2^2)^{-\alpha/2}|z_1-z_2|^{2\varepsilon}\,dz_1dz_2 =$$
$$c_1|\lambda|^{-(1-\varepsilon)}\langle X_\perp\rangle^{-2(\alpha-1-\varepsilon)} \tag{2.7}$$

where $\varepsilon\in(0,1]$, $\varepsilon<(\alpha-1)/2$, and $c_1$ is independent of $\lambda$ and $X_\perp$. Inserting (2.7) into (2.6), we obtain

$$n_\pm(\delta;G_2^{(1)}(\lambda,X_\perp)) \le \operatorname{ent}\left\{\delta^{-2}c_1|\lambda|^{-(1-\varepsilon)}\langle X_\perp\rangle^{-2(\alpha-1-\varepsilon)}\right\}. \tag{2.8}$$

Integrating both sides of (2.8) with respect to $X_\perp\in\mathbb{R}^{2d}$, we derive the estimate

$$\int_{\mathbb{R}^{2d}} n_\pm(\delta;G_2^{(1)}(\lambda,X_\perp))dX_\perp \le c_2|\lambda|^{-d(1-\varepsilon)/(\alpha-1-\varepsilon)}$$

where $\varepsilon\in(0,1]$, $\varepsilon<(\alpha-1)/2$, and $c_2=c_2(\delta)$ is independent of $\lambda$. Hence,

$$\int_{\mathbb{R}^{2d}} n_\pm(\delta;G_2^{(1)}(\lambda,X_\perp))\,dX_\perp = o(|\lambda|^{-d/(\alpha-1)}),\ \lambda\uparrow 0. \tag{2.9}$$

Further, we note that $G_1^{(1)}(\lambda, X_\perp)$ is a rank-one operator whose only non-zero eigenvalue coincides with $-\frac{1}{2|\lambda|^{1/2}}\int_{\mathbb{R}} V(X_\perp, z)dz$.

Introduce the Heaviside function

$$\theta(t) = \begin{cases} 0 \text{ if } t \leq 0, \\ 1 \text{ if } t > 0. \end{cases}$$

Then we have

$$n_+(1 \pm \delta; G_1^{(1)}(\lambda, X_\perp)) = \theta\left(-\int_{\mathbb{R}} V(X_\perp, z)dz - (1 \pm \delta)2|\lambda|^{1/2}\right)$$

and, hence,

$$\int_{\mathbb{R}^{2d}} n_+(1 \pm \delta; G_1^{(1)}(\lambda, X_\perp))dX_\perp = \nu_1((1 \pm \delta)2|\lambda|^{1/2}). \tag{2.10}$$

The combination of (2.4), (2.5), (2.9), and (2.10) entails

$$\nu_1((1+\delta)2|\lambda|^{1/2}) + o(|\lambda|^{-d/(\alpha-1)}) \leq \mathcal{D}(\lambda) \leq$$
$$\nu_1((1-\delta)2|\lambda|^{1/2}) + o(|\lambda|^{-d/(\alpha-1)}),\ \lambda \uparrow 0, \delta \in (0,1). \tag{2.11}$$

Bearing in mind (2.1) and (2.3), we find that (2.11) implies (2.2). ♢

**Theorem 2.2** *Let $k = 2$. Assume that $V$ satisfies* (1.7) *with $\alpha > 2$. In addition, suppose that* (2.1) *is valid. Then we have*

$$\mathcal{D}(\lambda) \sim \nu_1(4\pi|\ln|\lambda||^{-1}),\ \lambda \uparrow 0. \tag{2.12}$$

**Remark.** The assumptions of Theorem 2.2 entail

$$\nu_1(4\pi|\ln|\lambda||^{-1}) \asymp |\ln|\lambda||^{2d/(\alpha-2)},\ \lambda \uparrow 0.$$

**Proof of Theorem** 2.2. As in the proof of the preceding theorem we have

$$\mathcal{D}(\lambda) = \int_{\mathbb{R}^{2d}} n_+(1; G^{(2)}(\lambda, X_\perp))\, dX_\perp, \tag{2.13}$$

where $G^{(2)}(\lambda, X_\perp)$ is an integral operator with kernel

$$\mathcal{G}^{(2)}(z_1, z_2; \lambda, X_\perp) := |V(X_\perp, z_1)|^{1/2}\mathcal{R}^{(2)}(z_1 - z_2; \lambda)|V(X_\perp, z_2)|^{1/2}$$

and

$$\mathcal{R}^{(2)}(z; \lambda) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{iz\zeta}}{|\zeta|^2 + |\lambda|}\, d\zeta = \frac{1}{2\pi}K_0(|\lambda|^{1/2}|z|),$$

$K_0$ being the modified Bessel function of zeroth order (see below (4.19)). Set

$$\mathcal{R}_1^{(2)}(\lambda) := \frac{1}{(2\pi)^2}\int_{|\zeta|<1} \frac{d\zeta}{|\zeta|^2 + |\lambda|} = \frac{1}{4\pi}\lambda \uparrow 0 g((1 + |\lambda|)/|\lambda|),$$

$$\mathcal{R}_2^{(2)}(z;\lambda) := \frac{1}{(2\pi)^2}\int_{|\zeta|<1}\frac{e^{iz.\zeta}-1}{|\zeta|^2+|\lambda|}\,d\zeta = -\frac{1}{\pi^2}\int_0^1\int_0^\pi \frac{\sin^2(\cos\varphi|z|r/2)}{r^2+|\lambda|}\,rdrd\varphi,$$

$$\mathcal{R}_3^{(2)}(z;\lambda) := \frac{1}{(2\pi)^2}\int_{|\zeta|>1}\frac{e^{iz.\zeta}}{|\zeta|^2+|\lambda|}\,d\zeta,$$

$$\mathcal{G}_j^{(2)}(z_1,z_2;\lambda,X_\perp) := |V(X_\perp,z_1)|^{1/2}\mathcal{R}_j^{(2)}(z_1-z_2;\lambda)|V(X_\perp,z_2)|^{1/2},\ j=1,2,3,$$

and denote by $G_j^{(2)}(\lambda,X_\perp)$ the integral operator with kernel $\mathcal{G}_j^{(2)}(z_1,z_2;\lambda,X_\perp)$, $j=1,2,3$. Obviously,

$$G^{(2)}(\lambda,X_\perp) = \sum_{j=1,2,3} G_j^{(2)}(\lambda,X_\perp).$$

Therefore,

$$\begin{aligned} & n_+(1+\delta;G_1^{(2)}(\lambda,X_\perp)) - n_-(\delta/2;G_2^{(2)}(\lambda,X_\perp)) - n_-(\delta/2;G_3^{(2)}(\lambda,X_\perp)) \\ & \quad\le\quad n_+(1;G^{(2)}(\lambda,X_\perp)) \qquad (2.14)\\ & \quad\le\quad n_+(1-\delta;G_1^{(2)}(\lambda,X_\perp)) + n_+(\delta/2;G_2^{(2)}(\lambda,X_\perp)) + n_+(\delta/2;G_3^{(2)}(\lambda,X_\perp))\,, \end{aligned}$$

for all $\delta\in(0,1)$. By analogy with (2.6) write

$$n_\pm(\delta;G_j^{(2)}(\lambda,X_\perp)) \le \mathrm{ent}\left\{\delta^{-2}\|G_j^{(2)}(\lambda,X_\perp)\|_2^2\right\},\ \delta>0,\ j=2,3. \qquad (2.15)$$

It is easy to verify the estimates

$$\|G_2^{(2)}(\lambda,X_\perp)\|_2^2 \le c_3\langle X_\perp\rangle^{-2(\alpha-2-\varepsilon)},\ \varepsilon\in(0,\alpha-2), \qquad (2.16)$$

$$\|G_3^{(2)}(\lambda,X_\perp)\|_2^2 \le c_4\langle X_\perp\rangle^{-2(\alpha-1)}, \qquad (2.17)$$

where $c_3$ and $c_4$ are independent of $\lambda$ and $X_\perp$. Inserting (2.17) or (2.16) into (2.15), and integrating with respect to $X_\perp\in\mathbb{R}^{2d}$, we get

$$\int_{\mathbb{R}^{2d}} n_\pm(\delta;G_j^{(2)}(\lambda,X_\perp))\,dX_\perp = O(1),\ \lambda\uparrow 0,\ j=2,3. \qquad (2.18)$$

Further, $G_1^{(2)}(\lambda,X_\perp)$ is a rank-one operator whose only non-zero eigenvalue coincides with $-\frac{1}{4\pi}\ln((1+|\lambda|)/|\lambda|)\int_{\mathbb{R}^2}V(X_\perp,z)dz$. Hence,

$$\int_{\mathbb{R}^{2d}} n_+(1\pm\delta;G_1^{(2)}(\lambda,X_\perp))dX_\perp = \nu_1((1\pm\delta)4\pi/\ln((1+|\lambda|)/|\lambda|),\ \delta\in(0,1). \qquad (2.19)$$

Combining (2.13), (2.14), (2.18), and (2.19), and making use of (2.1), we come to (2.12). $\diamondsuit$

**Theorem 2.3** *Let $k\ge 3$. Assume that*

$$\int_{\mathbb{R}^k}|V(.,z)|^{k/2}\,dz\in L^1_{\mathrm{loc}}(\mathbb{R}^{2d}),\ \lim_{|X_\perp|\to\infty}\int_{\mathbb{R}^k}|V(X_\perp,z)|^{k/2}\,dz = 0.$$

*Then we have*

$$\mathcal{D}(\lambda) = O(1),\ \lambda\uparrow 0. \qquad (2.20)$$

**Remark.** If $k \geq 3$ and $V$ satisfies (1.7) with $\alpha > 2$, then the hypotheses of Theorem 2.3 hold.

**Proof of Theorem** 2.3. Since $k \geq 3$, we can apply the Rozenblyum-Lieb-Cwickel estimate and write

$$N(\lambda;\chi(X_\perp)) \leq \mathrm{ent}\left\{\int_{\mathbb{R}^k} |V(X_\perp,z)|^{k/2}dz\right\},\ \lambda \leq 0.$$

Integrating with respect to $X_\perp \in \mathbb{R}^{2d}$, we come to (2.20). $\diamondsuit$

As already mentioned in the introduction, the asymptotic behaviour of $N(\Lambda+\lambda;H(1))$ as $\lambda \uparrow 0$ has been studied in [9] and [10].

If we compare Theorems 2.1–2.2 with [9, Theorem 2.4 i)-ii)], we find that under the hypotheses of [9, Theorem 2.4 i)-ii)] which are quite similar although slightly more restrictive than those of Theorems 2.1–2.2, asymptotic relation (1.8) is valid.

Similarly, if we compare Theorem 2.3 with [9, Theorem 2.4 iii)], we find that under the hypotheses of [9, Theorem 2.4 iii)] both quantities $N(\Lambda+\lambda;H(1))$ and $\mathcal{D}(\lambda)$ remain bounded as $\lambda \uparrow 0$.

Moreover, the proofs of Theorems 2.1, 2.2, or 2.3 follow closely the ideas of the proof of [9, Theorem 2.4] (see also [8, Theorems 4.5-4.6]) but are much simpler. These circumstances would become clearer if we recall that in the proof of [9, Theorem 2.4] the study of the asymptotics of $N(\Lambda+\lambda;H(1))$ is reduced to the investigation of the behaviour as $\lambda \uparrow 0$ of $N(\lambda;\mathcal{H})$ where the operator

$$\mathcal{H} := -\sum_{l=1}^{k}\frac{\partial^2}{\partial z_l^2} + \int_{\mathbb{R}^k}^{\oplus} v(z)\,dz$$

acts in $L^2(\mathbb{R}^{d+k}_{y,z})$, and for each fixed $z \in \mathbb{R}^k$ the operator $v(z)$ acts in $L^2(\mathbb{R}^d_y)$ as a $\Psi$DO with anti-Wick symbol $V_B(-\eta,y,z)$, $(y,\eta) \in T^*\mathbb{R}^d = \mathbb{R}^{2d}$, and $V_B(X_\perp,z) := V(B_+^{-1/2}X_\perp,z)$, $X_\perp \in \mathrm{Ran}\,B = \mathbb{R}^{2d}$.

Comparing the operators $\mathcal{H}$ and $\chi(X_\perp)$, and, respectively, the quantities $N(\lambda,\mathcal{H})$ and $\mathcal{D}(\lambda)$, it is not difficult to understand the close similarity in the behaviour of the quantities $N(\Lambda+\lambda;H(1))$ and $(2\pi)^{-d}\sqrt{\det\, B_+}\,\mathcal{D}(\lambda)$ as $\lambda \uparrow 0$.

**2.2.** In this subsection we state a result concerning the intermediate case where $V$ satisfies (1.7) with $\alpha = 2$. More precisely, we assume that the estimate

$$|V(X) + g\langle X\rangle^{-2}| \leq c_5\langle X\rangle^{-2-\varepsilon} \tag{2.21}$$

holds with $g > 0$, $\varepsilon > 0$ and $c_5 > 0$.

On the Sobolev space $\mathrm{H}^2(\mathbb{R}^k)$ introduce the operator

$$h^{as}(g) := -\Delta - g\langle z\rangle^{-2}$$

whose negative spectrum is either empty or purely discrete. If $k = 1, 2$, then $h^{as}(g)$ has at least one negative eigenvalue for all $g > 0$, and if $k \geq 3$, the operator $h^{as}(g)$ is non-negative if and only if $g \leq (k-2)^2/4$.

Assume that the negative spectrum of $h^{as}(g)$ is not empty. Denote by $\{-\gamma_l(g)\}_{l\geq 1}$ the non-decreasing sequence of the negative eigenvalues of $h^{as}(g)$. Note that the estimate

$$\gamma_l(g) \leq c_6 e^{-c_7 l},\ l \geq 1,$$

holds with positive numbers $c_6$ and $c_7$ independent of $l$.

**Theorem 2.4** *Assume that $V$ satisfies* (2.21). *If the negative spectrum of $h^{as}(g)$ is non-empty, we have*

$$\mathcal{D}(\lambda) \sim \sum_{l\geq 1} \mathrm{vol}\left\{X_\perp \in \mathbb{R}^{2d} | \, |X_\perp|^2 < \gamma_l(g)|\lambda|^{-1}\right\} = \frac{\pi^d}{d!}\sum_{l\geq 1}\gamma_l(g)^d\,|\lambda|^{-d},\ \lambda \uparrow 0.$$

*If $k \geq 3$ and $g < (k-2)^2/4$, we have*

$$\mathcal{D}(\lambda) = O(1),\ \lambda \uparrow 0.$$

We omit the elementary proof of Theorem 2.4, but note that under its hypotheses which coincide with those of [10, Theorem 2.4], the asymptotic relation (1.8) still holds.

**2.3.** In this subsection we discuss briefly the case of slowly decaying potentials, i.e. potentials satisfying (1.7) with $\alpha \in (0,2)$.

**Theorem 2.5** *Let $k \geq 1$. Assume that $V \in C^1(\mathbb{R}^m)$ satisfies* (1.7) *with $\alpha \in (0,2)$ and, moreover,*

$$|\nabla V(X)| \leq C\langle X\rangle^{-\alpha-1},\ C > 0,\ X \in \mathbb{R}^m.$$

*Set*

$$\nu_2(\lambda) := (2\pi)^{-k}\int_{\mathbb{R}^{2d}} \mathrm{vol}\ \left\{(z,\zeta) \in T^*\mathbb{R}^k | \, |\zeta|^2 + V(X_\perp, z) < \lambda\right\}\, dX_\perp.$$

*Then we have*

$$\mathcal{D}(\lambda) \sim \nu_2(\lambda),\ \lambda \uparrow 0. \tag{2.22}$$

**Remark.** The assumptions of Theorem 2.5 entail

$$\nu_2(\lambda) \asymp |\lambda|^{-\frac{m}{\alpha}+\frac{k}{2}},\ \lambda \uparrow 0.$$

The proof of Theorem 2.5 is based on well-known standard techniques such as the covering of $\mathbb{R}^k$ by disjoint cubes of equal size, and the Dirichlet-Neumann bracketing (cf. [13, Theorems XIII.81-XIII.82]); that is why we omit the details. If we impose more restrictive assumptions and apply more sophisticated methods (see e.g. [6, Section 10.5]), we could obtain a sharp remainder estimate in (2.22).

Finally, we note that under the hypotheses of Theorem 2.5 which coincide with those of [9, Theorem 2.2], asymptotic relation (1.8) is valid again.

# 3 Asymptotic behaviour of $\mathcal{D}(\lambda)$ as $\lambda \downarrow \lambda_0$. The case $\lambda_0 > -\infty$

**3.1.** Set $T(X_\perp) := |V(X_\perp, .)|^{1/2}(-\Delta + 1)^{-1/2}$. Under the general hypotheses of Theorem 1.1 the family $T(X_\perp)$ of operators acting in $L^2(\mathbb{R}^k)$ is defined for almost every $X_\perp \in \mathbb{R}^{2d}$.

We shall say that assumption $\mathcal{H}_1$ holds if and only if $T(X_\perp)$ is a family of compact operators, continuous on $\mathbb{R}^{2d}$, such that $\|T(X_\perp)\| \to 0$ as $|X_\perp| \to \infty$.

Throughout the section we suppose that assumption $\mathcal{H}_1$ holds.

Recall the notations $\lambda_0$, $\mathcal{E}(X_\perp)$, $X_\perp \in \mathbb{R}^{2d}$, and $\mathcal{E}_0$ (see (1.6), (1.10), and (1.9)).

**Lemma 3.1** *Let assumption $\mathcal{H}_1$ hold. Then we have*

$$\mathcal{E}_0 > -\infty. \tag{3.1}$$

**Proof.** Fix $X_\perp \in \mathbb{R}^{2d}$, $E > 0$, and set $T_E(X_\perp){:=}|V(X_\perp, .)|^{1/2}(-\Delta + E)^{-1/2}$ so that $T_1(X_\perp) = T(X_\perp)$. Choose $E > -\mathcal{E}_0(X_\perp)$ and write

$$(\chi(X_\perp) + E)^{-1} = (-\Delta + E)^{-1/2}\left(1 - |T_E(X_\perp)|^2\right)^{-1}(-\Delta + E)^{-1/2}, \tag{3.2}$$

where

$$|T_E(X_\perp)| := \sqrt{T_E(X_\perp)^* T_E(X_\perp)} \equiv \sqrt{(-\Delta + E)^{-1/2}|V(X_\perp, .)|(-\Delta + E)^{-1/2}}.$$

It is not difficult to check that if $\mathcal{H}_1$ is fulfilled, then $\|T_E(X_\perp)\| \to 0$ as $E \to \infty$ uniformly with respect to $X_\perp \in \mathbb{R}^{2d}$. Choose $E$ large enough so that we have, say, $\|T_E(X_\perp)\|^2 < 1/2$ for all $X_\perp \in \mathbb{R}^{2d}$. Then, (3.2) entails

$$\operatorname{dist}(-E, \sigma(\chi(X_\perp)))^{-1} = \|(\chi(X_\perp) + E)^{-1}\| \leq 2/E, \ \forall X_\perp \in \mathbb{R}^{2d}.$$

Hence, $-E < \mathcal{E}(X_\perp)$ for all $X_\perp \in \mathbb{R}^{2d}$, or $-E < \mathcal{E}_0$ which implies (3.1). $\diamondsuit$

**Lemma 3.2** *Let assumption $\mathcal{H}_1$ hold. Suppose that $M \subset \mathbb{R}^{2d}$ and $\mathcal{M} \subset \mathbf{C}$ are compact sets such that $\inf_{X_\perp \in M} \operatorname{dist}(\mathcal{M}, \sigma(X_\perp)) > 0$. Then the operator family $(\chi(X_\perp) + E)^{-1}$ is uniformly continuous with respect to $(X_\perp, E) \in M \times \mathcal{M}$.*

**Proof.** Write the resolvent identity

$$\begin{gathered}(\chi(X'_\perp)+E')^{-1}-(\chi(X''_\perp)+E'')^{-1} = (E''-E')(\chi(X'_\perp)+E')^{-1}(\chi(X''_\perp)+E'')^{-1}+ \\ (\chi(X'_\perp) + E')^{-1}(V(X''_\perp, .) - V(X'_\perp, .))(\chi(X''_\perp) + E'')^{-1} = \\ (E'' - E')(\chi(X'_\perp) + E')^{-1}(\chi(X''_\perp) + E'')^{-1} + \\ (\chi(X'_\perp) + E')^{-1}(-\Delta + 1)^{1/2}(|T(X''_\perp)|^2 - |T(X'_\perp)|^2)(-\Delta + 1)^{1/2}(\chi(X''_\perp) + E'')^{-1}\end{gathered} \tag{3.3}$$

with $E', E'' \in \mathcal{M}$, $X'_\perp, X''_\perp \in M$. Obviously, the quantity $\|(\chi(X_\perp)+E)^{-1}\| = (\mathrm{dist}(-E,\sigma(X_\perp)))^{-1}$ is uniformly bounded with respect to $(X_\perp, E) \in (M,\mathcal{M})$. Applying Lemma 3.1, pick a number $E_0 > -\mathcal{E}_0$. Then the norm of the operator

$$\begin{aligned}&(-\Delta+1)^{1/2}(\chi(X_\perp)+E)^{-1}\\ &\quad = \quad (-\Delta+1)^{1/2}(-\Delta+E_0)^{-1/2}(-\Delta+E_0)^{1/2}\times\\ &\qquad\qquad (\chi(X_\perp)+E_0)^{-1/2}(\chi(X_\perp)+E_0)^{1/2}(\chi(X_\perp)+E)^{-1}\end{aligned}$$

is uniformly bounded with respect to $(X_\perp, E) \in (M,\mathcal{M})$. Since the operator $T(X_\perp)$ is continuous with respect to $X_\perp \in M$, we find that (3.3) implies the continuity of $(\chi(X_\perp)+E)^{-1}$ with respect to $(X_\perp, E) \in (M,\mathcal{M})$. $\diamondsuit$

**Proposition 3.1** *Suppose that assumption $\mathcal{H}_1$ holds. Then $\mathcal{E}(X_\perp)$ is continuous on $\mathbb{R}^{2d}$, and, moreover, $\mathcal{E}(X_\perp) \to 0$ as $|X_\perp| \to \infty$.*

**Proof.** Fix $E > \max\{1, -\mathcal{E}_0\}$. We have $\|(\chi(X_\perp)+E)^{-1}\| = (\mathcal{E}(X_\perp)+E)^{-1}$, $X_\perp \in \mathbb{R}^{2d}$. By Lemma 3.2, $(\mathcal{E}(X_\perp)+E)^{-1}$, and, hence, $\mathcal{E}(X_\perp)$ is continuous with respect to $X_\perp \in \mathbb{R}^{2d}$. Moreover, (3.2) entails $\lim_{|X_\perp|\to\infty}(\mathcal{E}(X_\perp)+E)^{-1} = 1/E$, and, therefore, $\lim_{|X_\perp|\to\infty}\mathcal{E}(X_\perp) = 0$. $\diamondsuit$

Set

$$\Phi := \left\{X_\perp \in \mathbb{R}^{2d} | \mathcal{E}(X_\perp) = \mathcal{E}_0\right\}.$$

In the sequel we assume that $\mathcal{E}_0 < 0$. Since $\mathcal{E}(X_\perp)$ is continuous and $\mathcal{E}(X_\perp) \to 0$ as $|X_\perp| \to \infty$, the set $\Phi$ is not empty and compact. Put

$$\Phi_\varepsilon := \left\{X_\perp \in \mathbb{R}^{2d} |\, \mathcal{E}(X_\perp) < \mathcal{E}_0 + \varepsilon\right\},\ \varepsilon > 0,$$

$$\tilde{\Phi}_\delta := \left\{X_\perp \in \mathbb{R}^{2d} |\, \mathrm{dist}\,(X_\perp, \Phi) < \delta\right\},\ \delta > 0.$$

The continuity of $\mathcal{E}(X_\perp)$ and implies that for each $\varepsilon > 0$ there exists $\delta$ such that $\tilde{\Phi}_\delta \subseteq \Phi_\varepsilon$. On the other hand, the continuity of $\mathcal{E}(X_\perp)$ combined with the fact that $\mathcal{E}(X_\perp) \to 0$ as $|X_\perp| \to \infty$ while $\mathcal{E}_0 < 0$, entails that for each $\delta > 0$ there exists an $\varepsilon$ such that $\Phi_\varepsilon \subseteq \tilde{\Phi}_\delta$.

**Proposition 3.2** *Let $\mathcal{E}_0 < 0$. For every sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$N(\mathcal{E}_0+\varepsilon; \chi(X_\perp)) = 1,\ \forall X_\perp \in \tilde{\Phi}_\delta. \tag{3.4}$$

**Proof.** The equality (3.4) follows from the continuity of $\mathcal{E}(X_\perp)$, and the fact that $\mathcal{E}_0 < 0$ is the first eigenvalue of $\chi(X_\perp)$, $X_\perp \in \Phi$, which is simple (see [13, Section XIII.12]). $\diamondsuit$

Fix $X_\perp \in \mathbb{R}^{2d}$ and assume $\mathcal{E}(X_\perp) < 0$. Denote by $P(X_\perp)$ the spectral projection of $\chi(X_\perp)$ corresponding to $\mathcal{E}(X_\perp)$.

**Corollary 3.1** *Let $\mathcal{E}_0 < 0$. Then the projection $P$ is uniformly continuous in a vicinity of $\Phi$.*

**Proof.** Fix sufficiently small $\varepsilon$, denote by $\Gamma_\varepsilon$ the circle of radius $\varepsilon$ centered at $\mathcal{E}_0$, and choose $\delta > 0$ so that $\operatorname{dist}(\Gamma_\varepsilon, \sigma(\chi(X_\perp)))^{-1}$ is uniformly bounded with respect to $X_\perp \in \overline{\tilde{\Phi}_\delta}$. Then we have

$$P(X_\perp) = -\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} (\chi(X_\perp) - E)^{-1} dE.$$

Applying Proposition 3.1, we easily deduce the continuity of $P(X_\perp)$ for $X_\perp \in \overline{\tilde{\Phi}_\delta}$. $\diamondsuit$

**Corollary 3.2** *Let assumption $\mathcal{H}_1$ hold. Then we have $\lambda_0 = \mathcal{E}_0$.*

**Proof.** The lemma is trivial if $\mathcal{E}_0 = 0$. Assume $\mathcal{E}_0 < 0$. Obviously, $\lambda_0 \geq \mathcal{E}_0$. Fix $\varepsilon > 0$ small enough, and applying Proposition 3.2 choose $\delta > 0$ such that $N(\mathcal{E}_0 + \varepsilon; \chi(X_\perp)) = 1$ for all $X_\perp \in \overline{\tilde{\Phi}_\delta}$. Therefore,

$$\mathcal{D}(\mathcal{E}_0 + \varepsilon) \geq \operatorname{vol} \tilde{\Phi}_\delta > 0.$$

Consequently, $\mathcal{E}_0 + \varepsilon \geq \lambda_0$ for all $\varepsilon > 0$. Hence, $\mathcal{E}_0 = \lambda_0$. $\diamondsuit$

Putting together the results of this subsection, we obtain our first general result on the behaviour of $\mathcal{D}(\lambda)$ as $\lambda \downarrow \lambda_0$.

**Theorem 3.1** *Let assumption $\mathcal{H}_1$ hold and $\mathcal{E}_0 < 0$. Then for sufficiently small $\eta > 0$ we have*

$$\mathcal{D}(\lambda_0 + \eta) \equiv \mathcal{D}(\mathcal{E}_0 + \eta) = \operatorname{vol} \Phi_\eta. \tag{3.5}$$

**3.2.** In this subsection we estimate the difference $\mathcal{E}(X'_\perp) - \mathcal{E}(X''_\perp)$ with $X'_\perp, X''_\perp \in \mathbb{R}^{2d}$.

Let $X_\perp \in \mathbb{R}^{2d}$. Assume $\mathcal{E}(X_\perp) < 0$. Denote by $\psi(X_\perp)$ the eigenfunction of $\chi(X_\perp)$ corresponding to $\mathcal{E}(X_\perp)$, normalized in $L^2(\mathbb{R}^k)$, such that $\psi(X_\perp, z) > 0$ for every $z \in \mathbb{R}^k$ (see [13, Section XIII.12]).

If we consider $V(X'_\perp) - V(X_\perp)$ as a perturbation to $\chi(X_\perp)$, then the intuition originating from analytic perturbation theory (see [7], [13, Chapter XII]) prompts us that

$$\mathcal{E}(X_\perp) - \mathcal{E}(X'_\perp) \sim \int_{\mathbb{R}^k} \psi(X_\perp, z)^2 (V(X_\perp, z) - V(X'_\perp, z))\, dz,\ X'_\perp \to X_\perp.$$

**Lemma 3.3** *Let assumption $\mathcal{H}_1$ hold. Let $X'_\perp, X''_\perp \in \mathbb{R}^{2d}$. Assume $\mathcal{E}(X'_\perp) < 0$, $\mathcal{E}(X''_\perp) < 0$. Then we have*

$$\int_{\mathbb{R}^k} \psi(X'_\perp, z)^2 (V(X'_\perp, z) - V(X''_\perp, z))\, dz \leq$$

$$\mathcal{E}(X'_\perp) - \mathcal{E}(X''_\perp) \leq \int_{\mathbb{R}^k} \psi(X''_\perp, z)^2 (V(X'_\perp, z) - V(X''_\perp, z))\, dz. \tag{3.6}$$

**Proof.** Since $X'_\perp$ and $X''_\perp$ enter (3.6) in a symmetric manner, it suffices to prove only the second inequality which is implied immediately from the following obvious relations

$$\mathcal{E}(X'_\perp) = \inf_{u \in \mathrm{H}^1(\mathbb{R}^k), u \neq 0} \frac{\int_{\mathbb{R}^k} \left( |\nabla u(z)|^2 + V(X'_\perp, z)|u(z)|^2 \right) dz}{\int_{\mathbb{R}^k} |u(z)|^2 \, dz} \leq$$

$$\int_{\mathbb{R}^k} \left( |\nabla \psi(X''_\perp, z)|^2 + V(X'_\perp, z)\psi(X''_\perp, z)^2 \right) \, dz,$$

$$\mathcal{E}(X''_\perp) = \int_{\mathbb{R}^k} \left( |\nabla \psi(X''_\perp, z)|^2 + V(X''_\perp, z)\psi(X''_\perp, z)^2 \right) \, dz. \quad \diamondsuit$$

**3.3.** In this and the following subsection we consider the special case where $\Phi = \{0\}$; in this case we shall say that assumption $\mathcal{H}_2$ holds.
Under some supplementary hypotheses we derive an asymptotic formula describing the behaviour of $\mathcal{D}(\lambda)$ as $\lambda \downarrow \lambda_0$, which is more explicit than (3.5). The results of this subsection could be extended automatically to the case where $\Phi$ consists of finitely many isolated points, and without any serious efforts to the case where $\Phi$ is a closed manifold of positive co-dimension. We leave these extensions to the interested reader.

Let assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. For $X_\perp \in \mathbb{R}^{2d}$ set

$$F(X_\perp) := \int_{\mathbb{R}^k} \psi(0, z)^2 \left( V(X_\perp, z) - V(0, z) \right) dz,$$

$$\tilde{F}(X_\perp) := \| |T(X_\perp)|^2 - |T(0)|^2 \| \equiv \|(-\Delta + 1)^{-1/2}(V(X_\perp) - V(0))(-\Delta + 1)^{-1/2}\|.$$

Note that (3.6) with $X'_\perp = X_\perp$ and $X''_\perp = 0$ implies $F(X_\perp) \geq 0$, $X_\perp \in \mathbb{R}^{2d}$, and $F(X_\perp) = 0$ implies $X_\perp = 0$.

We shall say that assumption $\mathcal{H}_3$ holds if and only if the estimate

$$\tilde{F}(X_\perp) \leq c_8 F(X_\perp) \tag{3.7}$$

holds for sufficiently small $|X_\perp|$ and $c_8$ independent of $X_\perp$.

**Remark.** Evidently, for some $c_9$ we have $F(X_\perp) \leq c_9 \tilde{F}(X_\perp)$, $X_\perp \in \mathbb{R}^{2d}$. Hence, the validity of $\mathcal{H}_3$ is equivalent to $F(X_\perp) \asymp \tilde{F}(X_\perp)$, $X_\perp \to 0$.

For $\eta > 0$ put $\nu_3(\eta) := \mathrm{vol} \left\{ X_\perp \in \mathbb{R}^{2d} | F(X_\perp) < \eta \right\}$.

**Theorem 3.2** *Let assumptions $\mathcal{H}_1$–$\mathcal{H}_3$ hold. Moreover, suppose that*

$$\lim_{\delta \downarrow 0} \limsup_{\eta \downarrow 0} \nu_3((1 + \delta)\eta)/\nu_3(\eta) = 1. \tag{3.8}$$

*Then we have*

$$\mathcal{D}(\mathcal{E}_0 + \eta) \sim \nu_3(\eta), \eta \downarrow 0. \tag{3.9}$$

**Proof.** By Lemma 3.3

$$F(X_\perp) + \int_{\mathbb{R}^k} (V(X_\perp, z) - V(0,z))(\psi(X_\perp, z)^2 - \psi(0,z)^2)\, dz \leq$$

$$\mathcal{E}(X_\perp) - \mathcal{E}_0 \leq F(X_\perp), \tag{3.10}$$

where $X_\perp \in \mathbb{R}^{2d}$ and $|X_\perp|$ is sufficiently small. Evidently,

$$\left| \int_{\mathbb{R}^k} (V(X_\perp, z) - V(0,z))(\psi(X_\perp, z)^2 - \psi(0,z)^2)\, dz \right| \leq$$

$$\tilde{F}(X_\perp)\, \|(-\Delta+1)^{1/2}(\psi(X_\perp) + \psi(0))\|\, \|(-\Delta+1)^{1/2}(\psi(X_\perp) - \psi(0))\|. \tag{3.11}$$

Further,

$$\begin{aligned} &\|(-\Delta+1)^{1/2}(\psi(X_\perp) + \psi(0))\| \\ &\quad\leq\ c_{10}(\|(\chi(X_\perp)+E)^{1/2}\psi(X_\perp)\| + \|(\chi(0)+E)^{1/2}\psi(0)\|) \\ &\quad=\ c_{10}((\mathcal{E}(X_\perp)+E)^{1/2} + (\mathcal{E}_0+E)^{1/2}) \\ &\quad\leq\ c_{11}\,, \end{aligned} \tag{3.12}$$

where $E > -\mathcal{E}_0$, and $c_{10}$, $c_{11}$ are independent of $X_\perp$. Analogously,

$$\|(-\Delta+1)^{1/2}(\psi(X_\perp) - \psi(0))\| \leq c_{10}\|(\chi(0)+E)^{1/2}(\psi(X_\perp) - \psi(0))\|. \tag{3.13}$$

Next,

$$\|(\chi(0)+E)^{1/2}(\psi(X_\perp)-\psi(0))\|^2 = \langle(\chi(0)+E)(\psi(X_\perp)-\psi(0)), \psi(X_\perp)-\psi(0)\rangle =$$

$$\langle(|T(0)|^2 - |T(X_\perp)|^2)(-\Delta+1)^{1/2}\psi(X_\perp), (-\Delta+1)^{1/2}(\psi(X_\perp)-\psi(0))\rangle +$$

$$\mathcal{E}(X_\perp)\langle\psi(X_\perp), \psi(X_\perp)-\psi(0)\rangle - \mathcal{E}_0\langle\psi(0), \psi(X_\perp)-\psi(0)\rangle + E\|\psi(X_\perp)-\psi(0)\|^2$$

where $\langle .,. \rangle$ denotes the scalar product in $L^2(\mathbb{R}^k)$.

Note that $\psi(X_\perp) = P(X_\perp)\psi(0)/\|P(X_\perp)\psi(0)\|$. Taking into account Corollary 3.1 and the fact that $T(X_\perp)$ is continuous, we conclude that

$$\lim_{X_\perp \to 0} \|(\chi(0)+E)^{1/2}(\psi(X_\perp)-\psi(0))\|^2 = 0$$

and hence, by (3.13),

$$\lim_{X_\perp \to 0} \|(-\Delta+1)^{1/2}(\psi(X_\perp)-\psi(0))\| = 0. \tag{3.14}$$

Fix $\varepsilon > 0$, and bearing in mind (3.11)-(3.14) and assumption $\mathcal{H}_3$, suppose that $X_\perp$ is so small that we have

$$-\varepsilon F(X_\perp) \leq \int_{\mathbb{R}^k} (V(X_\perp, z) - V(0,z))(\psi(X_\perp, z)^2 - \psi(0,z)^2)\, dz. \tag{3.15}$$

The combination of (3.10) and (3.15) yields

$$(1-\varepsilon)F(X_\perp) \leq \mathcal{E}_0 - \mathcal{E}(X_\perp) \leq F(X_\perp), \ \forall X_\perp \in \tilde{\Phi}_\delta, \tag{3.16}$$

where $\delta = \delta(\varepsilon)$ is small enough. Assume that $\eta > 0$ is so small that we have $\bar{\Phi}_\eta \subseteq \tilde{\Phi}_\delta$. Then (3.16) implies

$$\nu_3(\eta) \leq \operatorname{vol}\Phi_\eta \leq \nu_3((1-\varepsilon)^{-1}\eta). \tag{3.17}$$

Now, (3.9) follows directly from (3.17), (3.5), and (3.8). ♢

**3.4.** In this subsection we assume $m = 3$ (i.e. $k = 1$ and $d = 1$), and deal with homogeneous potentials

$$V(X) := -g|X|^{-\alpha}, \ g > 0, \ \alpha \in (0,1), \ X \in \mathbb{R}^3. \tag{3.18}$$

Note that if $V$ satisfies (3.18), then it belongs to the class $\mathcal{L}_{3/2}$ for *all* $\alpha \in (0,2)$. However, there is an essential difference between the case $\alpha \in (0,1)$ considered in this subsection, and the case $\alpha \in [1,2)$ which will be dealt with in the following section.

Fix $\varepsilon > 0$ and set

$$V_1(X) := \begin{cases} V(X), \text{ if } |V(X)| > \varepsilon, \\ 0, \text{ otherwise,} \end{cases}$$

$$V_2 := V - V_1.$$

In the case $\alpha \in (0,1)$ we have $V_1(X_\perp, .) \in L^1(\mathbb{R})$ for all $X_\perp \in \mathbb{R}^2$ and, in particular, for $X_\perp = 0$. Consequently, the operator $T(X_\perp)$ is compact, and the operator $\chi(X_\perp)$ is well-defined for all $X_\perp \in \mathbb{R}^2$. Since $V(X_\perp, z) \geq V(X_\perp, 0)$ for all $X_\perp \in \mathbb{R}^2$, $z \in \mathbb{R}$, it is clear that $\mathcal{E}_0$ coincides with $\mathcal{E}(0)$, i.e. with the first eigenvalue of the operator $\chi(0)$.

In the case $\alpha \in [1,2)$ the potential $V_1(0,.)$ is not in $L^1(\mathbb{R})$, the operator $\chi(0)$ is not well-defined and, as we shall see in the next section, $\lambda_0 = -\infty$.

**Proposition 3.3** *Let $V$ satisfy* (3.18) *with $\alpha \in (0,1)$. Then the operator family $T(X_\perp)$ satisfies $\mathcal{H}_1$.*

**Proof.** For $X_\perp \neq 0$ we have $\| |V(X_\perp,.)|^{1/2} \| = \sqrt{g}|X_\perp|^{-\alpha/2}$. Moreover, the multiplier $|V(X_\perp,.)|^{1/2}$ is uniformly continuous with respect to $|X_\perp| \geq \varepsilon$, $\varepsilon > 0$. Since $\|T(X_\perp)\| \leq \| |V(X_\perp,.)|^{1/2}\|$ for all $X_\perp \in \mathbb{R}^2$, it remains only to show that $T(X_\perp)$ is continuous at $X_\perp = 0$. To this end, we write

$$|V(0,z)|^{1/2} - |V(X_\perp,z)|^{1/2} = \frac{\sqrt{g}\alpha}{4}\int_0^{|X_\perp|^2} (t+z^2)^{-1-\alpha/4}dt,$$

and easily find that

$$\|T(X_\perp)-T(0)\|^2 \leq \|T(X_\perp)-T(0)\|_2^2 = \frac{g\alpha^2}{32}\int_{\mathbb{R}} \left\{ \int_0^{|X_\perp|^2} (t+z^2)^{-1-\alpha/4}dt \right\}^2 dz.$$

Changing the variables $t = |X_\perp|^2 s$, $z = |X_\perp| y$, we get

$$\|T(X_\perp) - T(0)\|_2^2 = \frac{g\alpha^2}{32}|X_\perp|^{1-\alpha} \int_{\mathbb{R}} \left\{ \int_0^1 (s+y^2)^{-1-\alpha/4} ds \right\}^2 dy. \qquad (3.19)$$

Since $\alpha \in (0,1)$, the right-hand side of (3.19) vanishes as $|X_\perp| \to 0$. $\diamondsuit$

Evidently, if (3.18) holds, then assumption $\mathcal{H}_2$ is fulfilled.

**Proposition 3.4** *Let $V$ satisfy* (3.18) *with $\alpha \in (0,1)$. Then assumption $\mathcal{H}_3$ holds.*

**Proof.** Obviously

$$F(X_\perp) = g \int_{\mathbb{R}} \left( |z|^{-\alpha} - (|X_\perp|^2 + z^2)^{-\alpha/2} \right) \psi(0,z)^2 \, dz =$$

$$\frac{g\alpha}{2} \int_{\mathbb{R}} \int_0^{|X_\perp|^2} (t+z^2)^{-1-\alpha/2} \psi(0,z)^2 \, dt dz =$$

$$\frac{g\alpha}{2}|X_\perp|^{1-\alpha} \int_{\mathbb{R}} \psi(0, |X_\perp| y)^2 \int_0^1 (s+y^2)^{-1-\alpha/2} \, ds dy. \qquad (3.20)$$

Recall that $\mathrm{H}^1(\mathbb{R})$ is continuously imbedded in $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Assume $|X_\perp| < 1$. Then (3.20) implies

$$F(X_\perp) \geq \frac{g\alpha}{2}|X_\perp|^{1-\alpha} \min_{|z|\leq 1} \psi(0,z)^2 \int_{|y|<1} dy \int_0^1 (s+y^2)^{-1-\alpha/2} ds. \qquad (3.21)$$

On the other hand, if (3.18) holds, we have

$$\tilde{F}(X_\perp) = \sup_{u \in \mathrm{H}^1(\mathbb{R}), u \neq 0} \frac{\int_{\mathbb{R}} (V(X_\perp, z) - V(0,z)) |u(z)|^2 \, dz}{\|u\|^2_{\mathrm{H}^1(\mathbb{R})}} =$$

$$\frac{g\alpha}{2}|X_\perp|^{1-\alpha} \sup_{u \in \mathrm{H}^1(\mathbb{R}), u \neq 0} \frac{\int_{\mathbb{R}} \int_0^1 (s+y^2)^{-1-\alpha/2} |u(|X_\perp| y)|^2 \, dy ds}{\|u\|^2_{\mathrm{H}^1(\mathbb{R})}} \leq$$

$$\frac{g\alpha}{2}|X_\perp|^{1-\alpha} \int_{\mathbb{R}} \int_0^1 (s+y^2)^{-1-\alpha/2} dy ds \sup_{u \in \mathrm{H}^1(\mathbb{R}), u \neq 0} \frac{\|u\|^2_{L^\infty(\mathbb{R})}}{\|u\|^2_{\mathrm{H}^1(\mathbb{R})}}. \qquad (3.22)$$

Comparing (3.21) and (3.22), we find that (3.7) holds with $c_8$ independent of $X_\perp \in \tilde{\Phi}_1$. $\diamondsuit$

**Proposition 3.5** *Let $V$ satisfy* (3.18) *with $\alpha \in (0,1)$. Then we have*

$$F(X_\perp) \sim \frac{g\sqrt{\pi}\alpha\Gamma((\alpha+1)/2)}{(1-\alpha)\Gamma(1+\alpha/2)} \psi(0,0)^2 |X_\perp|^{1-\alpha}, \quad X_\perp \to 0. \qquad (3.23)$$

**Proof.** Recall (3.20). Since the function $\int_0^1 (s+y^2)^{-1-\alpha/2} ds$ is in $L^1(\mathbb{R}_y)$, the dominated convergence theorem yields

$$\begin{aligned} & \lim_{X_\perp \to 0} \int_{\mathbb{R}} \psi(0, |X_\perp| y)^2 \int_0^1 (s+y^2)^{-1-\alpha/2} ds \\ = \; & \psi(0,0)^2 \int_{\mathbb{R}} \int_0^1 (s+y^2)^{-1-\alpha/2} \, dy \, ds \\ = \; & \frac{2\sqrt{\pi}\Gamma((\alpha+1)/2)}{(1-\alpha)\Gamma(1+\alpha/2)} \psi(0,0)^2. \end{aligned} \tag{3.24}$$

Inserting (3.24) into (3.20), we get (3.23). $\diamondsuit$

Combining Theorem 3.2 and Propositions 3.3 – 3.5, we obtain the following result.

**Theorem 3.3** *Let $V$ satisfy* (3.18) *with $\alpha \in (0,1)$. Then we have*

$$\mathcal{D}(\mathcal{E}_0 + \eta) \sim \pi \left\{ \frac{g\sqrt{\pi}\alpha\Gamma((\alpha+1)/2)}{(1-\alpha)\Gamma(1+\alpha/2)} \psi(0,0)^2 \right\}^{2/(\alpha-1)} \eta^{2/(1-\alpha)}, \; \eta \downarrow 0.$$

# 4 Asymptotic behaviour of $\mathcal{D}(\lambda)$ as $\lambda \downarrow \lambda_0$. The case $\lambda_0 = -\infty$

**4.1.** Throughout the section we assume $m = 3$, i.e. $d = 1$, $k = 1$. Moreover, we suppose

$$V = U + W \tag{4.1}$$

where

$$U(X) := -g|X|^{-\alpha}, \; g > 0, \; \alpha \in [1, 2), \tag{4.2}$$

(cf. (3.18)), and $W$ is a perturbation which is less singular at the origin than $U$, and does not contribute to the main asymptotic term of $\mathcal{D}(\lambda)$ as $\lambda \to -\infty$.

In this and the following subsection we assume $W = 0$. Put $r := |X_\perp|$, and

$$\mathcal{N}_g(\lambda; r) := N\left(\lambda; -\frac{d^2}{dz^2} - g(z^2 + r^2)^{-\alpha/2}\right), \; \lambda < 0.$$

Then we have

$$\mathcal{D}(\lambda) \equiv \mathcal{D}(\lambda, g) = 2\pi \int_0^\infty \mathcal{N}_g(\lambda; r) \, r dr. \tag{4.3}$$

Moreover,

$$\mathcal{D}(\lambda, g) = g^{-2/(2-\alpha)} \mathcal{D}(g^{-2/(2-\alpha)}\lambda, 1). \tag{4.4}$$

**Proposition 4.1** *Let $V$ satisfy* (4.1)-(4.2) *with $W = 0$ and $\alpha \in (1,2)$. Then we have*

$$\lim_{\lambda \to -\infty} |\lambda|^{1/(\alpha-1)} \mathcal{D}(\lambda; g) = \mathcal{C}_\alpha \equiv \mathcal{C}_\alpha(g) := \pi \left( \frac{g\sqrt{\pi}\Gamma((\alpha-1)/2)}{2\Gamma(\alpha/2)} \right)^{2/(\alpha-1)}. \tag{4.5}$$

**Proof.** Bearing in mind (4.4), we assume $g = 1$ without any loss of generality. Moreover, we write $\mathcal{N}(\lambda; r)$ instead of $\mathcal{N}_1(\lambda; r)$.

Introduce the semi-classical parameter $h = |\lambda|^{-1/2}$, and change the variables $r = h^2\varrho$ in (4.3). Thus we get

$$\mathcal{D}(\lambda) = 2\pi h^4 \int_0^\infty \mathcal{N}(-h^{-2}; h^2\varrho)\, \varrho d\varrho. \tag{4.6}$$

Applying the minimax principle, we find that the quantity $\mathcal{N}(-h^{-2}; h^2\varrho)$ coincides with the maximal dimension of the linear subsets of $C_0^\infty(\mathbf{R})$ whose nonzero elements $u$ satisfy the inequality

$$\int_{\mathbb{R}} \left\{|u'|^2 + h^{-2}|u|^2\right\} dz < h^{-2\alpha} \int_{\mathbb{R}} (h^{-4}z^2 + \varrho^2)^{-\alpha/2}|u|^2\, dz.$$

Scaling $z = h^2 t$ and multiplying by $h^4$, we find that $\mathcal{N}(-h^{-2}; h^2\varrho)$ is equal to the maximal dimension of the linear subsets of $C_0^\infty(\mathbf{R})$ whose nonzero elements $w$ satisfy the inequality

$$\int_{\mathbb{R}} \left\{|w'|^2 + h^2|w|^2\right\} dt < h^{2(2-\alpha)} \int_{\mathbb{R}} (t^2 + \varrho^2)^{-\alpha/2}|w|^2\, dt. \tag{4.7}$$

Denote by $G_{h,\varrho}$ the integral operator with kernel

$$\mathcal{G}_{h,\varrho}(s,t) := \frac{h}{2}(s^2 + \varrho^2)^{-\alpha/4} e^{-h|s-t|} (t^2 + \varrho^2)^{-\alpha/4}.$$

Note that $\frac{h}{2}e^{-h|s-t|}$ is the integral kernel of the operator $h^2\left(-\frac{d^2}{dt^2} + h^2\right)^{-1}$. Applying the Birman-Schwinger principle, we find that (4.7) entails

$$\mathcal{N}(-h^{-2}; h^2\varrho) = n_+(h^{2(\alpha-1)}; G_{h,\varrho}). \tag{4.8}$$

Further, set

$$\mathcal{G}^{(1)}_{h,\varrho}(s,t) := \frac{h}{2}(s^2 + \varrho^2)^{-\alpha/4} (t^2 + \varrho^2)^{-\alpha/4},$$

$$\mathcal{G}^{(2)}_{h,\varrho}(s,t) := \frac{h}{2}(s^2 + \varrho^2)^{-\alpha/4} \left(e^{-h|s-t|} - 1\right) (t^2 + \varrho^2)^{-\alpha/4},$$

and denote by $G^{(j)}_{h,\varrho}$ the integral operator with kernel $\mathcal{G}^{(j)}_{h,\varrho}(s,t)$, $j = 1, 2$. Then we have

$$G_{h,\varrho} = G^{(1)}_{h,\varrho} + G^{(2)}_{h,\varrho},$$

and therefore the estimates

$$\pm n_+(h^{2(\alpha-1)}; G_{h,\varrho}) \le \pm n_+((1 \mp \varepsilon)h^{2(\alpha-1)}; G^{(1)}_{h,\varrho}) + n_\pm(\varepsilon h^{2(\alpha-1)}; G^{(2)}_{h,\varrho}) \tag{4.9}$$

hold for each $\varepsilon \in (0, 1)$.

Next, $G^{(1)}_{h,\varrho}$ is a rank-one operator whose unique non-zero eigenvalue equals

$$\frac{h}{2}\int_{\mathbb{R}} \frac{dx}{(x^2 + \varrho^2)^{\alpha/2}} = \varrho^{1-\alpha} h \tilde{\mathcal{C}}_\alpha,\ \tilde{\mathcal{C}}_\alpha := \frac{\sqrt{\pi}\,\Gamma((\alpha-1)/2)}{2\Gamma(\alpha/2)}.$$

Hence, for $\varepsilon \in (0,1)$,

$$n_+((1\mp\varepsilon)h^{2(\alpha-1)}; G^{(1)}_{h,\varrho}) = \begin{cases} 1, \text{ if } \varrho < (1\mp\varepsilon)^{-1/(\alpha-1)}\tilde{\mathcal{C}}_\alpha^{1/(\alpha-1)} h^{-2+1/(\alpha-1)}, \\ 0, \text{ otherwise.}\end{cases}$$

Thus we get

$$2\pi h^4 \int_0^\infty n_+((1\mp\varepsilon)h^{2(\alpha-1)}; G^{(1)}_{h,\varrho})\, \varrho d\varrho =$$

$$(1\mp\varepsilon)^{-2/(\alpha-1)}\pi\tilde{\mathcal{C}}_\alpha^{2/(\alpha-1)} h^{2/(\alpha-1)} = (1\mp\varepsilon)^{-2/(\alpha-1)}\mathcal{C}_\alpha|\lambda|^{-1/(\alpha-1)}. \tag{4.10}$$

Further, for $\delta \in (0,(\alpha-1)/2)$ we have

$$\|G^{(2)}_{h,\varrho}\|_2^2 = \frac{h^2}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} (s^2+\varrho^2)^{-\alpha/2}(t^2+\varrho^2)^{-\alpha/2}\left(e^{-h|s-t|}-1\right)^2 dsdt \le$$

$$c_{12}h^{2(1+\delta)}\int_{\mathbb{R}} |s|^{2\delta}(s^2+\varrho^2)^{-\alpha/2}\,ds \int_{\mathbb{R}}(t^2+\varrho^2)^{-\alpha/2}\,dt = c_{13}h^{2(1+\delta)}\varrho^{2(1-\alpha+\delta)}$$

where $c_{12}$ and $c_{13}$ are independent of $h$ and $\varrho$. Using the estimate

$$n_\pm(\varepsilon h^{2(\alpha-1)}; G^{(2)}_{h,\varrho}) \le \text{ent}\left\{\varepsilon^{-2}h^{4(1-\alpha)}\|G^{(2)}_{h,\varrho}\|_2^2\right\},$$

we find that

$$n_\pm(\varepsilon h^{2(\alpha-1)}; G^{(2)}_{h,\varrho}) = 0$$

if $\varrho > c_{14}h^{(3-2\alpha+\delta)/(\alpha-1-\delta)}$ with $c_{14} := (\varepsilon^{-2}c_{13})^{1/2(1-\alpha+\delta)}$, and

$$n_\pm(\varepsilon h^{2(\alpha-1)}; G^{(2)}_{h,\varrho}) \le c_{13}\varepsilon^{-2}h^{2(3+\delta-2\alpha)}\varrho^{2(1-\alpha+\delta)}$$

if $\varrho \le c_{14}h^{(3-2\alpha+\delta)/(\alpha-1-\delta)}$. Consequently,

$$2\pi h^4\int_0^\infty n_\pm(\varepsilon h^{2(\alpha-1)}; G^{(2)}_{h,\varrho})\,\varrho d\varrho \le c_{15}h^{2(1-\delta)/(\alpha-1-\delta)} = c_{15}|\lambda|^{-(1-\delta)/(\alpha-1-\delta)}$$

where $c_{15}$ depends on $\varepsilon$ but is independent of $h$ and $\varrho$. Since $\alpha < 2$, we have $\frac{1-\delta}{\alpha-1-\delta} > \frac{1}{\alpha-1}$ for $\delta > 0$ small enough. Therefore,

$$2\pi h^4\int_0^\infty n_\pm(\varepsilon h^{2(\alpha-1)}; G^{(2)}_{h,\varrho})\,\varrho d\varrho = o(|\lambda|^{-1/(\alpha-1)}),\ \lambda \to -\infty. \tag{4.11}$$

The combination of (4.6) and (4.8)–(4.11) yields

$$\liminf_{\lambda\to-\infty} |\lambda|^{-1/(\alpha-1)}\mathcal{D}(\lambda) \ge (1+\varepsilon)^{-2/(\alpha-1)}\mathcal{C}_\alpha, \tag{4.12}$$

$$\limsup_{\lambda\to-\infty} |\lambda|^{-1/(\alpha-1)}\mathcal{D}(\lambda) \le (1-\varepsilon)^{-2/(\alpha-1)}\mathcal{C}_\alpha. \tag{4.13}$$

Letting $\varepsilon \downarrow 0$ in (4.12) and (4.13), we get (4.5). $\diamondsuit$

**4.2.** In this section we consider the potential satisfying (4.1)–(4.2) with $\alpha = 1$ and $W = 0$, i.e. the Coulomb potential.

**Proposition 4.2** *Let $V$ satisfy* (4.1) - (4.2) *with $\alpha = 1$ and $W = 0$. Then we have*

$$\lim_{\lambda\to-\infty} |\lambda| e^{2\sqrt{|\lambda|}/g}\mathcal{D}(\lambda; g) = \pi e^{-2\gamma_E} \tag{4.14}$$

*where $\gamma_E$ is the Euler constant.*

**Proof.** As in the proof of Proposition 4.1 we assume $g = 1$ without any loss of generality. Arguing as in the derivation of (4.6), we get

$$\mathcal{D}(\lambda) = 2\pi h^4 \int_0^1 \mathcal{N}(-h^{-2}; h^2\varrho)\, \varrho d\varrho, \quad h = |\lambda|^{-1/2}. \tag{4.15}$$

Here we have also taken into account the fact that $(z^2+r^2)^{-1/2} < |\lambda|$ if $r > |\lambda|^{-1}$, $z \in \mathbb{R}$, and, hence, $\mathcal{N}(\lambda; r) = 0$ for $r > |\lambda|^{-1}$.

However, now we need a modification of the proof of Proposition 4.1 since the function

$$f_\varrho(t) := (t^2 + \varrho^2)^{-1/2}, \quad \varrho > 0, \tag{4.16}$$

does not belong to $L^1(\mathbb{R}_t)$.

Define the unitary operator $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\mathcal{F}_h u)(\xi) := \left(\frac{h}{2\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-ix\xi h} u(x)\, dx, \; h > 0.$$

Arguing as in the derivation of (4.8), we get

$$\mathcal{N}(-h^{-2}; h^2\varrho) = n_+(1; f_\varrho^{1/2}\mathcal{F}_h^* f^2 \mathcal{F}_h f_\varrho^{1/2}), \tag{4.17}$$

where $f := f_1$ (see (4.16)). The operator $f_\varrho^{1/2}\mathcal{F}_h^* f^2 \mathcal{F}_h f_\varrho^{1/2}$ is unitarily equivalent to $\mathcal{F}_h f_\varrho^{1/2}\mathcal{F}_h^* f^2 \mathcal{F}_h f_\varrho^{1/2}\mathcal{F}_h^*$, whose non-zero eigenvalues coincide with the non-zero eigenvalues of the operator $J_{h,\varrho} := f\mathcal{F}_h f_\varrho \mathcal{F}_h^* f$. Therefore,

$$n_+(1; f_\varrho^{1/2}\mathcal{F}_h^* f^2 \mathcal{F}_h f_\varrho^{1/2}) = n_+(1; J_{h,\varrho}). \tag{4.18}$$

Note that the operator $J_{h,\varrho}$ is an integral operator with kernel

$$\mathcal{J}_{h,\varrho}(s,t) := \frac{h}{\pi} f(s) R_{h,\varrho}(s-t) f(t)$$

where

$$R_{h,\varrho}(z) := K_0(h\varrho|z|), \; z \in \mathbb{R} \setminus \{0\},$$

and

$$K_0(t) := \int_0^\infty \frac{\cos(tx)}{(x^2+1)^{1/2}} dx = -\frac{1}{\pi}\int_0^\pi e^{t\cos\theta} \left\{\gamma_E + \ln(2|t|\sin^2\theta)\right\} d\theta, \tag{4.19}$$

for $t$ in $\mathbb{R} \setminus \{0\}$ is the modified Bessel (McDonald) function (see [1, (9.6.21), (9.6.17)]). Obviously,

$$K_0(t) = \varphi(t) \ln|t| + \omega(t)$$

with

$$\varphi(t) := -\frac{1}{\pi}\int_0^\pi e^{t\cos\theta}\, d\theta,$$

$$\omega(t) := -\frac{1}{\pi}\int_0^\pi e^{t\cos\theta} \left(\gamma_E + \ln 2 + \ln\sin^2\theta\right) d\theta.$$

Moreover, $\varphi(0) = -1$, $\omega(0) = \ln 2 - \gamma_E$. Finally, [1, (9.6.27)] entails

$$K_0(t) = \int_0^\infty e^{-|t|\cosh x}\,dx \le e^{-|t|}\int_0^\infty e^{-\frac{|t|x^2}{2}}\,dx = \sqrt{\frac{\pi}{2|t|}}e^{-|t|}.$$

Introduce the operators $J^{(l)}_{h,\varrho}$, $l = 1,2,3$, as integral operators with kernels

$$\mathcal{J}^{(l)}(s,t) := \frac{h}{\pi}f(s)R^{(l)}_{h,\varrho}(s-t)f(t)$$

where

$$R^{(1)}_{h,\varrho}(z) := \varphi(0)\ln(h\varrho) + \omega(0) = -\ln(h\varrho) + \ln 2 - \gamma_E,$$

$$R^{(2)}_{h,\varrho}(z) := \varphi(0)\ln|z| = -\ln|z|,$$

$$R^{(3)}_{h,\varrho}(z) := (\varphi(h\varrho|z|) - \varphi(0))\ln(h\varrho|z|) + \omega(h\varrho|z|) - \omega(0) = K_0(h\varrho|z|) - \ln 2 + \gamma_E + \ln(h\varrho|z|).$$

Then we have $R_{h,\varrho}(z) = \sum_{l=1}^3 R^{(l)}_{h,\varrho}(z)$ and, therefore, $J_{h,\varrho} = \sum_{l=1}^3 J^{(l)}_{h,\varrho}$.

Note that the quantity $J^{(1)}_{h,\varrho}(z)$ is in fact independent of $z$, while $J^{(2)}_{h,\varrho}(z)$ is independent of $\varrho$ and $h$. Moreover, for each $\delta \in (0,1)$ there exits a constant $c = c(\delta)$ independent of $h$ and $\varrho$ such that the estimate

$$|R^{(3)}_{h,\varrho}(z)| \le c(h\varrho)^\delta|z|^\delta \tag{4.20}$$

holds for each $h > 0$, $\varrho > 0$, and $z \in \mathbb{R}$.

Define the orthogonal projection $\mathcal{P} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\mathcal{P}u)(t) := \frac{1}{\pi}f(t)\int_{\mathbb{R}} u(s)f(s)\,ds,\ u \in L^2(\mathbb{R}).$$

Set $\mathcal{Q} := \mathrm{Id} - \mathcal{P}$. Since $J^{(1)}_{h,\varrho}\mathcal{Q} = 0$, we have

$$J_{h,\varrho} = \mathcal{P}J_{h,\varrho}\mathcal{P} + \mathcal{Q}(J^{(2)}_{h,\varrho} + J^{(3)}_{h,\varrho})\mathcal{Q} + 2\mathrm{Re}\mathcal{P}(J^{(2)}_{h,\varrho} + J^{(3)}_{h,\varrho})\mathcal{Q}.$$

Completing the squares, we derive the estimates

$$\begin{aligned} \pm J_{h,\varrho} \quad \le \quad & \pm\mathcal{P}\left(J_{h,\varrho} \pm \varepsilon|J^{(2)}_{h,\varrho}| \pm |J^{(3)}_{h,\varrho}|\right)\mathcal{P} \\ & \pm\mathcal{Q}\left(J^{(2)}_{h,\varrho} \pm \varepsilon^{-1}|J^{(2)}_{h,\varrho}| + J^{(3)}_{h,\varrho} \pm |J^{(3)}_{h,\varrho}|\right)\mathcal{Q},\ \forall\varepsilon > 0\,. \end{aligned} \tag{4.21}$$

For $\varepsilon > 0$ set

$$\mathcal{Z}_1^\pm \equiv \mathcal{Z}_1^\pm(\varepsilon,\varrho,h) := \mathcal{P}\left(J_{h,\varrho} \pm \varepsilon|J^{(2)}_{h,\varrho}| \pm |J^{(3)}_{h,\varrho}|\right)\mathcal{P}_{|\mathcal{P}L^2(\mathbb{R})},$$

$$\mathcal{Z}_2^\pm \equiv \mathcal{Z}_2^\pm(\varepsilon,\varrho,h) := \mathcal{Q}\left(J^{(2)}_{h,\varrho} \pm \varepsilon^{-1}|J^{(2)}_{h,\varrho}| + J^{(3)}_{h,\varrho} \pm |J^{(3)}_{h,\varrho}|\right)\mathcal{Q}_{|\mathcal{Q}L^2(\mathbb{R})}.$$

Since $L^2(\mathbb{R}) = \mathcal{P}L^2(\mathbb{R}) \oplus \mathcal{Q}L^2(\mathbb{R})$, (4.21) implies that the inequalities

$$n_+(1; J_{h,\varrho}) \geq n_+(1; \mathcal{Z}_1^-(\varepsilon)) + n_+(1; \mathcal{Z}_2^-(\varepsilon)), \tag{4.22}$$

$$n_+(1; J_{h,\varrho}) \leq n_+(1; \mathcal{Z}_1^+(\varepsilon)) + n_+(1; \mathcal{Z}_2^+(\varepsilon)), \tag{4.23}$$

hold for each $\varepsilon > 0$. Note that if $\varepsilon \in (0,1)$, then the operator $\mathcal{Z}_2^-(\varepsilon)$ is non-positive, and hence

$$n_+(1; \mathcal{Z}_2^-(\varepsilon)) = 0. \tag{4.24}$$

Next, we show that for $\varrho \in (0,1]$, every fixed $\varepsilon > 0$, and $h$ small enough we have

$$n_+(1; \mathcal{Z}_2^+(\varepsilon)) = 0. \tag{4.25}$$

First, we apply the elementary estimates

$$n_+(1; \mathcal{Z}_2^+(\varepsilon)) \leq 4(1+\varepsilon^{-2})\|J_{h,\varrho}^{(2)}\|_2^2 + 2\|J_{h,\varrho}^{(3)}\|_2^2. \tag{4.26}$$

Further,

$$\|J_{h,\varrho}^{(2)}\|_2^2 = \frac{h^2}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} (\ln|s-t|)^2(1+s^2)^{-1}(1+t^2)^{-1}\, dsdt = c_{16}h^2 \tag{4.27}$$

with $c_{16}$ independent of $h$ and $\varrho$. Similarly, (4.20) implies

$$\|J_{h,\varrho}^{(3)}\|_2^2 \leq c(\delta)^2\frac{h^2}{\pi^2}(h\varrho)^{2\delta}\int_{\mathbb{R}}\int_{\mathbb{R}} |s-t|^{2\delta}(1+s^2)^{-1}(1+t^2)^{-1}\, dsdt =$$

$$c_{17}h^2(h\varrho)^{2\delta} \leq c_{17}h^{2+2\delta} \tag{4.28}$$

with $\delta \in (0,1/2)$, and $c_{17}$ independent of $h$ and $\varrho \in (0,1]$.

Fix $\varepsilon \in (0,1)$ and choose $h$ so small that we have $4c_{16}(1+\varepsilon^{-2})h^2 + 2c_{17}h^{2+2\delta} < 1$. Then (4.26)–(4.28) entails (4.25).

Next the operator $\mathcal{Z}_1^{\pm}(\varepsilon)$ is unitarily equivalent to "the operator"

$$\mathbf{j}_{h,\varrho}^{\pm}(\varepsilon) := \frac{1}{\pi}\left(\langle J_{h,\varrho}f, f\rangle_{L^2(\mathbb{R})} \pm \varepsilon\langle |J_{h,\varrho}^{(2)}|f, f\rangle_{L^2(\mathbb{R})} \pm \langle |J_{h,\varrho}^{(3)}|f, f\rangle_{L^2(\mathbb{R})}\right) =$$

$$h\left(-\ln(h\varrho) - \gamma_E\right) \pm \frac{\varepsilon}{\pi}\langle |J_{h,\varrho}^{(2)}|f, f\rangle_{L^2(\mathbb{R})} + \frac{1}{\pi}\langle\left(J_{h,\varrho}^{(3)} \pm |J_{h,\varrho}^{(3)}|\right) f, f\rangle_{L^2(\mathbb{R})}$$

acting in $\mathbf{C}$. Here we have taken into account the identities

$$\frac{1}{\pi}\langle J_{h,\varrho}^{(1)}f, f\rangle_{L^2(\mathbb{R})} = h(-\ln(h\varrho) - \gamma_E + \ln 2),$$

$$\frac{1}{\pi}\langle J_{h,\varrho}^{(2)}f, f\rangle_{L^2(\mathbb{R})} = -\frac{h}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \ln|s-t|(1+s^2)^{-1}(1+t^2)^{-1}\, dsdt = -h\ln 2.$$

Therefore, the identities

$$n_+(1; \mathcal{Z}_1^{\pm}(\varepsilon)) = n_+(1; \mathbf{j}_{h,\varrho}^{\pm}(\varepsilon)). \tag{4.29}$$

are valid for each $h > 0$, $\varrho \in (0,1]$, and $\varepsilon \in (0,1)$.

Fix an arbitrary $\eta > 0$, and bearing in mind (4.27)–(4.28), choose $\varepsilon > 0$ and $h > 0$ so small that we have

$$\pm \mathbf{j}^{\pm}_{h,\varrho}(\varepsilon) \leq \pm \tilde{\mathbf{j}}^{\pm}_{h,\varrho}(\eta) := \pm h \left(-\ln(h\varrho) - \gamma_E \pm \eta\right).$$

Thus we get

$$\pm n_+(1; \mathbf{j}^{\pm}_{h,\varrho}(\varepsilon)) \leq \pm n_+(1; \tilde{\mathbf{j}}^{\pm}_{h,\varrho}(\eta)). \tag{4.30}$$

Finally, we take into account the elementary relation

$$n_+(1; \tilde{\mathbf{j}}^{\pm}_{h,\varrho}(\eta)) = \begin{cases} 1, & \text{if} \quad \varrho < \frac{1}{h} e^{-1/h - \gamma_E \pm \eta}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.31}$$

Combining (4.5), (4.17), (4.18), (4.22)–(4.25), (4.29)–(4.31), we find that the estimates

$$\pm \mathcal{D}(\lambda) \leq \pm 2\pi h^4 \int_0^{\frac{1}{h} e^{-1/h - \gamma_E \pm \eta}} \varrho d\varrho = \pm \pi |\lambda|^{-1} e^{-2\sqrt{|\lambda|} - 2\gamma_E \pm 2\eta} \tag{4.32}$$

are valid for each fixed $\eta > 0$ and $|\lambda|$ large enough. Multiplying (4.32) by $|\lambda| e^{2\sqrt{|\lambda|}}$, letting at first $\lambda \to -\infty$, and then $\eta \downarrow 0$, we come to (4.14). $\diamondsuit$

**4.3.** In this subsection we consider potentials $V$ satisfying (4.1)–(4.2) with $\alpha \in [1,2)$ and general $W$.

**Theorem 4.1** *Let $V$ satisfy* (4.1)–(4.2) *with $\alpha \in (1,2)$ and $|W| \leq W_1 + W_2$ where $W_1(X) = c|X|^{-\alpha+\varepsilon_0}$, $c > 0$, $\varepsilon_0 \in (0, \alpha - 1)$, $W_2 \geq 0$, and the operator family $W_2(X_\perp)^{1/2} \left(-\frac{d^2}{dz^2} + 1\right)^{-1/2}$ satisfies assumption $\mathcal{H}_1$. Then* (4.5) *remains valid.*

**Proof.** Fix $\delta \in (0,1)$. The minimax principle yields

$$\pm N(\lambda; \chi(V(X_\perp))) \leq \pm N(\lambda; \chi((1 \mp \delta)^{-1} U(X_\perp))) +$$
$$N(\lambda; \chi(-2\delta^{-1} W_1(X_\perp))) + N(\lambda; \chi(-2\delta^{-1} W_2(X_\perp))). \tag{4.33}$$

On the other hand, Proposition 4.1 entails, for all $\delta \in (0,1)$,

$$\lim_{\lambda \to -\infty} |\lambda|^{1/(\alpha-1)} \int_{\mathbb{R}^2} N(\lambda; \chi((1 \mp \delta)^{-1} U(X_\perp)))\, dX_\perp = \mathcal{C}_\alpha((1 \mp \delta)^{-1} g), \tag{4.34}$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} N(\lambda; \chi(-\eta W_1(X_\perp)))\, dX_\perp &= O(|\lambda|^{-1/(\alpha - \varepsilon_0 - 1)}) \qquad (4.35) \\ &= o(|\lambda|^{-1/(\alpha-1)}),\ \lambda \to -\infty, \forall \eta > 0, \end{aligned}$$

Moreover, Lemma 3.1 implies

$$N(\lambda; \chi(-\eta W_2(X_\perp))) = 0,\ \forall X_\perp \in \mathbb{R}^2. \tag{4.36}$$

for every fixed $\eta > 0$ and sufficiently large $|\lambda|$.

Integrating (4.33) with respect to $X_\perp \in \mathbb{R}^2$, and bearing in mind (4.34)-(4.36), we get

$$\limsup_{\lambda \to -\infty} |\lambda|^{1/(\alpha-1)} \mathcal{D}(\lambda, g) \leq \mathcal{C}_\alpha((1-\delta)^{-1}g),$$

$$\liminf_{\lambda \to -\infty} |\lambda|^{1/(\alpha-1)} \mathcal{D}(\lambda, g) \geq \mathcal{C}_\alpha((1+\delta)^{-1}g),$$

with arbitrary $\delta \in (0,1)$. Letting $\delta \downarrow 0$, we come to (4.5). $\diamondsuit$

**Theorem 4.2** *Let $V$ satisfy* (4.1)–(4.2) *with $\alpha = 1$ and let the asymptotic estimate*

$$\left\| |W(X_\perp)|^{1/2} \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} \right\| = O\left(|\lambda|^{-\frac{1}{4}-\epsilon_0}\right), \ \lambda \to -\infty, \tag{4.37}$$

*hold with some $\varepsilon_0 > 0$ uniformly with respect to $X_\perp \in \mathbb{R}^2$. Then* (4.14) *remains valid.*

**Remark.** Let $|W| \leq W_1 + W_2$ where $W_1(X) = c|X|^{-\beta}$, $c > 0$, $\beta \in (0,1)$, and $W_2$ is non-negative, bounded and decays at infinity. Obviously,

$$\left\| W_1(X_\perp)^{1/2} \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} \right\| \leq \left\| W_1(0)^{1/2} \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} \right\|, \ \forall X_\perp \in \mathbb{R}^2.$$

Utilizing [12, Theorem XI.20], we easily get

$$\left\| W_1(0)^{1/2} \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} \right\| = O\left(|\lambda|^{\frac{1}{2}-\frac{1}{\beta}}\right), \ \lambda \to -\infty.$$

On the other hand,

$$\left\| W_2(X_\perp)^{1/2} \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} \right\| \leq \|W_2\|_{L^\infty(\mathbb{R}^3)} |\lambda|^{-1/2}, \ \lambda \to -\infty.$$

Hence, in this case $W$ satisfies (4.37).

**Proof of Theorem** 4.1**.** Assume $\lambda < -1$. Set $\delta(\lambda) := |\lambda|^{-\frac{1}{2}-\varepsilon_1}$ with $\varepsilon_1 \in (0, 2\varepsilon_0)$.

Similarly to (4.33) we write

$$\pm N(\lambda; \chi(V(X_\perp))) \leq \pm N(\lambda; \chi((1 \mp \delta(\lambda))^{-1} U(X_\perp))) + N(\lambda; \chi(\delta(\lambda)^{-1} W)(X_\perp))). \tag{4.38}$$

Taking into account (4.3) and (4.14) with $V = U = -g|X|^{-1}$, we get

$$\lim_{\lambda \to -\infty} |\lambda| e^{2\sqrt{|\lambda|}/g} \int_{\mathbb{R}^2} N(\lambda; \chi((1 \mp \delta(\lambda))^{-1} U(X_\perp)))\, dX_\perp =$$

$$\lim_{\lambda\to-\infty}(1\mp\delta)^2|\lambda|e^{2\sqrt{|\lambda|}/g}\int_{\mathbb{R}^2}N((1\mp\delta)^2\lambda;\chi(U(X_\perp))\,dX_\perp =$$

$$\lim_{t\to\infty}e^{\mp 2t^{-\varepsilon_1}/g}\lim_{s\to\infty}se^{2\sqrt{s}/g}\int_{\mathbb{R}^2}N(-s;\chi(U(X_\perp)))\,dX_\perp = \pi e^{-2\gamma_E}. \tag{4.39}$$

On the other hand, the Birman–Schwinger principle and the minimax principle entail

$$N(\lambda;\chi(\delta(\lambda)^{-1}W(X_\perp))) \leq$$

$$n_+\left(\delta(\lambda);\left(-\frac{d^2}{dz^2}-\lambda\right)^{-1/2}|W(X_\perp)|\left(-\frac{d^2}{dz^2}-\lambda\right)^{-1/2}\right)$$

Taking into account (4.37), we get

$$\begin{aligned}\|\left(-\frac{d^2}{dz^2}-\lambda\right)^{-1/2}|W(X_\perp)|\left(-\frac{d^2}{dz^2}-\lambda\right)^{-1/2}\| &= O\left(|\lambda|^{-\frac{1}{2}-2\varepsilon_0}\right)\\ &= o(\delta(\lambda)),\ \lambda\to-\infty\,.\end{aligned}$$

Hence, we have

$$N(\lambda;\chi(\delta(\lambda)^{-1}W(X_\perp))) = 0,\ \forall X_\perp\in\mathbb{R}^2, \tag{4.40}$$

for $|\lambda|$ large enough. Integrating (4.38) with respect to $X_\perp\in\mathbb{R}^2$, and taking into account (4.39) and (4.40), we obtain (4.14). $\diamondsuit$

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