Electronic Journal of Differential Equations, Vol. 2007(2007), No. 123, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SOLVABILITY OF A FOUR-POINT BOUNDARY-VALUE PROBLEM FOR FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

JING GE, CHUANZHI BAI

ABSTRACT. In this paper we investigate the existence of solutions of a class of four-point boundary-value problems for fourth-order ordinary differential equations. Our analysis relies on a fixed point theorem due to Krasnoselskii and Zabreiko.

1. Introduction

In recent years, boundary-value problems for second and higher order differential equations have been extensively studied. The monographs of Agarwal [1] and Agarwal, O'Regan, and Wong [2] contain excellent surveys of known results. Recently an increasing interest in studying the existence of solutions and positive solutions to boundary-value problems for higher order differential equations is observed; see for example [3, 4, 5, 6, 7, 8].

Very recently, Zhang, Chen and Lü [10] by using the upper and lower solution method investigated the fourth order nonlinear ordinary differential equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \tag{1.1}$$

with the four-point boundary conditions

$$\begin{gathered} u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0, \end{gathered} \tag{1.2}$$

where a, b, c, d are nonnegative constants satisfying $ad + bc + ac(\xi_2 - \xi_1) > 0$, $0 \le \xi_1 < \xi_2 \le 1$ and $f \in C([0,1] \times \mathbb{R} \times \mathbb{R})$. They proved the following Lemma (a key lemma):

Lemma 1.1 ([10, Lemma 2.2])**.** *Suppose a, b, c, d, ξ_1, ξ_2 are nonnegative constants satisfying $0 \le \xi_1 < \xi_2 \le 1$, $b - a\xi_1 \ge 0$, $d - c + c\xi_2 \ge 0$ and $\delta = ad + bc + ac(\xi_2 - \xi_1) \ne 0$. If $u(t) \in C^4[0,1]$ satisfies*

- $u^{(4)}(t) \ge 0$, $t \in (0,1)$,
- $u(0) \ge 0$, $u(1) \ge 0$,

2000 *Mathematics Subject Classification.* 34B10, 34B15.
Key words and phrases. Four-point boundary-value problem; existence of solutions; fixed point theorem.

Submitted May 10, 2007. Published September 21, 2007.
Supported by grants NNSFC-10771212 and 06KJB110010.

- $au''(\xi_1) - bu'''(\xi_1) \le 0$, $cu''(\xi_2) + du'''(\xi_2) \le 0$,

then $u(t) \ge 0$ *and* $u''(t) \le 0$ *for* $t \in [0,1]$.

Unfortunately this Lemma is wrong as shown below.

Counterexample to [10, Lemma 2.2]**.** Let $u(t) = \frac{1}{3}t^4 + \frac{1}{4}t^3 - \frac{4}{3}t^2 + \frac{3}{4}t$ which belongs to $C^4[0,1]$, $\xi_1 = \frac{1}{10}$, $\xi_2 = \frac{1}{8}$, a, b, c, d be nonnegative constants satisfying $b \ge \frac{1}{10}a = a\xi_1$, $d = \frac{15}{16}c > \frac{7}{8}c = (1-\xi_2)c$ and $\delta = ad + bc + \frac{1}{40}ac \neq 0$. Then we have

$$
\begin{gathered}
u^{(4)}(t) = 8 > 0, \quad t \in (0,1), \\
u(0) = 0, \quad u(1) = 0, \\
\begin{aligned}
au''(\xi_1) - bu'''(\xi_1) &= a\Big[4t^2 + \frac{3}{2}t - \frac{8}{3}\Big]_{t=1/10} - b\Big[8t + \frac{3}{2}\Big]_{t=1/10} \\
&= -2\frac{143}{300}a - 2\frac{3}{10}b \le 0,
\end{aligned}
\end{gathered}
$$

and

$$
\begin{aligned}
cu''(\xi_2) + du'''(\xi_2) &= c\Big[4t^2 + \frac{3}{2}t - \frac{8}{3}\Big]_{t=1/8} + d\Big[8t + \frac{3}{2}\Big]_{t=1/8} \\
&= -\frac{29}{12}c + \frac{5}{2}d = -\frac{29}{12}c + \frac{5}{2}\cdot\frac{15}{16}c = -\frac{7}{96}c \le 0.
\end{aligned}
$$

But

$$
u(\frac{8}{9}) = -0.0031 < 0;
$$

that is, [10, Lemma 2.2] is incorrect.

So the conclusions of [10] should be reconsidered. The aim of this paper is to investigate the existence of solutions of the BVP (1.1)-(1.2) by using a fixed point theorem due to Krasnoselskii and Zabreiko in [9].

2. Main result

First, we give some lemmas which are needed in our discussion of the main results.

Lemma 2.1. *Suppose* a, b, c, d, ξ_1, ξ_2 *are nonnegative constants satisfying* $0 \le \xi_1 < \xi_2 \le 1$ *and* $\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0$. *If* $h \in C[0,1]$, *then the boundary-value problem*

$$
v''(t) = h(t), \quad t \in [0,1], \tag{2.1}
$$

$$
av(\xi_1) - bv'(\xi_1) = 0, \quad cv(\xi_2) + dv'(\xi_2) = 0, \tag{2.2}
$$

has a unique solution

$$
v(t) = \int_{\xi_1}^{t} (t-s)h(s)ds + \frac{1}{\delta}\int_{\xi_1}^{\xi_2} (a(\xi_1 - t) - b)(c(\xi_2 - s) + d)h(s)ds. \tag{2.3}
$$

Proof. By (2.1), it is easy to know that

$$
v(t) = C_1 + C_2 t + \int_0^t (t-s)h(s)ds, \tag{2.4}
$$

where C_1, C_2 are any two constants. Substituting (2.4) into boundary conditions (2.2), by a routine calculation, we get

$$C_1 = \int_0^{\xi_1} sh(s)ds + \frac{1}{\delta}(a\xi_1 - b)\int_{\xi_1}^{\xi_2}(c(\xi_2 - s) + d)h(s)ds, \tag{2.5}$$

$$C_2 = -\int_0^{\xi_1} h(s)ds - \frac{a}{\delta}\int_{\xi_1}^{\xi_2}(c(\xi_2 - s) + d)h(s)ds. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we obtain (2.3) which implies lemma. □

Remark 2.2. Let $\xi_1 = 0$, $\xi_2 = 1$, then (2.3) reduces to

$$v(t) = -\int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{\delta}\begin{cases}(as+b)(d+c(1-t)), & 0 \le s \le t \le 1,\\ (at+b)(d+c(1-s)), & 0 \le t < s \le 1.\end{cases}$$

Remark 2.3. Let

$$R(t) = \frac{1}{\delta}((a(t-\xi_1)+b)x_3 + (c(\xi_2 - t)+d)x_2),$$

$$G_1(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1,\\ t(1-s), & 0 \le t < s \le 1,\end{cases} \tag{2.7}$$

$$G_2(t,s) = \frac{1}{\delta}\begin{cases}(a(s-\xi_1)+b)(d+c(\xi_2 - t)), & \xi_1 \le s \le t \le \xi_2,\\ (a(t-\xi_1)+b)(d+c(\xi_2 - s)), & \xi_1 \le t < s \le \xi_2.\end{cases}$$

In [10, Lemma 2.2] it is claimed that

$$u(t) = tx_1 + (1-t)x_0 - \int_0^1 G_1(t,\xi)R(\xi)d\xi + \int_0^1 G_1(t,\xi)\int_{\xi_1}^{\xi_2} G_2(\xi,s)h(s)dsd\xi, \tag{2.8}$$

is the solution of the boundary-value problem

$$u^{(4)}(t) = h(t), \quad 0 < t < 1,$$
$$u(0) = x_0, \quad u(1) = x_1,$$
$$au''(\xi_1) - bu'''(\xi_1) = x_2, \quad cu''(\xi_2) + du'''(\xi_2) = x_3.$$

However (2.8) is wrong. Indeed, by Lemma 2.1, (2.8) should be replaced by

$$u(t) = tx_1 + (1-t)x_0 - \int_0^1 G_1(t,\xi)R(\xi)d\xi - \int_0^1 G_1(t,\eta)v(\eta)d\eta,$$

where

$$v(\eta) = \int_{\xi_1}^{\eta}(\eta - s)h(s)ds + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(a(\xi_1 - \eta) - b)(c(\xi_2 - s) + d)h(s)ds.$$

Remark 2.4. In [10, Theorem 3.1], the operator $T : C[0,1] \to C[0,1]$ is defined as

$$Tu(t) = \int_0^1 G_1(t,\eta)\int_{\xi_1}^{\xi_2} G_2(\eta,s)f(s,u(s),u''(s))dsd\eta,$$

where $G_1(t,s)$ and $G_2(t,s)$ are as in Remark 2.2. By 2.1 and Remark 2.2, the definition of T is incorrect. In fact, the operator T should be defined as

$$\begin{aligned} Tu(t) &= \int_0^1 G_1(t,\eta)\int_{\xi_1}^{\eta}(s-\eta)f(s,u(s),u''(s))dsd\eta \\ &\quad + \frac{1}{\delta}\int_0^1 G_1(t,\eta)\int_{\xi_1}^{\xi_2}(b-a(\xi_1-\eta))(c(\xi_2-s)+d)f(s,u(s),u''(s))dsd\eta. \end{aligned}$$

The following well-known fixed point theorem [9] will play an important role in the proof of our theorem.

Lemma 2.5. *Let X be a Banach space, and $F : X \to X$ be completely continuous. Assume that $A : X \to X$ is a bounded linear operator such that 1 is not an eigenvalue of A and*

$$\lim_{\|x\|\to\infty}\frac{\|F(x)-A(x)\|}{\|x\|}=0.$$

Then F has a fixed point in X.

Let $X = C^2[0,1]$ be endowed with the norm by

$$\|u\|_0 = \max\{\|u\|, \|u''\|\},$$

where $\|u\| = \max_{0\le t\le 1}|u(t)|$.

We are now in a position to present and prove our main result. Let

(H1) a, b, c, d, ξ_1, ξ_2 are nonnegative constants satisfying $0 \le \xi_1 < \xi_2 \le 1$, $b - a\xi_1 \ge 0$ and $\delta = ad + bc + ac(\xi_2-\xi_1) \ne 0$,

(H2) $f(t,u,v) = p(t)g(u) + q(t)h(v)$, where $g, h : \mathbb{R} \to \mathbb{R}$ are continuous with

$$\lim_{u\to\infty}\frac{g(u)}{u}=\lambda, \quad \lim_{v\to\infty}\frac{h(v)}{v}=\mu,$$

where $p, q \in C[0,1]$. Moreover, there exists some $t_0 \in [0,1]$ such that $p(t_0)g(0)+q(t_0)h(0) \ne 0$, and there exists a continuous nonnegative function $w : [0,1] \to \mathbb{R}^+$ such that $|p(s)| + |q(s)| \le w(s)$ for each $s \in [0,1]$.

Theorem 2.6. *Assume* (H1)–(H2). *If* $\max\{|\lambda|, |\mu|\} < \min\{\frac{1}{L_1}, \frac{1}{L_2}\}$, *where*

$$\begin{aligned} L_1 = \frac{1}{12}\Big[&\int_0^{\xi_1}\tau^3(2-\tau)w(\tau)d\tau + \int_{\xi_1}^1(1-\tau)^3(1+\tau)w(\tau)d\tau \\ &+ \frac{2(b-a\xi_1)+a}{\delta}\int_{\xi_1}^{\xi_2}(c(\xi_2-\tau)+d)w(\tau)d\tau\Big], \end{aligned}$$

and

$$L_2 = \int_{\xi_1}^1(1-\tau)w(\tau)d\tau + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b+a(1-\xi_1))(c(\xi_2-\tau)+d)w(\tau)d\tau,$$

then BVP (1.1) *and* (1.2) *has at least one nontrivial solution* $u \in C^2[0,1]$.

Proof. Define an operator $F : C^2[0,1] \to C^2[0,1]$ by

$$\begin{aligned} Fu(t) := &\int_0^1 G_1(t,s)\int_{\xi_1}^{s}(\tau-s)[p(\tau)g(u(\tau))+q(\tau)h(u''(\tau)]d\tau ds \\ &+ \frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2}(b-a(\xi_1-s))(c(\xi_2-\tau)+d) \\ &\times \big[p(\tau)g(u(\tau))+q(\tau)h(u''(\tau)\big]d\tau ds, \end{aligned} \tag{2.9}$$

where $G_1(t,s)$ is as in (2.7). Then by Lemma 2.1 and Remark 2.4, we easily know that the fixed points of F are the solutions to the boundary-value problem (1.1) and (1.2). It is well known that the operator F is a completely continuous operator. Now, we consider the following boundary-value problem

$$\begin{gathered} u^{(4)}(t) = \lambda p(t)u(t) + \mu q(t)u''(t), \quad 0 < t < 1 \\ u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0. \end{gathered} \tag{2.10}$$

Define

$$\begin{aligned} Au(t) := &\int_0^1 G_1(t,s)\int_{\xi_1}^s (\tau - s)[\lambda p(\tau)u(\tau) + \mu q(\tau)u''(\tau)d\eta ds \\ &+ \frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} (b - a(\xi_1 - s))(c(\xi_2 - \tau) + d) \\ &[\lambda p(\tau)u(\tau) + \mu q(\tau)u''(\tau)]d\eta ds. \end{aligned} \tag{2.11}$$

Obviously, A is a bounded linear operator. Furthermore, the fixed point of A is a solution of the BVP (2.10) and conversely.

We now assert that 1 is not an eigenvalue of A. In fact, if $\lambda = 0$ and $\mu = 0$, then the BVP (2.10) has no nontrivial solution. If $\lambda \neq 0$ or $\mu \neq 0$, suppose the BVP (2.10) has a nontrivial solution u and $\|u\|_0 > 0$, then

$$\begin{aligned} |Au(t)| \le &\int_0^1 G_1(t,s)\int_{\xi_1}^s |(\tau - s)[\lambda p(\tau)u(\tau) + \mu q(\tau)u''(\tau)]|d\tau ds \\ &+ \frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} \big|(b - a(\xi_1 - s))(c(\xi_2 - \tau) + d) \\ &\times [\lambda p(\tau)u(\tau) + \mu q(\tau)u''(\tau)]\big|d\tau ds \\ \le &\int_0^1 s(1-s)\int_{\xi_1}^s (s-\tau)[|\lambda||p(\tau)||u(\tau)| + |\mu||q(\tau)||u''(\tau)|]d\tau ds \\ &+ \frac{1}{\delta}\int_0^1 s(1-s)\int_{\xi_1}^{\xi_2} (b - a(\xi_1 - s))(c(\xi_2 - \tau) + d) \\ &\times [|\lambda||p(\tau)||u(\tau)| + |\mu||q(\tau)||u''(\tau)|]d\tau ds \\ \le &\Big[\int_0^1 s(1-s)\int_{\xi_1}^s (s-\tau)[|\lambda||p(\tau)| + |\mu||q(\tau)|]d\tau ds + \frac{1}{\delta}\int_0^1 s(1-s) \\ &\times \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - s))(c(\xi_2 - \tau) + d)[|\lambda||p(\tau)| + |\mu||q(\tau)|]d\tau ds\Big]\|u\|_0 \\ = &\frac{1}{12}\Big[\int_0^{\xi_1} \tau^3(2-\tau)(|\lambda||p(\tau)| + |\mu||q(\tau)|)d\tau \\ &+ \int_{\xi_1}^1 (1-\tau)^3(1+\tau)(|\lambda||p(\tau)| + |\mu||q(\tau)|)d\tau \\ &+ \frac{2(b - a\xi_1) + a}{\delta}\int_{\xi_1}^{\xi_2} (c(\xi_2 - \tau) + d)(|\lambda||p(\tau)| + |\mu||q(\tau)|)d\tau\Big]\|u\|_0 \\ \le &\max\{|\lambda|, |\mu|\}\frac{1}{12}\Big[\int_0^{\xi_1} \tau^3(2-\tau)w(\tau)d\tau + \int_{\xi_1}^1 (1-\tau)^3(1+\tau)w(\tau)d\tau \end{aligned}$$

$$+ \frac{2(b-a\xi_1)+a}{\delta}\int_{\xi_1}^{\xi_2}(c(\xi_2-\tau)+d)w(\tau)d\tau\Big]\|u\|_0, \quad t\in[0,1],$$

which implies that

$$|Au(t)| \le \max\{|\lambda|,|\mu|\}L_1\|u\|_0 < \frac{1}{L_1}L_1\|u\|_0 = \|u\|_0.$$

On the other hand, we have

$$\begin{aligned}
|(Au)''(t)| &= \Big|\int_{\xi_1}^{t}(s-t)[\lambda p(s)u(s)+\mu q(s)u''(s)]ds \\
&\quad + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b-a(\xi_1-t))(c(\xi_2-s)+d)[\lambda p(s)u(s)+\mu q(s)u''(s)]ds\Big| \\
&\le \Big[\int_{\xi_1}^{1}(1-s)(|\lambda||p(s)|+|\mu||q(s)|)ds \\
&\quad + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b+a(1-\xi_1))(c(\xi_2-s)+d)(|\lambda||p(s)|+|\mu||q(s)|)ds\Big]\|u\|_0 \\
&\le \max\{|\lambda|,|\mu|\}L_2\|u\|_0 < \frac{1}{L_2}L_2\|u\|_0 = \|u\|_0, \quad t\in[0,1].
\end{aligned}$$

Then $\|Au\|_0 < \|u\|_0$. This contradiction means that BVP (2.10) has no nontrivial solution. Hence, 1 is not an eigenvalue of A.

Finally, we prove that

$$\lim_{\|u\|_0\to\infty}\frac{\|Fu-Au\|_0}{\|u\|_0} = 0.$$

According to $\lim_{u\to\infty}\frac{g(u)}{u}=\lambda$ and $\lim_{v\to\infty}\frac{h(v)}{v}=\mu$, for any $\varepsilon>0$, there must be $R>0$ such that

$$|g(u)-\lambda u| < \varepsilon|u|, \quad |h(v)-\mu v| < \varepsilon|v|, \quad |u|,|v| > R.$$

Set $R^* = \max\{\max_{|u|\le R}|g(u)|, \max_{|v|\le R}|h(v)|\}$ and select $M>0$ such that $R^* + \max\{|\lambda|,|\mu|\} < \varepsilon M$. Denote

$$\begin{aligned}
E_1 &= \{t\in[0,1] : |u(t)|\le R,\ |v(t)|>R\},\\
E_2 &= \{t\in[0,1] : |u(t)|>R,\ |v(t)|\le R\},\\
E_3 &= \{t\in[0,1] : \max\{|u(t)|,|v(t)|\}\le R\},\\
E_4 &= \{t\in[0,1] : \min\{|u(t)|,|v(t)|\}>R\}.
\end{aligned}$$

Thus for any $u\in C^2[0,1]$ with $\|u\|_0>M$, when $t\in E_1$, we have

$$|g(u(t))-\lambda u(t)| \le |g((u(t))| + |\lambda||u(t)| \le R^* + |\lambda|R < \varepsilon M < \varepsilon\|u\|_0,$$

and

$$|h(v(t))-\mu v(t)| < \varepsilon|v(t)| \le \varepsilon\|v\|_0.$$

Similarly, we conclude that for any $u\in C^2[0,1]$ with $\|u\|_0>M$, when $t\in E_i$ $(i=2,3,4)$, we also have that

$$|g(u(t))-\lambda u(t)| < \varepsilon\|u\|_0, \quad |h(v(t))-\mu v(t)| < \varepsilon\|v\|_0.$$

Hence, we get

$$
\begin{aligned}
&|Fu(t) - Au(t)| \\
&= \Big| \int_0^1 G_1(t,s) \int_{\xi_1}^s (\tau - s)(p(\tau)[g(u(\tau)) - \lambda u(\tau)] + q(\tau)[h(u''(\tau)) - \mu u''(\tau)])d\tau ds \\
&\quad + \frac{1}{\delta} \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - s))(c(\xi_2 - \tau) + d) \\
&\quad \times (p(\tau)[g(u(\tau)) - \lambda u(\tau)] + q(\tau)[h(u''(\tau)) - \mu u''(\tau)])d\tau ds\Big| \\
&\le \Big[\int_0^1 G_1(s,s) \int_{\xi_1}^s (s - \tau)(|p(\tau)| + |q(\tau)|)d\tau ds \\
&\quad + \frac{1}{\delta} \int_0^1 G_1(s,s) \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - s))(c(\xi_2 - \tau) + d)(|p(\tau)| + |q(\tau)|)d\tau ds\Big] \varepsilon \|u\|_0 \\
&\le \frac{1}{12} \Big[\int_0^{\xi_1} \tau^3 (2 - \tau) w(\tau) d\tau + \int_{\xi_1}^1 (1 - \tau)^3 (1 + \tau) w(\tau) d\tau \\
&\quad + \frac{2(b - a\xi_1) + a}{\delta} \int_{\xi_1}^{\xi_2} (c(\xi_2 - \tau) + d) w(\tau) d\tau \Big] \varepsilon \|u\|_0. \\
&= \varepsilon L_1 \|u\|_0.
\end{aligned}
\tag{2.12}
$$

On the other hand, we have

$$
\begin{aligned}
&|(Fu - Au)''(t)| \\
&= \Big| \int_{\xi_1}^t (s - t)(p(s)[g(u(s)) - \lambda u(s)] + q(s)[h(u''(s)) - \mu u''(s)])ds \\
&\quad + \frac{1}{\delta} \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - t))(c(\xi_2 - s) + d) \\
&\quad \times (p(s)[g(u(s)) - \lambda u(s)] + q(s)[h(u''(s)) - \mu u''(s)])ds\Big| \\
&\le \Big[\int_{\xi_1}^1 (1 - s) w(s) ds + \frac{1}{\delta} \int_{\xi_1}^{\xi_2} (b + a(1 - \xi_1))(c(\xi_2 - s) + d) w(s) ds \Big] \varepsilon \|u\|_0 \\
&= \varepsilon L_2 \|u\|_0.
\end{aligned}
$$

Combining the above inequality with (2.12), we have

$$
\lim_{\|u\|_0 \to \infty} \frac{\|Fu - Au\|_0}{\|u\|_0} = 0.
$$

Lemma 2.5 now guarantees that BVP (1.1) and (1.2) has a solution $u^* \in C^2[0,1]$. Obviously, $u^* \neq 0$ when $p(t_0)g(0) + q(t_0)h(0) \neq 0$ for some $t_0 \in [0,1]$. In fact, if $u^* = 0$, then $(0)^{(4)} = p(t_0)g(0) + q(t_0)h(0) \neq 0$ will lead to a contradiction. This completes the proof. □

Example 2.7. Consider the fourth-order four-point boundary-value problem

$$
\begin{gathered}
u^{(4)}(t) = \frac{t \sin 2\pi t}{t^2 + 1} u(t) - \frac{1}{2} t e^{\cos t} \cos u''(t), \quad 0 < t < 1, \\
u(0) = u(1) = 0, \\
u''(1/3) - u'''(1/3) = 0, \quad u''(2/3) + u'''(2/3) = 0.
\end{gathered}
\tag{2.13}
$$

To show (2.13) has at least one nontrivial solution we apply Theorem 2.6 with $p(t) = \frac{t \sin 2\pi t}{t^2+1}$, $q(t) = \frac{1}{2} t e^{\cos t}$, $g(u) = u$, $h(u) = \cos u$, $a = b = c = d = 1$, $\xi_1 = 1/3$ and $\xi_2 = 2/3$. Clearly (H1) is satisfied. Obviously,

$$p(t_0)g(0) + q(t_0)h(0) = \frac{1}{2} t_0 e^{\cos t_0} \neq 0, \quad t_0 \in (0, 1].$$

Since $|p(s)| + |q(s)| \le (\frac{e}{2} + 1)s := w(s)$ for each $s \in [0, 1]$, we have

$$\begin{aligned} L_1 = \frac{\frac{e}{2}+1}{12} \Big[\int_0^{1/3} \tau^4 (2-\tau) d\tau + (e+1) \int_{1/3}^{1} (1-\tau)^3 (1+\tau) \tau d\tau \\ + \int_{1/3}^{2/3} (\frac{5}{3} - \tau) \tau d\tau \Big], \end{aligned}$$

$$L_2 = \big(\frac{e}{2} + 1\big) \Big[\int_{1/3}^{1} \tau (1-\tau) d\tau + \frac{5}{7} \int_{1/3}^{2/3} \big(\frac{5}{3} - \tau\big) \tau d\tau \Big].$$

By simple calculation we easily know that

$$L_1 < L_2 < \frac{1}{3} \big(\frac{e}{2} + 1\big) < 1.$$

Notice

$$\lambda = \lim_{u \to \infty} \frac{g(u)}{u} = 1, \quad \mu = \lim_{u \to \infty} \frac{h(u)}{u} = 0,$$

we have

$$\max\{\lambda, \mu\} < 1 < \min\{\frac{1}{L_1}, \frac{1}{L_2}\}.$$

So (H2) is satisfied. Thus, Theorem 2.6 now guarantees that BVP (2.13) has at least one nontrivial solution $u \in C^2[0, 1]$.

References

[1] R. P. Agarwal, *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer Academic, Dordrecht, 1998.

[2] R. P. Agarwal, D. O'Regan, P. J.Y . Wong; *Positive Solutions of Differential, Difference, and Integral Equations*, Kluwer Academic, Dordrecht, 1998.

[3] D. R. Anderson, J. M. Davis; *Multiple solutions and eigenvalues for third-order right focal boundary-value problem*, J. Math. Anal. Appl. 267 (2002), 135C157.

[4] Z. Bai, H. Wang; *On positive solutions of some nonlinear fourth-ordr beam equations*, J. Math. Anal. Appl. 270 (2002) 357-368.

[5] J. R. Graef, B. Yang; *On a nonlinear boundary-value problem for fourth order equations*, Appl. Anal. 72 (1999) 439-448.

[6] J. R. Graef, B. Yang; *Positive solutions to a multi-point higher order boundary-value problem*, J. Math. Anal. Appl. 316 (2006), 409-421.

[7] J. R. Graef, J. Henderson, B. Yang; *Positive solutions of a nonlinear higher order boundary-value problem*, Electron. J. Differential Equations, Vol. 2007(2007), No. 45, pp. 1-10.

[8] Z. Hao, L. Liu, L. Debnath; *A necessary and sufficiently condition for the existence of positive solution of fourth-order singular boundary-value problems*, Appl. Math. Lett. 16 (2003) 279-285.

[9] M. A. Krasnoselskii, P. P. Zabreiko; *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, New York, 1984.

[10] Q. Zhang, S. Chen, J. Lü; *Upper and lower solution method for fourth-order four-point boundary-value problems*, J. Comput. Appl. Math. 196 (2006) 387-393.

Jing Ge
Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsi 223300, China
E-mail address: gejing0512@163.com

Chuanzhi Bai
Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsi 223300, China
E-mail address: czbai8@sohu.com