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UNIFORMLY ISOCHRONOUS POLYNOMIAL CENTERS

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Abstract. We study a specific family of uniformly isochronous polynomial systems. Our results permit us to solve a problem about centers of such systems. We consider the composition conjecture for uniformly isochronous polynomial systems.

1. Introduction

Consider the planar autonomous system of ordinary differential equations

$$\begin{aligned} \dot{x} &= -y + xH(x,y), \\ \dot{y} &= x + yH(x,y), \end{aligned} \tag{1.1}$$

where $H(x,y)$ is a polynomial in x and y of degree n, and $H(0,0) = 0$. This system has only one singular point at $O(0,0)$ which is the center of the linear part of the system. The solutions of this system move around the origin with constant angular speed, and the origin is so a uniformly isochronous singular point.

The problem of characterizing uniformly isochronous centers has attracted attention of several authors; see [1]–[3], [8] and the references therein. In particular, the following problem was posed:

> It is true that all centers for uniformly isochronous polynomial systems are either reversible or admit a nontrivial polynomial commuting system?

The problem appeared in [3] and was mentioned as an open question in [2]. We prove the following proposition which permits to give a negative answer to the question.

Theorem 1.1. *Let a uniformly isochronous polynomial system have the form*

$$\begin{aligned} \dot{x} &= -y + xQ(x,y)\sum_{i=0}^{m} a_i(x^2+y^2)^i, \\ \dot{y} &= x + yQ(x,y)\sum_{i=0}^{m} a_i(x^2+y^2)^i, \end{aligned} \tag{1.2}$$

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where $Q(x, y)$ is a homogeneous polynomial in x, y of degree k and

$$\int_0^{2\pi} Q(\cos\vartheta, \sin\vartheta)d\vartheta = 0. \tag{1.3}$$

Then the origin is a center of (1.2). The center is of type B^ν with $\nu \le k$, and a "generic" center is of type B^1 if k is odd or of type B^2 if k is even.

Proof. System (1.2) can be written as the single separable equation

$$\frac{d\varrho}{d\vartheta} = \varrho^{k+1} Q(\cos\vartheta, \sin\vartheta) R(\varrho) \tag{1.4}$$

with ϱ, ϑ polar coordinates and $R(\varrho) = \sum_{i=0}^{m} a_i \varrho^{2i}$.

Equation (1.4) has a solution $\varrho \equiv 0$ defined for all ϑ. Therefore every solution $\varrho(\vartheta)$ with the initial value $\varrho(0) = \varrho_0$ where $\varrho_0 > 0$ is small enough is defined for $\vartheta \in [0, 2\pi]$ and satisfies the condition

$$\int_0^{\vartheta} Q(\cos\varphi, \sin\varphi)d\varphi = \int_{\varrho_0}^{\varrho(\vartheta)} \frac{dr}{r^{k+1}R(r)}. \tag{1.5}$$

From (1.5) we conclude that the solution is 2π-periodic, and so the origin is a center. The first part of the theorem is proved.

The center of a planar system is said to be of type B^ν if the boundary of the center region is the union of ν open unbounded trajectories [11]. By [12], the center of (1.2) is of type B^ν with $\nu \le n = k + 2m$.

The circles $x^2 + y^2 = \varrho_i^2$, with ϱ_i the roots of the equations $R(\varrho) = 0$, are trajectories of (1.2). All of them lie in the center region.

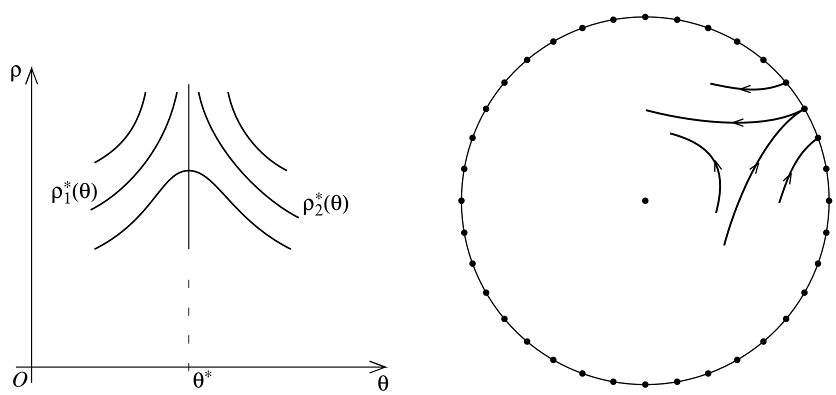

Figure 1. Solution curves of (1.4) in the (ϑ, ϱ)-plane (left) and trajectories of (1.2) in the Poincaré disc (right)

The unbounded trajectories of (1.2) correspond to unbounded solutions of (1.4). Studying the behaviour of solution curves of (1.4) at large ϱ, we can show that for every null isocline $\vartheta = \vartheta^*$ where solutions have a maximum there exist two solutions $\varrho_1^*(\vartheta)$, $\varrho_2^*(\vartheta)$, such that the isocline is a vertical asymptote

$$\lim_{\vartheta\to\vartheta^*-0} \varrho_1^*(\vartheta) = +\infty, \lim_{\vartheta\to\vartheta^*+0} \varrho_2^*(\vartheta) = +\infty,$$

(see Fig. 1).

In this situation, there is a relevant equilibrium point at infinity in the intersection of the equator of the Poincaré sphere with the ray $x = \varrho\cos\vartheta^*$, $y = \varrho\sin\vartheta^*$, $\varrho > 0$. The point has one hyperbolic sector with two separatrices corresponding to the solutions $\varrho_1^*(\vartheta)$, $\varrho_2^*(\vartheta)$ (see Fig. 1). The boundary of the center region consists of such separatrices. The number ν of these equilibrium points coincides with the number of the null isoclines of direction field (1.4) where solutions has a maximum for large ϱ. These isoclines are vertical lines $\vartheta = \vartheta_i^*$, where the values of ϑ_i^* are determined from the conditions $Q(\cos\vartheta, \sin\vartheta) = 0$, $0 \le \vartheta < 2\pi$. Hence, we have the estimate $\nu \le k$ at describing type B^ν of the center of (1.2).

If k is even our trigonometric polynomial $Q(\cos\vartheta, \sin\vartheta)$ has a period equal to π (but not 2π as it happens for odd k). Therefore (1.4) has an even number of the blocks discussed above and the relevant equilibrium points lie at the diameters of the Poincaré sphere. It may be noted that (1.2) is O-symmetric in this case. The upper bound k can be attained by ν. As an example we can consider (1.2) with $Q(\cos\vartheta, \sin\vartheta) = \sin k\vartheta$ and a_i arbitrary real numbers.

In a "generic" situation, (1.4) has no solution for which two different null isoclines are its asymptotes. Therefore, in such a situation the solution curve separating bounded and unbounded solutions has a minimum number of discontinuity points within $[0, 2\pi]$: one point if k is odd, and two points if k is even.

Hence a "generic" center is of type B^1 when k is odd or type B^2 when k is even (see Fig. 2). The theorem is proved. □

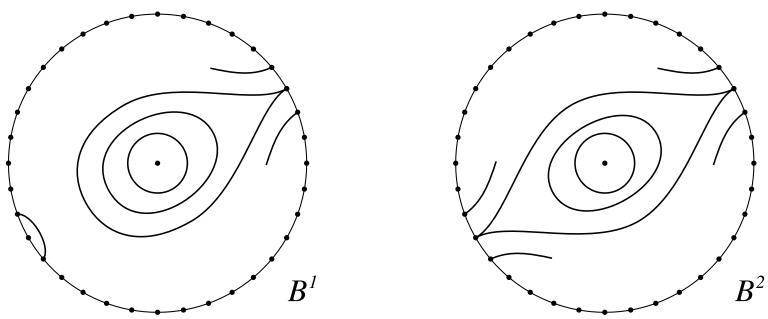

Figure 2. Phase portraits of Equation (1.2)

Remarks. In [12, Theorem 2.1] about the centers of homogeneous systems is a particular case of Theorem 1.1 set $m = 0, a_0 = 1$.

We can generalize the first part of Theorem 1.1 as follows. Let the polynomial $H(x, y)$ in (1.1) take the form

$$H(x, y) = (xp_y(x, y) - yp_x(x, y))h(x^2 + y^2, p(x, y)),$$

where $h(u, v)$ is a polynomial, $p(x, y)$ is a homogeneous polynomial of degree k. Then the origin is a center of (1.1). Indeed, the case under study (1.1) can be written as a single equation of the form

$$\frac{d\varrho}{d\vartheta} = \varrho^{k+1}h(\varrho^2, \varrho^k f(\vartheta))f'(\vartheta)$$

with ϱ, ϑ polar coordinates and $f(\vartheta) = p(\cos\vartheta, \sin\vartheta)$. The solutions of this equation are clearly some functions of $f(\vartheta)$. The function $f(\vartheta)$ is 2π-periodic. Then, the solutions with small enough initial values are 2π-periodic functions as well. So, the origin is a center.

The functions

$$f_1(x,y) = x^2 + y^2, \quad f_2(x,y) = \sum_{i=0}^{m} a_i (x^2+y^2)^i$$

are invariants for (1.2) with the respective cofactors

$$\begin{aligned} K_1(x,y) &= 2Q(x,y) \sum_{i=0}^{m} a_i (x^2+y^2)^i, \\ K_2(x,y) &= 2Q(x,y) \sum_{i=0}^{m} i a_i (x^2+y^2)^i. \end{aligned}$$

We have

$$\frac{k+2}{2} K_1(x,y) + K_2(x,y) = \operatorname{div},$$

where div is the divergence of (1.2). In this case the function $\mu(x,y) = f_1^{(k+2)/2} f_2$ is a reciprocal integrating factor of the Darboux form. For algebraic invariants and Darboux's method of integration see [15], [16], for example. The factor gives information about our system. For instance, it may be used to find a first Darboux integral of (1.2), [7]. A first integral of (1.2) may be found from (1.5) as well.

It is obvious that (1.2) commutes with the system

$$\begin{aligned} \dot{x} &= x(x^2+y^2)^{k/2} \sum_{i=0}^{m} a_i (x^2+y^2)^i, \\ \dot{y} &= y(x^2+y^2)^{k/2} \sum_{i=0}^{m} a_i (x^2+y^2)^i. \end{aligned} \tag{1.6}$$

If k is even (1.6) gives a polynomial commuting system without a linear part. If k is odd then we have a non-polynomial commuting system. Nevertheless a polynomial commuting system may exist in the case of odd k. For example, if (1.2) is homogeneous ($m = 0$, $a_0 = 1$) then there exists a polynomial system commuting with (1.2), [14].

Using Theorem 1.1, we may construct an example of an uniformly isochronous system that is not reversible and commutes with no polynomial system.

A planar differential system is said to be reversible if its corresponding direction field is symmetric with respect to a straight line passing through the origin (a symmetric line). If a system is reversible then its trajectories are symmetric with respect to a symmetric line (Necessary and sufficient conditions for reversibility of planar analytic vector fields were derived in [10]). If a symmetric line of (1.2) is $x \sin\vartheta^* - y \cos\vartheta^* = 0$, then the vertical line $\vartheta = \vartheta^*$ is the symmetric axis of the graph of the trigonometric polynomial $Q(\cos\vartheta, \sin\vartheta)$ and the symmetric axis of solution curves of (1.4).

Conditions for the existence of a polynomial commuting systems for uniformly isochronous polynomial systems was considered in [1], [3]. In particular, it was proved that (1.1) commutes with a polynomial system if and only if the function $H(x,y)$ satisfies one of the following two conditions:

$$H(x,y) = P_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j \tag{1.7}$$

where $P_{2l}(x,y)$ is a homogeneous polynomial of degree $2l, l \geq 0$. Or there are homogeneous polynomials α_l, β_l of order l ($l \leq n$, l divides n), satisfying $x\partial_y\beta_l - y\partial_x\beta_l = l\alpha_l$ such that

$$H(x,y) = \alpha_l \sum_{k=0}^{n/l-1} a_k\beta_l^k. \tag{1.8}$$

So, to construct the example in question it suffices to take a system of the form (1.2) where the homogeneous polynomial $Q(x,y)$ is of an odd degree (In this case (1.5) is fulfilled and the function $H(x,y)$ is not of the form (1.7)), the graph of the trigonometric polynomial $Q(\cos\vartheta,\sin\vartheta)$ has no symmetric axes, and the numbers m, a_i are such that the function

$$H(x,y) = Q(x,y)\sum_{i=0}^{m} a_i(x^2+y^2)^i$$

is not of the form (1.8). Put

$$Q(x,y) = y^3 - 3xy^2 + 2x^2y = y(x-y)(2x-y), \quad m=1, \quad a_0 = a_1 = 1.$$

Then

$$\begin{gathered}\dot{x} = -y + x(y^3 - 3xy^2 + 2x^2y)(1+x^2+y^2),\\ \dot{y} = x + y(y^3 - 3xy^2 + 2x^2y)(1+x^2+y^2).\end{gathered} \tag{1.9}$$

According to Theorem 1.1, (1.9) has a (isochronous) center at the origin. The function

$$I(x,y) = \frac{r^6}{(1-3r^2-4x^3-3xy^2-3y^3-3r^3\arctan r)^2}, \quad r^2 = x^2+y^2.$$

is a first integral of (1.9) obtained from (1.5). Fig. 3 depicts the phase portrait of (1.9). The center is clearly of type B^1.

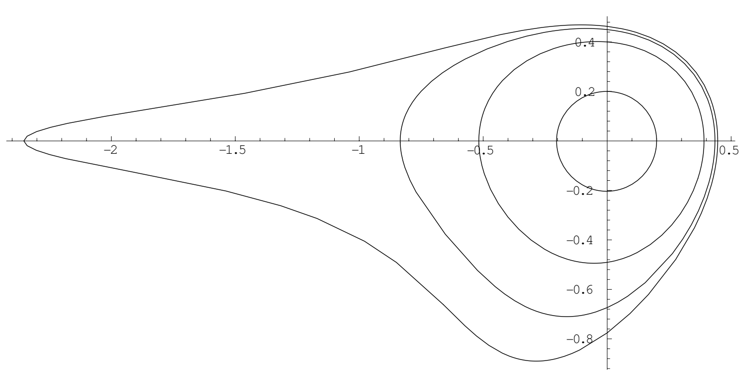

FIGURE 3. Phase portrait for Equation (1.9)

Evidently, the graph of $Q(\cos\vartheta, \sin\vartheta)$ has no symmetric axes and therefore system (1.9) is nonreversible. We may show that if the graph of the homogeneous trigonometric polynomial

$$T_3(\vartheta) = a_1 \cos\vartheta + a_3 \cos 3\vartheta + b_1 \sin\vartheta + b_3 \sin 3\vartheta$$

has a symmetric axis then its coefficients satisfy the condition

$$a_1 b_3 (a_1^2 - 3b_1^2) = a_3 b_1 (3a_1^2 - b_1^2).$$

It easy to verify that system (1.9) may fail to commute with any non-proportional polynomial systems. This fact follows from the impossibility of presenting the function

$$H(x,y) = (y^3 - 3xy^2 + 2x^2y)(1 + x^2 + y^2) \equiv H_3(x,y) + H_5(x,y)$$

in the form (1.8) but we can also prove it in a different way.

Indeed, assume that (1.9) commutes with a polynomial system of degree n

$$\begin{aligned} \dot{x} &= R(x,y) \equiv R_1(x,y) + R_2(x,y) + \cdots + R_n(x,y), \\ \dot{y} &= S(x,y) \equiv S_1(x,y) + S_2(x,y) + \cdots + S_n(x,y), \end{aligned} \tag{1.10}$$

where $R_i(x,y)$, $S_i(x,y)$ are homogeneous polynomials of degree i.

Then the Lie bracket between vector fields (1.9), (1.10) is equal to zero:

$$[(-y + xH(x,y), x + yH(x,y))^T, (R(x,y), S(x,y))^T] = (0,0)^T.$$

In particular, we have the terms of highest degree equal to zero:

$$[(xH_5(x,y), yH_5(x,y))^T, (R_n(x,y), S_n(x,y))^T] = (0,0)^T.$$

After transformations using Euler's theorem for homogeneous functions, this equality may be written as

$$\begin{gathered} (xH_{5x}(x,y) + (1-n)H_5(x,y))R_n(x,y) + xH_{5y}(x,y)S_n(x,y) = 0, \\ yH_{5x}(x,y)R_n(x,y) + (yH_{5y}(x,y) + (1-n)H_5(x,y))S_n(x,y) = 0. \end{gathered}$$

The linear system for the polynomials $R_n(x,y)$, $S_n(x,y)$ has a nontrivial solution if its determinant Δ is equal to zero:

$$\begin{aligned} \Delta \equiv &(xH_{5x}(x,y) + (1-n)H_5(x,y))\big(yH_{5y}(x,y) + (1-n)H_5(x,y)\big) \\ &- xyH_{5x}(x,y)H_{5y}(x,y) = 0. \end{aligned}$$

Since $xH_{5x}(x,y) + yH_{5y}(x,y) = 5H_5(x,y)$, we infer

$$\Delta = (1-n)(6-n)H_5^2(x,y) = 0.$$

Therefore, either $n = 6$ or $n = 1$. It is obvious that $n \neq 1$. We now compute the Lie brackets of (1.9) and (1.10) with $n = 6$. We obtain a polynomial vector field. Equating to zero the coefficients of the polynomials we derive a system of linear equations for coefficients of (1.10). The system is simple and may be solved by successive substitutions. We used the software package *Mathematica* here. Our calculations show that in the case under study the polynomial commuting system (1.10) is proportional to (1.9). According to (1.6), system (1.9) commutes with the system

$$\begin{aligned} \dot{x} &= x(x^2 + y^2)\sqrt{x^2 + y^2}(1 + x^2 + y^2), \\ \dot{y} &= y(x^2 + y^2)\sqrt{x^2 + y^2}(1 + x^2 + y^2). \end{aligned}$$

Hence (1.9) has a center but is nonreversible and commutes with no polynomial system nonproportional to it.

We derive that the answer to the question from [2], [3] is negative.

2. Results

Note that the following condition is fulfilled for central systems of the form (1.1) which are reversible, or have a polynomial commuting system, or are described in Theorem 1.1: In polar coordinates these systems may be written as

$$\begin{gathered}\dot\varrho = \varrho H(\varrho\cos\vartheta, \varrho\sin\vartheta) = h(\varrho, f(\vartheta))f'(\vartheta),\\ \dot\vartheta = 1.\end{gathered} \tag{2.1}$$

Indeed, let (1.1) be reversible. Without loss of generality we can take the y-axes as its symmetric line. Then

$$-y + xH(x,y) = -y - xH(-x,y), \quad x + yH(x,y) = -(-x + yH(-x,y)),$$

and the polynomial $H(x,y)$ satisfies the condition

$$H(x,y) = -H(-x,y).$$

Hence, the polynomial may be written as

$$H(x,y) = x\tilde H(x^2,y).$$

Passing to polar coordinates, we find that (1.1) is transformed into

$$\begin{gathered}\dot\varrho = \varrho^2\cos\vartheta\tilde H(\varrho^2\cos^2\vartheta, \varrho\sin\vartheta) = h(\varrho,\sin\vartheta))\cos\vartheta,\\ \dot\vartheta = 1,\end{gathered}$$

and the system has the form (2.1) for $f(\vartheta) = \sin\vartheta$.

It was proved in [3] that if (1.1) has a polynomial commuting system then it may be written in the form (2.1). The systems in Theorem 1.1 obviously satisfy our condition. Therefore, we supposed naturally that the following statement is true:

> System (1.1) has a center at the origin if and only if the function $H(x,y)$ may be written in the form (2.1).

We called this statement the composition claim for uniformly isochronous systems, by analogy with the composition conjecture for the center problem for the Abel equation (see [4]–[6], [17]).

Now we recall some definitions and results (see for example [4]). Consider the Abel equation

$$\dot z = a_2(t)z^2 + \dots + a_n(t)z^n,$$

where $a_i(t)$ are homogeneous 2π-periodic functions.

Then the equation has center $z = 0$ if all solutions $z(t)$, starting near the origin, satisfy the condition $z(0) = z(2\pi)$.

The composition conjecture for the center problem for the Abel equation appeared in [6]. It reads that the center problem is equivalent to the fact that the functions $a_i(t)$ may be presented in the form

$$a_i(t) = a_{i,1}(s(t))s'(t), \tag{2.2}$$

for some continuous functions $a_{i,1}(\cdot)$ and a differentiable function $s(t)$ with $s(0) = s(2\pi)$. Condition 2.2 is the composition condition for $a_i(t)$ or the composition condition for the Abel equation.

It was shown in [4] that the conjecture is not true if $a_i(t)$ are polynomials in $\cos t$ and $\sin t$. The following Abel equation was presented as the counterexample

$$\frac{dz}{d\vartheta} = A(\vartheta)z^3 + B(\vartheta)z^2, \tag{2.3}$$

where

$$\begin{gathered} A(\vartheta) = -f(\vartheta)g(\vartheta), \quad B(\vartheta) = f(\vartheta) - g'(\vartheta), \\ f(\vartheta) = h\cos^3\vartheta + 3\cos^2\vartheta\sin\vartheta + (3h+6k)\cos\vartheta\sin^2\vartheta - \sin^3\vartheta, \\ g(\vartheta) = \cos^3\vartheta + (2h+5k)\cos^2\vartheta\sin\vartheta - 3\cos\vartheta\sin^2\vartheta - k\sin^3\vartheta \end{gathered} \tag{2.4}$$

with $1 + hk + 2k^2 = 0$.

The choice of functions was motivated by the following arguments. Consider the quadratic system

$$\begin{aligned} \dot{x} &= -y - bx^2 - Cxy - dy^2, \\ \dot{y} &= x + ax^2 + Axy - ay^2, \end{aligned}$$

for which the conditions

$$2A + C = A + 3b + 5d = a^2 + bd + 2d^2 = 0$$

are sufficient for the origin to be a center (see for example [13]).

Let $b = -h$, $C = -2$, $d = -k$, $a = 1$, $A = 3h + 5k$. Then the system

$$\begin{aligned} \dot{x} &= -y + hx^2 + 2xy + ky^2, \\ \dot{y} &= x + x^2 + (3h+5k)xy - y^2 \end{aligned} \tag{2.5}$$

has a center at the origin if $1 + hk + 2k^2 = 0$. The first integral of (2.5) is

$$I(x,y) = \frac{(k^2 - 3k\kappa y + 3k\kappa xy + 3k^2\kappa y^2 - \kappa(x+ky)^3)^2}{(k^2 - 2k\kappa y + \kappa(x+ky)^2)^3}, \quad \kappa = 1 + k^2.$$

In polar coordinates $x = \varrho\cos\vartheta$, $y = \varrho\sin\vartheta$, (2.5) is transformed into the system

$$\begin{aligned} \dot{\varrho} &= \varrho^2 f(\vartheta), \\ \dot{\vartheta} &= 1 + \varrho g(\vartheta), \end{aligned}$$

where the functions $f(\vartheta)$ and $g(\vartheta)$ are from (2.4). By the Cherkas transformation [9], $r = \varrho/(1 + g(\vartheta))$, we reduce this system to the equation

$$\frac{dz}{d\vartheta} = A(\vartheta)z^3 + B(\vartheta)z^2,$$

where $A(\vartheta)$, $B(\vartheta)$ are from (2.4). The closed trajectories of (2.5) correspond to 2π-periodic solutions of (2.4). So, $z = 0$ is a center for (2.3).

It was shown in [4] that $A(\vartheta), B(\vartheta)$ are not of the form $A(\vartheta) = a_1(s(\vartheta))s'(\vartheta)$, $B(\vartheta) = b_1(s(\vartheta))s'(\vartheta)$, and hence the composition conjecture for (2.3) is not true.

Our composition claim for uniformly isochronous systems was based on the assumption that such systems are transformed to the specific Abel equations such that the composition conjecture is true. However, this assumption is false.

Consider the uniformly isochronous system

$$\begin{aligned} \dot{x} &= -y + x(h_3(x,y) + h_6(x,y))/3, \\ \dot{y} &= x + y(h_3(x,y) + h_6(x,y))/3, \end{aligned} \tag{2.6}$$

where the functions $h_3(x,y)$, $h_6(x,y)$ are defined as follows: we take the functions $\tilde{A}(\varrho,\vartheta) = \varrho^6 A(\vartheta)$, $\tilde{B}(\varrho,\vartheta) = \varrho^3 B(\vartheta)$, where $A(\vartheta)$, $B(\vartheta)$ are from (2.4) and put $\varrho\cos\vartheta = x$ and $\varrho\sin\vartheta = y$. We obtain

$$h_3(x,y) = ((h+5k)x^3 - 12x^2y + (-7h-19k)xy^2 + 4y^3),$$
$$h_6(x,y) = (-hx^3 - 3x^2y - (3h+6k)xy^2 + y^3)(x^3 + (2h+5k)x^2y - 3xy^2 - ky^3).$$

System (2.6) is transformed into

$$\begin{aligned} \dot{\varrho} &= \varrho(A(\vartheta)\varrho^6 + B(\vartheta)\varrho^3)/3, \\ \dot{\vartheta} &= 1, \end{aligned}$$

and the substitution $z = \varrho^3$ makes the latter into the above discussed equation

$$\frac{dz}{d\vartheta} = A(\vartheta)z^3 + B(\vartheta)z^2.$$

It follows that the uniformly isochronous system (2.6) has a center but contradicts our composition claim.

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