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BERNSTEIN APPROXIMATIONS OF DIRICHLET PROBLEMS FOR ELLIPTIC OPERATORS ON THE PLANE

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ABSTRACT. We study the finitely dimensional approximations of the elliptic problem

$$
\begin{gathered}
(Lu)(x,y) + \varphi(\lambda,(x,y),u(x,y)) = 0 \quad \text{for } (x,y) \in \Omega \\
u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega,
\end{gathered}
$$

defined for a smooth bounded domain Ω on a plane. The approximations are derived from Bernstein polynomials on a triangle or on a rectangle containing Ω. We deal with approximations of global bifurcation branches of nontrivial solutions as well as certain existence facts.

1. Preliminaries

We consider the elliptic problem

$$
\begin{gathered}
(Lu)(x,y) + \varphi(\lambda,(x,y),u(x,y)) = 0 \quad \text{for } (x,y) \in \Omega \\
u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^2$ is a smooth, bounded domain and the uniformly elliptic operator L is

$$
Lu = a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy},
$$

where $a,b,c : \overline{\Omega} \to \mathbb{R}$ are continuous. We assume as well, that the map $\varphi : A \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, and $A \subset \mathbb{R}$ is an open interval.

We are going to define two finitely dimensional approximations of the problem (1.1). These approximations will be based on the Bernstein polynomials on the triangle (see [1], [5], [9], [15]) and on the rectangle (see [3], [9]).

So, let us start with some basic information about Bernstein polynomials on a triangle and on a rectangle. Let $\mathbb{T} \subset \mathbb{R}^2$ be a closed triangle with the vertices $P,Q,R \in \mathbb{R}^2$. Let $(p(x,y), q(x,y), r(x,y)) \in [0,1]^3$ denote the barycentric coordinates of the point $(x,y) \in \mathbb{T}$ with respect to the triangle $\mathbb{T}$ (when it is not confusing we will write p,q,r instead of $p(x,y), q(x,y), r(x,y)$). So each point $(x,y) \in \mathbb{T}$ is uniquely expressed by coordinates (p,q,r) such that $p,q,r \geq 0$ and $p+q+r = 1$. The relation between the coordinates is given by

$$
(x,y) = pP + qQ + rR.
$$

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Let now $n \in \mathbb{N}$ be fixed and i, j, k be nonnegative integers such that $i+j+k = n$. We call the set of functions $T^n_{i,j,k} : \mathbb{T} \to \mathbb{R}$, given by

$$T^n_{i,j,k}(x, y) = \frac{n!}{i!j!k!} p^i q^j r^k,$$

the Bernstein basis polynomials on the triangle.

For the continuous function $f : \mathbb{T} \to \mathbb{R}$ we call the function

$$(B^1_n f)(x, y) = \sum_{i,j,k\geq 0; i+j+k=n} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) T^n_{i,j,k}(x, y) \tag{1.2}$$

the Bernstein polynomial of degree n of the function f. In the above formula the triple $(\frac{i}{n}, \frac{j}{n}, \frac{k}{n})$ expresses the barycentric coordinates of the point in the triangle $\mathbb{T}$.

On the other hand, let $\mathbb{S} = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \mathbb{R}^2$ be the rectangle. Let now $n \in \mathbb{N}$ be fixed and let $i, j \in \{0, 1, \dots, n\}$. We call the set of functions $S^n_{i,j} : \mathbb{S} \to \mathbb{R}$ given by

$$S^n_{i,j}(x, y) = \binom{n}{i}\binom{n}{j}\Big(\frac{x-\alpha_1}{\alpha_2-\alpha_1}\Big)^i \Big(\frac{\alpha_2 - x}{\alpha_2-\alpha_1}\Big)^{n-i} \Big(\frac{y-\beta_1}{\beta_2-\beta_1}\Big)^j \Big(\frac{\beta_2 - y}{\beta_2-\beta_1}\Big)^{n-j}$$

the Bernstein basis polynomials on the rectangle. Similarly as above, for the continuous function $f : \mathbb{S} \to \mathbb{R}$ we call the function

$$(B^2_n f)(x, y) = \sum_{0\leq i,j\leq n} f(\alpha_1 + \frac{i}{n}(\alpha_2 - \alpha_1), \beta_1 + \frac{j}{n}(\beta_2 - \beta_1)) S^n_{i,j}(x, y) \tag{1.3}$$

the Bernstein polynomial of degree n of the function f.

The most important properties of Bernstein polynomials on a triangle and a rectangle will be given in the lemmas below. Here ω_u denotes the modulus of continuity of the function u, i.e.

$$\omega_u(\delta) = \max\{|u(x_1, y_1) - u(x_2, y_2)| : |x_1 - x_2| + |y_1 - y_2| \leq \delta\}.$$

The first lemma (see [15]) refers to Bernstein polynomials on a triangle.

Lemma 1.1. *If $u : \mathbb{T} \to \mathbb{R}$ is continuous, then $B^1_n u$ converges uniformly to u. Moreover, the estimation holds*

$$\|u - B^1_n u\|_0 \leq 2\omega_u(\frac{1}{\sqrt{n}}).$$

The second lemma refers to Bernstein polynomials on a rectangle and will be proved below.

Lemma 1.2. *If $u : \mathbb{S} \to \mathbb{R}$ is continuous, then $B^2_n u$ converges uniformly to u. Moreover, the estimation holds*

$$\|u - B^2_n u\|_0 \leq \frac{5}{2}\omega_u(\frac{\gamma}{\sqrt{n}}),$$

where $\gamma = \max\{\alpha_2 - \alpha_1, \beta_2 - \beta_1\}$.

Proof. As we can see in [9] the Bernstein approximation of the continuous function $v_0 : [0, 1] \to \mathbb{R}$ may be estimated by $|(B_n v_0)(t) - v_0(t)| \leq \frac{5}{4}\omega_{v_0}(\frac{1}{\sqrt{n}})$. The similar estimation may be obtained for a continuous function $v : [a, b] \to \mathbb{R}$, for any interval $[a, b] \subset \mathbb{R}$. Let us define $v_0(t) = v(tb + (1 - t)a) = v(x)$.

$$\begin{aligned}(B_n v)(x) &= \sum_{k=0}^{n} \binom{n}{k} v(a + \frac{k}{n}(b - a))\Big(\frac{x - a}{b - a}\Big)^k \Big(\frac{b - x}{b - a}\Big)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} v_0(\frac{k}{n}) t^k (1 - t)^{n-k} = (B_n v_0)(t).\end{aligned}$$

So, we have

$$|(B_n v)(x) - v(x)| = |(B_n v_0)(t) - v_0(t)| \le \omega_{v_0}(\frac{1}{\sqrt{n}}) = \omega_v(\frac{b - a}{\sqrt{n}}). \tag{1.4}$$

To prove the estimation for the interval in $\mathbb{R}^2$ we will repeat the reasoning given in [10]. Let $v_y(x) = u(x, y)$ for the fixed $y \in [\beta_1, \beta_2]$, and $w_x(y) = u(x, y)$ for fixed $x \in [\alpha_1, \alpha_2]$. Let ω_v and ω_w denote the moduli of continuity of v and w respectively. Then $\omega_v(\delta) \le \omega_u(\delta)$ and $\omega_w(\delta) \le \omega_u(\delta)$.

The functions

$$\begin{aligned}B_n^v(x, y) &= \sum_{i=0}^{n} \binom{n}{i} u(\alpha_1 + \frac{i}{n}(\alpha_2 - \alpha_1), y)\Big(\frac{x - \alpha_1}{\alpha_2 - \alpha_1}\Big)^i \Big(\frac{\alpha_2 - x}{\alpha_2 - \alpha_1}\Big)^{n-i}, \\ B_n^w(x, y) &= \sum_{j=0}^{n} \binom{n}{j} u(x, \beta_1 + \frac{j}{n}(\beta_2 - \beta_1))\Big(\frac{y - \beta_1}{\beta_2 - \beta_1}\Big)^j \Big(\frac{\beta_2 - y}{\beta_2 - \beta_1}\Big)^{n-j}\end{aligned}$$

are Bernstein polynomials of v_y and w_x respectively.

So from estimation (1.4) we have

$$\begin{aligned}|B_n^v(x, y) - u(x, y)| &\le \frac{5}{4}\omega_v(\frac{\alpha_2 - \alpha_1}{\sqrt{n}}) \le \frac{5}{4}\omega_u(\frac{\gamma}{\sqrt{n}}), \\ |B_n^w(x, y) - u(x, y)| &\le \frac{5}{4}\omega_w(\frac{\beta_2 - \beta_1}{\sqrt{n}}) \le \frac{5}{4}\omega_u(\frac{\gamma}{\sqrt{n}}).\end{aligned}$$

We can see that

$$(B_n u)(x, y) = \sum_{i=0} n B_n^w(\frac{i}{n}, y)\Big(\frac{x - \alpha_1}{\alpha_2 - \alpha_1}\Big)^i \Big(\frac{\alpha_2 - x}{\alpha_2 - \alpha_1}\Big)^{n-i}$$

So,

$$\begin{aligned}&|(B_n u)(x, y) - u(x, y)| \le |(B_n u)(x, y) - B_n^v(x, y)| + |B_n^v(x, y) - u(x, y)| \\ &\le \sum_{i=0} n|B_n^w(\frac{i}{n}, y) - u(\frac{i}{n}, y)|\Big(\frac{x - \alpha_1}{\alpha_2 - \alpha_1}\Big)^i \Big(\frac{\alpha_2 - x}{\alpha_2 - \alpha_1}\Big)^{n-i} + \frac{5}{4}\omega_u(\frac{\gamma}{\sqrt{n}}) \\ &\le \frac{5}{4}\omega_u(\frac{\gamma}{\sqrt{n}}) + \frac{5}{4}\omega_u(\frac{\gamma}{\sqrt{n}}).\end{aligned}$$

This completes the proof. □

Now we return to problem (1.1). First let us consider the linear spectral problem

$$\begin{gathered}(Lu)(x, y) + \lambda u(x, y) = 0 \quad \text{for } (x, y) \in \Omega \\ u(x, y) = 0 \quad \text{for } (x, y) \in \partial\Omega\end{gathered} \tag{1.5}$$

It is well known (see [6]) that there exists the minimal eigenvalue μ_0 of the problem (1.5). This eigenvalue is positive, simple and the associated eigenvector has constant sign. Moreover, μ_0 is the only eigenvalue with the corresponding eigenvector having constant sign.

Let $m > 0$ be fixed and $A \subset (0, +\infty)$ be an open interval such that $\frac{\mu_0}{m} \in A$. Let $\varphi : A \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{(x,y)\in\overline{\Omega}, \lambda\in B, s\in\mathbb{R}} 0 \le s \le \delta \Rightarrow |\varphi(\lambda,(x,y),s) - \lambda m s| \le \varepsilon |s|, \tag{1.6}$$

for any bounded $B \subset A$;

$$\forall_{(x,y)\in\overline{\Omega}, \lambda\in A, s<0} \varphi(\lambda,(x,y),s) > 0. \tag{1.7}$$

Note that from (1.6) the following conclusion may be drawn

$$\varphi(\lambda,(x,y),0) = 0, \quad \text{for } (x,y) \in \overline{\Omega}, \lambda \in A. \tag{1.8}$$

Let us now fix the closed triangle $\mathbb{T} \subset \mathbb{R}^2$, such that $\Omega \subset \mathbb{T}$. Assume, that continuous $u : \overline{\Omega} \to \mathbb{R}$ satisfies boundary conditions, i.e. $u(x,y) = 0$ for $(x,y) \in \partial\Omega$. Because of (1.8) we may continuously extend the superposition $\varphi(\lambda,\cdot,u(\cdot))$ to the triangle $\mathbb{T}$, in such a way that this extension achieves 0 on $\mathbb{T} \setminus \Omega$. Let $\tilde{\varphi}(\lambda,\cdot,u(\cdot))$ denote this extension. So we may consider the boundary-value problem

$$\begin{gathered} (Lu)(x,y) + (B_n^1 \tilde{\varphi}(\lambda,\cdot,u(\cdot)))(x,y) = 0 \quad \text{for } (x,y) \in \Omega \\ u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega. \end{gathered} \tag{1.9}$$

Let us now fix the closed rectangle $\mathbb{S} \subset \mathbb{R}^2$, such that $\Omega \subset \mathbb{S}$. As above, we may continuously extend the function $\varphi(\lambda,\cdot,u(\cdot))$ to the rectangle $\mathbb{S}$, in such a way that this extension achieves 0 on $\mathbb{S} \setminus \Omega$. Let $\hat{\varphi}(\lambda,\cdot,u(\cdot))$ denote such extension on the rectangle $\mathbb{S}$. So let us consider the problem

$$\begin{gathered} (Lu)(x,y) + (B_n^2 \hat{\varphi}(\lambda,\cdot,u(\cdot)))(x,y) = 0 \quad \text{for } (x,y) \in \Omega \\ u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega. \end{gathered} \tag{1.10}$$

For the above boundary-value problems (1.1), (1.9), (1.10), we are looking for the weak solutions $(\lambda, u) \in A \times W^{2,2}(\Omega)$. From Sobolev embedding theorem we have $W^{2,2}(\Omega) \subset C(\overline{\Omega})$ (see [6]), so $B_n^1 \tilde{\varphi}(\lambda,\cdot,u(\cdot))$ and $B_n^2 \hat{\varphi}(\lambda,\cdot,u(\cdot))$ are well defined.

Because of (1.8) each pair $(\lambda, 0) \in A \times W^{2,2}(\Omega)$ is the solution of problems (1.1), (1.9) and (1.10). We call such pairs *trivial solutions*.

Let $\mathcal{R}$ denote the closure (in $A \times C(\overline{\Omega})$) of the set of nontrivial solutions of the problem (1.1).

Let $\mathcal{R}_n^i$ $(i = 1, 2)$ denote the closure (in $A \times C(\overline{\Omega})$) of the set of nontrivial solutions of the problems (1.9) and (1.10) respectively.

The classical global bifurcation theorem given by Rabinowitz (see [12] and also [2, 11]) may be applied to elliptic boundary-value problems. Such applications were considered by many authors (see e.g. [12, 13, 14, 7]). Below we prove a similar result

Theorem 1.3. *There exists the noncompact component C of $\mathcal{R}$ such that $(\frac{\mu_0}{m}, 0) \in C$.*

We are also going to show that the similar thesis may be proved for approximating problems (1.9) and (1.10). We are going to prove that the connected component C of $\mathcal{R}$ is, in a sense explained below, approximated by the branches of sets $\mathcal{R}_n^i$ $(i = 1, 2)$. Let us further assume that $i \in \{1, 2\}$ is fixed.

Theorem 1.4. *Let $\varepsilon > 0$, $\frac{\mu_0}{m} \in (a, b) \subset [a, b] \subset A$ and $R > 0$. Then, for almost all $n \in \mathbb{N}$, there exists a component C_n^i of the set $\mathcal{R}_n^i$, such that*

(i) $C_n^i \cap \Big((\frac{\mu_0}{m} - \varepsilon, \frac{\mu_0}{m} + \varepsilon) \times \{0\} \Big) \neq \emptyset$;
(ii) $C_n^i \cap \partial([a,b] \times \overline{B(0,R)}) \neq \emptyset$.

The relation between components C_n^i and C will be established in Theorem 1.5 given below. But first we need to define some notation. For a set $U \subset A \times C(\overline{\Omega})$ let us denote

$$O_\varepsilon(U) = \{(\lambda, u) \in A \times C(\overline{\Omega}) : \exists_{(\mu,v)\in U} |\lambda - \mu| + \|u - v\|_0 < \varepsilon\}.$$

where $\|\cdot\|_0$ denotes the norm in $C(\overline{\Omega})$.

Theorem 1.5. *Let $\varepsilon > 0$, $\frac{\mu_0}{m} \in (a,b) \subset [a,b] \subset A$ and $R > 0$ be fixed. Assume that*

(i) *$C \subset \mathcal{R}$ is a noncompact component, such that $(\frac{\mu_0}{m}, 0) \in C$;*
(ii) *$C_n^i \subset \mathcal{R}_n^i$ is a component, such that $C_i^n \cap \Big((\frac{\mu_0}{m} - \varepsilon, \frac{\mu_0}{m} + \varepsilon) \times \{0\} \Big) \neq \emptyset$ and $C_n^i \cap \partial([a,b] \times \overline{B(0,R)}) \neq \emptyset$;*
(iii) *S is a component of $C \cap \Big([a,b] \times \overline{B(0,R)} \Big)$, such that $(\frac{\mu_0}{m}, 0) \in S$;*
(iv) *S_n^i is a component of $C_n^i \cap \Big([a,b] \times \overline{B(0,R)} \Big)$, such that $S_n^i \cap \Big((\frac{\mu_0}{m} - \varepsilon, \frac{\mu_0}{m} + \varepsilon) \times \{0\} \Big) \neq \emptyset$.*

Then, for almost all $n \in \mathbb{N}$, the relation $S_n^i \subset O_\varepsilon(S)$ holds.

Remark 1.6. From Theorem 1.5 we may conclude that

$$\operatorname*{Li}_{n\to+\infty} S_n^i \subset \operatorname*{Ls}_{n\to+\infty} S_n^i \subset S,$$

where Li and Ls denote Kuratowski lower and upper limit respectively (see [8]).

2. Approximation of global bifurcation branches

We are now going to introduce the necessary notation. First of all, let $C(\overline{\Omega})$ denote the space of all continuous functions $u : \overline{\Omega} \to \mathbb{R}$, with the norm $\|u\|_0 = \sup_{(x,y)\in\overline{\Omega}} |u(x,y)|$. Let $C_0(\overline{\Omega})$ denote the subspace of $C(\overline{\Omega})$ consisting of all functions $u : \overline{\Omega} \to \mathbb{R}$ satisfying boundary conditions $u(x,y) = 0$ for $(x,y) \in \partial\Omega$.

Let the maps $B_n^i : C_0(\overline{\Omega}) \to C(\overline{\Omega})$ $(i = 1,2)$ be given by (1.2) and (1.3) respectively (here we assume that the appropriate formulas are applied to extensions $\tilde{f} : \mathbb{T} \to \mathbb{R}$ and $\hat{f} : \mathbb{S} \to \mathbb{R}$, of the function $f \in C_0(\overline{\Omega})$). It is easy to observe that both maps are bounded linear maps and $\|B_n^i\| \leq 1$, for $n \in \mathbb{N}$ and $i = 1,2$.

It is also well known (see [6]), that there exists a continuous map $\hat{T} : W^{0,2}(\Omega) \to W_0^{2,2}(\Omega)$, such that

$$\hat{T}h = u \Leftrightarrow \begin{cases} (Lu)(x,y) + h(x,y) = 0 & \text{for } (x,y) \in \Omega \\ u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega. \end{cases}$$

Let $\Phi : A \times C(\overline{\Omega}) \to C(\overline{\Omega})$ be given by $\Phi(\lambda,u)(x,y) = \varphi(\lambda,(x,y),u(x,y))$. From (1.8) we may conclude that $\Phi(A \times C_0(\overline{\Omega})) \subset C_0(\overline{\Omega})$. Additionally, let $j : W_0^{2,2}(\Omega) \to C_0(\overline{\Omega})$ be the inclusion. From the Sobolev embedding theorem (see [6]) we have the compactness of j. Let us denote $T = j \circ \hat{T}$. Hence the superposition $T \circ \Phi : A \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ is completely continuous. The same conclusion may be drawn for maps $T \circ B_n^i \circ \Phi : A \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ where $n \in \mathbb{N}$ and $i = 1, 2$. The situation is described by the following diagrams

$$\begin{array}{ccccccc} A \times C_0(\overline{\Omega}) & & & & & & \\ \downarrow \Phi & & & & & & \\ C(\overline{\Omega}) & \xrightarrow{i_0} & W^{0,2}(\Omega) & \xrightarrow{\hat{T}} & W_0^{2,2}(\Omega) & \xrightarrow{j} & C_0(\overline{\Omega}) \end{array}$$

$$\begin{array}{ccccccccc} A \times C_0(\overline{\Omega}) & & & & & & & & \\ \downarrow \Phi & & & & & & & & \\ C_0(\overline{\Omega}) & \xrightarrow{B_n^i} & C(\overline{\Omega}) & \xrightarrow{i_0} & W^{0,2}(\Omega) & \xrightarrow{\hat{T}} & W_0^{2,2}(\Omega) & \xrightarrow{j} & C_0(\overline{\Omega}) \end{array}$$

Here i_0 denotes the natural inclusion $i_0 : C(\overline{\Omega}) \to W^{0,2}(\Omega)$.

Let $f_n^1 : A \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ be given by

$$f_n^1(\lambda, u) = u - TB_n^1\Phi(\lambda, u).$$

The zeros of the map f_n^1 correspond to the solutions of the problem (1.9). Similarly let $f_n^2 : A \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ be given by

$$f_n^2(\lambda, u) = u - TB_n^2\Phi(\lambda, u).$$

The zeros of the map f_n^2 correspond to the solutions of the problem (1.10).

We are going to prove Theorems 1.3, 1.4 and 1.5 in the sequence of lemmas.

Lemma 2.1. *If $(\lambda, u) \in \mathcal{R}$, then $u \geq 0$.*

Proof. Assume that $U = \{(x, y) \in \Omega | u(x, y) < 0\} \neq \emptyset$. Because of (1.7) the relation $(Lu)(x, y) \leq 0$ holds for all $(x, y) \in U$. That is why, by the maximum principle (see [6]), there exists $(x, y) \in \partial U$ such that $u(x, y) < 0$. On the other hand the definition of U implies that if $(x, y) \in \partial U$ then either $u(x, y) = 0$ or $(x, y) \in \partial\Omega$. The latter and boundary conditions imply that $u(x, y) = 0$ as well. So we have the contradiction with the maximum principle. $\square$

Corollary 2.2. *If $u = \lambda mT|u|$ and $u \neq 0$, then $\lambda = \frac{\mu_0}{m}$.*

Proof. Because of Lemma 2.1, we can see that $|u| = u$. Hence λ is the eigenvalue of the problem (1.5) with the corresponding nonnegative eigenvector. So $\lambda m = \mu_0$. $\square$

Remark 2.3. Without loss of generality we may assume that $\varphi(\lambda, (x, y), s) = \lambda m|s|$ for $s < 0$, $\lambda \in A$ and $(x, y) \in \overline{\Omega}$. Because of Lemma 2.1, both the original and modified problem have the same set of solutions.

Hence, we may assume that the strenghtened version of (1.6) holds

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{(x,y)\in\overline{\Omega},\lambda\in B,s\in\mathbb{R}}|s| \leq \delta \Rightarrow \Big|\varphi(\lambda, (x, y), s) - \lambda m|s|\Big| \leq \varepsilon|s|, \tag{2.1}$$

for any bounded $B \subset A$.

Lemma 2.4. *For any compact $B \subset A \setminus \{\frac{\mu_0}{m}\}$, there exist $\gamma > 0$ and $\delta > 0$, such that*

$$\|f(\lambda, u)\|_0 \geq \gamma\|u\|_0 \quad \textit{for } \lambda \in B, \|u\|_0 \leq \delta. \tag{2.2}$$

Proof. Let us first observe, that the inequality holds

$$\gamma_0 = \inf_{\lambda \in B, \|u\|_0 = 1} \Big\| u - \lambda m T|u| \Big\|_0 > 0.$$

Assume, contrary to our claim, that there exists the sequence $\{(\lambda_n, u_n)\} \subset B \times C_0(\overline{\Omega})$, such that $\|u_n\|_0 = 1$, $(\lambda_n, u_n) \in B \times C_0(\overline{\Omega})$ and $u_n - \lambda_n m T|u_n| \to 0$. Because T is completely continuous, we can see that $\{T|u_n|\}$ contains convergent subsequence, so we may assume that $u_n \to u_0 \in C_0(\overline{\Omega})$. Of course, we may also assume, that $\lambda_n \to \lambda_0 \in B$. Hence, without the loss of generality, we have $u_0 = \lambda_0 m T|u_0|$, so by corollary 2.2 $\lambda_0 m = \mu_0$, a contradiction.

Because of (2.1) there exists $\delta_0 > 0$, such that for $\lambda \in B$,

$$\|u\|_0 \le \delta_0 \Rightarrow \Big\| T\Phi(\lambda, u) - m\lambda T|u| \Big\|_0 \le \frac{\gamma_0}{2} \|u\|_0.$$

So, for $\|u\|_0 \le \delta_0$ and $\lambda \in B$, the relation holds

$$\begin{aligned} \|f(\lambda, u)\|_0 &\ge \Big\| u - \lambda m T|u| \Big\|_0 - \Big\| T\Phi(\lambda, u) - m\lambda T|u| \Big\|_0 \\ &\ge \gamma_0 \|u\|_0 - \frac{\gamma_0}{2}\|u\|_0 \\ &= \frac{\gamma_0}{2}\|u\|_0 > 0. \end{aligned}$$

This completes the proof. $\square$

Lemma 2.5. (i) *If $\lambda < \frac{\mu_0}{m}$, then* $\deg(f(\lambda, \cdot), B(0, r), 0) = 1$ *for $r > 0$ small enough.*

(ii) *If $\lambda > \frac{\mu_0}{m}$, then* $\deg(f(\lambda, \cdot), B(0, r), 0) = 0$ *for $r > 0$ small enough.*

Proof. First, let us observe that for any $[a, b] \subset A \setminus \{\frac{\mu_0}{m}\}$ there exists $r > 0$, such that for $\lambda \in [a, b]$ the map $f(\lambda, \cdot) : \overline{B(0, r)} \to C_0(\overline{\Omega})$ may be joined by homotopy with $f_0(\lambda, \cdot) : \overline{B(0, r)} \to C_0(\overline{\Omega})$ given by $f_0(\lambda, u) = u - \lambda m T|u|$. By Lemma 2.4 there exist $r_1 > 0$ and $\gamma > 0$ such that

$$\|f(\lambda, u)\|_0 \ge \gamma \|u\|_0$$

for $\lambda \in [a, b]$ and $\|u\|_0 \le r_1$.

From (2.1) we may conclude that for $\|u\|_0 \le r_2$ the inequality holds

$$\|\Phi(\lambda, u) - m\lambda|u|\|_0 \le \frac{\gamma}{2\|T\|}\|u\|_0.$$

Let us take $r = \min\{r_1, r_2\}$ and define the homotopy $h : [0, 1] \times \overline{B(0, r)} \to C_0(\overline{\Omega})$, by

$$h(\tau, u) = f_0(\lambda, u) - \tau[f(\lambda, u) - f_0(\lambda, u)].$$

Then for $\|u\|_0 = r$ we have

$$\|h(\tau, u)\|_0 \ge \gamma\|u\|_0 - \frac{\gamma}{2}\|u\|_0 > 0,$$

so the homotopy is well defined.

Moreover, if $\lambda < \frac{\mu_0}{m}$ the homotopy $h_1 : [0, 1] \times \overline{B(0, r)} \to C_0(\overline{\Omega})$, given by $h_1(\tau, u) = u - \tau\lambda m T|u|$, joins $f_0(\lambda, \cdot)$ with identity map. As we can see from corollary 2.2 the homotopy may have nontrivial zero only when $\tau\lambda = \frac{\mu_0}{m}$, which is not possible. So (i) is proved.

On the other hand if $\lambda > \frac{\mu_0}{m}$, then the map $f_0(\lambda, \cdot)$ may be joined by homotopy with the map $f_1 : \overline{B(0, r)} \to C_0(\overline{\Omega})$ given by $f_1(u) = f_0(\lambda, u) - u_0$, where u_0 is positive eigenvector of (1.5), associated with the eigenvalue μ_0. The homotopy may be given by

$$h_1(\tau,u) = u - \lambda mT|u| - \tau u_0.$$

So, let us now assume that $h_1(\tau,u)=0$ for $\|u\|_0 \le r$ and $\tau\in[0,1]$. Then we have $u=\lambda mT|u|+\tau u_0 \ge 0$ and

$$\int_\Omega uu_0 = \lambda m\int_\Omega (Tu)u_0 + \tau\int_\Omega u_0^2.$$

As we can see from the definition of a weak solution (see [6]) of the Dirichlet problem, for each $u,v\in W_0^{2,2}(\Omega)$ the relation $\int_\Omega (Tu)v = \int_\Omega (Tv)u$ holds, so

$$\int_\Omega (Tu)u_0 = \int_\Omega u(Tu_0) = \frac{1}{\mu_0}\int_\Omega uu_0.$$

Hence

$$(1-\frac{m\lambda}{\mu_0})\int_\Omega uu_0 = \tau\int_\Omega u_0^2 > 0,$$

what implies $\lambda < \frac{\mu_0}{m}$, and contradicts our assumption. That is why (ii) holds true. $\square$

Proof of Theorem 1.3. We are going to refer to the generalization of Rabinowitz global bifurcation theorem given in [4]. This theorem refers to the more general case of convex-valued maps, but may be applied to the single valued case, as in Theorem 1.3. What we need is the interval $[a,b]$, such that the set of all bifurcation points of f is contained in that interval, and the change of local topological degree of the maps $f(\lambda,\cdot):\overline{B(0,r)}\to C_0(\overline{\Omega})$ on the small balls around zero. As we can see, because of Lemma 2.1, the only bifurcation point is $(\frac{\mu_0}{m},0)$, so we may take $[a,b]=[\frac{\mu_0}{m},\frac{\mu_0}{m}]$. Additionally, by Lemma 2.5, there is the degree change in the neighborhood of $\frac{\mu_0}{m}$. $\square$

Lemma 2.6. *Let $\delta>0$ and $\mathcal{K}=(a,\frac{\mu_0}{m}-\delta)\cup(\frac{\mu_0}{m}+\delta,b)\subset[a,b]\subset A$. Then there exists $r>0$, such that $(\mathcal{K}\times\overline{B(0,r)})\cap\mathcal{R}_n^i=\emptyset$ for almost all $n\in\mathbb{N}$.*

Proof. Let us have $i\in\{1,2\}$ fixed. Assume, contrary to our claim, that there exists the increasing sequence $\{\gamma(n)\}\subset\mathbb{N}$ and points $(\lambda_n,u_n)\in\mathcal{K}\times C_0(\overline{\Omega})$, such that $f^i_{\gamma(n)}(\lambda_n,u_n)=0$, $u_n\ne0$, $u_n\to0$ and $\lambda_n\to\lambda_0\in\overline{\mathcal{K}}$. Then

$$u_n = TB^i_{\gamma(n)}\Phi(\lambda_n,u_n),$$
$$u_n = \lambda_n mTB^i_{\gamma(n)}|u_n| + TB^i_{\gamma(n)}[\Phi(\lambda_n,u_n)-m\lambda_n|u_n|].$$

Let us now denote $v_n=u_n/\|u_n\|_0$. Then

$$v_n = \lambda_n mTB^i_{\gamma(n)}|v_n| + TB^i_{\gamma(n)}\frac{\Phi(\lambda_n,u_n)-m\lambda_n|u_n|}{\|u_n\|_0}.$$

Because of (2.1) there is $\Big\|\frac{\Phi(\lambda_n,u_n)-m\lambda_n|u_n|}{\|u_n\|_0}\Big\|_0\to0$. Then, because of $\|T\circ B^i_{\gamma(n)}\|\le\|T\|$, letting $n\to+\infty$ gives

$$TB^i_{\gamma(n)}\frac{\Phi(\lambda_n,u_n)-m\lambda_n|u_n|}{\|u_n\|_0}\to0.$$

Additionally, the sequence $\{B^i_{\gamma(n)}v_n\}$ is bounded, so taking appropriate subsequence of $\{v_n\}$ we may assume that $v_n \to v_0 \in C_0(\overline{\Omega})$. Moreover, we can see that

$$\|B^i_{\gamma(n)}v_n - v_0\|_0 \leq \|B^i_{\gamma(n)}(v_n - v_0)\|_0 + \|B^i_{\gamma(n)}v_0 - v_0\|_0.$$

Because $\|B^i_{\gamma(n)}\| \leq 1$ and $B^i_{\gamma(n)}v_0 \to v_0$, we can see that $B^i_{\gamma(n)}v_n \to v_0$. So, letting $n \to +\infty$ we have $v_0 = \lambda_0 mT|v_0|$. This, for $\|v_0\|_0 = 1$, implies $\lambda_0 = \frac{\mu_0}{m} \notin \overline{\mathcal{K}}$, a contradiction. $\square$

Lemma 2.7. *Let $\delta > 0$ and $\mathcal{K} = (a, \frac{\mu_0}{m} - \delta) \cup (\frac{\mu_0}{m} + \delta, b) \subset [a,b] \subset A$. Then there exists $r_0 > 0$, such that*

$$\deg(f(\lambda,\cdot), B(0,r), 0) = \deg(f_n(\lambda,\cdot), B(0,r), 0),$$

for all $r \in (0, r_0)$, almost all $n \in \mathbb{N}$ and for all $\lambda \in \mathcal{K}$.

Proof. Let us now take $\gamma > 0$ and $\delta > 0$ as in Lemma 2.4, for $B = \overline{\mathcal{K}}$. Let us also take $r \in (0,\delta)$, such that for all $\lambda \in \mathcal{K}$, the implication holds

$$\|u\|_0 \leq r \Rightarrow \|\Phi(\lambda,u) - m\lambda|u|\|_0 \leq \frac{\gamma}{6\|T\|}\|u\|_0.$$

Let us, for the fixed $n \in \mathbb{N}$, $\lambda \in \mathcal{K}$ and $i \in \{1,2\}$ take, the homotopy $h^i_{n,\lambda} : [0,1] \times \overline{B(0,r)} \to C_0(\overline{\Omega})$, given by

$$h^i_{n,\lambda}(\tau,u) = f(\lambda,u) - \tau[f^i_n(\lambda,u) - f(\lambda,u)].$$

Assume now that $h^i_{n,\lambda}(\tau,u) = 0$ for $\|u\|_0 = r$ and $\tau \in [0,1]$. Then

$$u = T[\Phi(\lambda,u) + \tau(B^i_n\Phi(\lambda,u) - \Phi(\lambda,u))].$$

Because the set

$$\mathcal{A} = \{\Phi(\lambda,u) + \tau(B^i_n\Phi(\lambda,u) - \Phi(\lambda,u)) : u \in C_0(\overline{\Omega}), \|u\|_0 = r, \tau \in [0,1]\}$$

is bounded, the set $T(\mathcal{A})$ is relatively compact, so all functions $u \in C_0(\overline{\Omega})$, such that $h^i_{n,\lambda}(\tau,u) = 0$ and $\|u\|_0 = r$, are uniformly continuous. Hence, by Lemma 1.1 and 1.2, we may assume, that for $n \in \mathbb{N}$ large enough, $\lambda \in \mathcal{K}$ and $u \in \mathcal{A}$ the inequality holds $\|m\lambda T(|u| - B^i_n|u|)\|_0 \leq \frac{\gamma}{6}r = \frac{\gamma}{6}\|u\|_0$. That is why, for all functions $u \in C_0(\overline{\Omega})$, such that $h^i_{n,\lambda}(\tau,u) = 0$ and $\|u\|_0 = r$, and $n \in \mathbb{N}$ large enough

$$\begin{aligned}
&\|f(\lambda,u) - f^i_n(\lambda,u)\|_0 \\
&\leq \|T(\Phi(\lambda,u) - m\lambda|u|)\|_0 + \|TB^i_n(m\lambda|u| - \Phi(\lambda,u))\|_0 + \|m\lambda T(|u| - B^i_n|u|)\|_0 \\
&\leq \frac{\gamma}{6\|T\|}\|T\|\|u\|_0 + \frac{\gamma}{6\|T\|}\|T\|\|u\|_0 + \frac{\gamma}{6}\|u\|_0 = \frac{\gamma}{2}\|u\|_0.
\end{aligned}$$

Consequently

$$\|h^i_{n,\lambda}(\tau,u)\|_0 \geq \gamma\|u\|_0 - \frac{\gamma}{2}\|u\|_0 > 0,$$

which is a contradiction. $\square$

Proof of Theorem 1.4. Let us take any $\varepsilon > 0$ and the interval $[a,b] \subset A$, such that $\frac{\mu_0}{m} \in (a,b)$. From Lemma 2.6 we can see that for almost all $n \in \mathbb{N}$ the relation holds $\mathcal{R}^i_n \cap (([a, \frac{\mu_0}{m} - \varepsilon] \cup [\frac{\mu_0}{m} + \varepsilon, b]) \times \{0\}) = \emptyset$. So the set of bifurcation points of $f^i_n|_{(a,b)\times C_0(\overline{\Omega})}$ is contained in the interval $[\frac{\mu_0}{m} - \varepsilon, \frac{\mu_0}{m} + \varepsilon]$. Moreover, by Lemmas 2.5 and 2.7, there is the change of topological degree for $\lambda < \frac{\mu_0}{m} - \varepsilon$ and $\lambda > \frac{\mu_0}{m} + \varepsilon$.

So, as in the proof of Theorem 1.3, we may apply the global bifurcation theorem given in [4]. □

For the rest of this article, let us have an interval $[a,b] \subset A$, such that $\frac{\mu_0}{m} \in (a,b)$, and a constant $R > 0$ fixed. Moreover, let us assume, according to Theorem 1.5, that

(i) $C \subset \mathcal{R}$ is noncompact component, such that $(\frac{\mu_0}{m}, 0) \in C$;
(ii) $C_n^i \subset \mathcal{R}_n^i$ is a component, such that $C_i^n \cap \Big((\frac{\mu_0}{m} - \varepsilon, \frac{\mu_0}{m} + \varepsilon) \times \{0\}\Big) \neq \emptyset$ and $C_n^i \cap \partial([a,b] \times \overline{B(0,R)}) \neq \emptyset$;
(iii) S is a component of $C \cap \Big([a,b] \times \overline{B(0,R)}\Big)$, such that $(\frac{\mu_0}{m}, 0) \in S$;
(iv) S_n^i is a component of $C_n^i \cap \Big([a,b] \times \overline{B(0,R)}\Big)$, such that $S_n^i \cap \Big((\frac{\mu_0}{m} - \varepsilon, \frac{\mu_0}{m} + \varepsilon) \times \{0\}\Big) \neq \emptyset$.

Lemma 2.8. *For almost all $n \in \mathbb{N}$ the inclusion $S_n^i \subset O_\varepsilon(S)$ holds.*

Proof. Let us fix $i \in \{1,2\}$. Let us observe that

$$\forall_{\eta>0} \exists_{n_0 \in \mathbb{N}} \forall_{n>n_0} (f_n^i)^{-1}(0) \cap ([a,b] \times \overline{B(0,R)}) \subset O_\eta\Big(f^{-1}(0) \cap ([a,b] \times \overline{B(0,R)})\Big). \tag{2.3}$$

Assume, contrary to our claim, that there exists $\eta_0 > 0$ and the sequence $(\lambda_n, u_n) \in (f_{\gamma(n)}^i)^{-1}(0) \cap ([a,b] \times \overline{B(0,R)})$, where $\{\gamma(n)\} \subset \mathbb{N}$, such that $(\lambda_n, u_n) \notin O_{\eta_0}(f^{-1}(0) \cap ([a,b] \times \overline{B(0,R)}))$.

We may assume that $\lambda_n \to \lambda_0 \in [a,b]$. Moreover, the sequence $\{B_{\gamma(n)}^i \Phi(\lambda_n, u_n)\}$ is bounded, so $u_n = TB_{\gamma(n)}^i \Phi(\lambda_n, u_n)$, contains convergent subsequence. So we may also assume that $u_n \to u_0 \in \overline{B(0,R)}$. We can see that

$$\begin{aligned}
&\|TB_{\gamma(n)}^i \Phi(\lambda_n, u_n) - T\Phi(\lambda_n, u_n)\|_0 \\
&\leq \|TB_{\gamma(n)}^i \Phi(\lambda_n, u_n) - TB_{\gamma(n)}^i \Phi(\lambda_0, u_0)\|_0 + \|TB_{\gamma(n)}^i \Phi(\lambda_0, u_0) - T\Phi(\lambda_0, u_0)\|_0.
\end{aligned}$$

As we can see the above sum converges to zero, what gives $u_0 = T\Phi(\lambda_0, u_0)$, a contradiction.

Let us now show that

$$\forall_{\eta>0} \exists_{n_0 \in \mathbb{N}} \forall_{n>n_0}\ S_n^i \subset O_\eta(\mathcal{R}_0), \tag{2.4}$$

where $\mathcal{R}_0 = \mathcal{R} \cap ([a,b] \times \overline{B(0,R)})$.

Assume, contrary to our claim, that there exists the sequence $S_{\gamma(n)}^i$ and positive number $\eta_0 > 0$, satisfying $S_{\gamma(n)}^i \not\subset O_{\eta_0}(\mathcal{R}_0)$. Let $(\lambda_n, u_n) \in S_{\gamma(n)}$ satisfy $(\lambda_n, u_n) \notin O_{\eta_0}(\mathcal{R}_0)$. By Lemma 2.6 (applied for an open open neighbourhood of $\mathcal{K}$) there exists $r_0 > 0$, such that for almost all $n \in \mathbb{N}$,

$$S_{\gamma(n)} \cap \big(\mathcal{K} \times \overline{B(0, r_0)}\big) = \emptyset,$$

where $\mathcal{K} = [a, \frac{\mu_0}{m} - \frac{\eta_0}{2}] \cup [\frac{\mu_0}{m} + \frac{\eta_0}{2}, b]$. We may assume that $r_0 < \frac{\eta_0}{2}$.

If $\|u_n\|_0 < r_0$, then $\lambda_n \in (\frac{\mu_0}{m} - \frac{\eta_0}{2}, \frac{\mu_0}{m} + \frac{\eta_0}{2})$ and consequently $(\lambda_n, u_n) \in B((\frac{\mu_0}{m}, 0), \eta_0) \subset O_{\eta_0}(\mathcal{R}_0)$, what contradicts our assumption. That is why there must be $\|u_n\|_0 \geq r_0$ and $|\lambda_n - \lambda| + \|u_n\|_0 > r_0$, for any $\lambda \in [a,b]$.

So $(\lambda_n, u_n) \notin O_{r_0}([a,b] \times \{0\})$ and $(\lambda_n, u_n) \notin O_{r_0}(\mathcal{R}_0)$, what contradicts (2.3), because $f^{-1}(0) \cap ([a,b] \times \overline{B(0,R)}) = ([a,b] \times \{0\}) \cup \mathcal{R}_0$. The contradiction proves (2.4).

Now we are going to prove

$$\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n>n_0}\ S_n^i\subset O_\varepsilon(S). \tag{2.5}$$

Assume that there exists $\eta_0>0$ and the subsequence $S^i_{\gamma(n)}$ of S^i_n such that

$$S^i_{\gamma(n)}\not\subset O_{\eta_0}(S).$$

But for almost all $n\in\mathbb{N}$ we have

$$\emptyset\neq S^i_{\gamma(n)}\cap([a,b]\times\{0\})\subset(\frac{\mu_0}{m}-\eta_0,\frac{\mu_0}{m}+\eta_0)\times\{0\}\subset O_{\eta_0}(S)$$

and consequently $S^i_{\gamma(n)}\cap O_{\eta_0}(S)\neq\emptyset$.

Assume that $(\lambda_n,u_n)\in S^i_{\gamma(n)}$ are points such that $(\lambda_n,u_n)\notin O_{\eta_0}(S)$. Because, by (2.4)

$$\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n>n_0}(\lambda_n,u_n)\in O_\varepsilon(\mathcal{R}_0),$$

and $\mathcal{R}_0$ is compact, there exists the subsequence of $\{(\lambda_n,u_n)\}$ converging to (λ_0,u_0) in $\mathcal{R}_0$. Because $(\lambda_n,u_n)\notin O_{\eta_0}(S)$ the relation $(\lambda_0,u_0)\notin O_{\eta_0}(S)$ holds as well. The set $\mathcal{R}_0$ is a compact metric space, $X=S\cap\mathcal{R}_0$ and $Y=\{(\lambda_0,u_0)\}$ are its closed subsets, not belonging to the same component of $\mathcal{R}_0$. By separation lemma (see [16]) there exists the separation $\mathcal{R}_0=\mathcal{R}_x\cup\mathcal{R}_y$ of $\mathcal{R}_0$, where $\mathcal{R}_x$ and $\mathcal{R}_y$ are closed and disjoint, and such that $S\cap\mathcal{R}_0\subset\mathcal{R}_x$ and $(\lambda_0,u_0)\in\mathcal{R}_y$.

This implies, that there exist open and disjoint subsets $U_x,U_y\subset[a,b]\times\overline{B(0,R)}$, such that $(\lambda_0,u_0)\in U_y$ and $S\cap\mathcal{R}_0\subset U_x$ and $\mathcal{R}_0\subset U_x\cup U_y$.

Because $\mathcal{R}_0$ is compact, there exists $\eta>0$, such that $O_\eta(\mathcal{R}_0)\subset U_x\cup U_y$. Let us observe that, by (2.4), for almost all $n\in\mathbb{N}$ the relation holds $S^i_{\gamma(n)}\subset O_\eta(\mathcal{R}_0)\subset U_x\cup U_y$. Moreover, $S^i_{\gamma(n)}\cap U_x\neq\emptyset$ and $S^i_{\gamma(n)}\cap U_y\neq\emptyset$, what (because of $U_x\cap U_y=\emptyset$) contradicts the connectedness of $S^i_{\gamma(n)}$. The contradiction proves (2.5) and finishes the proof of the lemma. $\square$

Now Theorem 1.5 follows as a corollary of the above result.

Remark 2.9. Both approximations are, in fact, the finitely dimensional ones. Let us take, as an example, the approximation on the triangle. Let us denote $N=\frac{n(n+1)}{2}$ and associate the coordinates of the point $\xi\in\mathbb{R}^N$ with $\xi_{i,j,k}=u(\frac{i}{n},\frac{j}{n},\frac{k}{n})$, where $i,j,k\geq 0$ and $i+j+k=n$. Then we have

$$u(x,y)=T[B^1_n\Phi(\lambda,u)],$$

and substituting $(x,y)=\frac{i_0}{n}P+\frac{j_0}{n}Q+\frac{k_0}{n}R$ we have

$$\xi_{i_0,j_0,k_0}=\sum_{i,j,k\geq 0;i+j+k=n}\varphi(\lambda,\xi_{i,j,k})[T(T^n_{i,j,k})](\frac{i_0}{n},\frac{j_0}{n},\frac{k_0}{n}). \tag{2.6}$$

where $[T(T^n_{i,j,k})](\frac{i_0}{n},\frac{j_0}{n},\frac{k_0}{n})$ are known coefficients depending on $\mathbb{T}$ and Ω only. In the above formula we should treat the function $T(T^n_{i,j,k})\in C(\overline{\Omega})$ as extended by zero to the whole triangle $\mathbb{T}$. That is why the dimension of the problem (2.6) is generally smaller then N.

3. Existence theorem

In this section we are going to show how the approximation theorem for global bifurcation branches given above, may be applied in the proof of the existence theorem for the boundary-value problem

$$\begin{gathered}(Lu)(x,y)+\varphi(x,y,u(x,y))=0 \quad \text{for } (x,y)\in\Omega\\ u(x,y)=0 \quad \text{for } (x,y)\in\partial\Omega.\end{gathered} \tag{3.1}$$

We will show, that the solution of the above problem exists and is approximated by solutions of

$$\begin{gathered}(Lu)(x,y)+(B^1_n\tilde{\varphi}(\cdot,u(\cdot)))(x,y)=0 \quad \text{for } (x,y)\in\Omega\\ u(x,y)=0 \quad \text{for } (x,y)\in\partial\Omega.\end{gathered} \tag{3.2}$$

and

$$\begin{gathered}(Lu)(x,y)+(B^2_n\hat{\varphi}(\cdot,u(\cdot)))(x,y)=0 \quad \text{for } (x,y)\in\Omega\\ u(x,y)=0 \quad \text{for } (x,y)\in\partial\Omega.\end{gathered} \tag{3.3}$$

Theorem 3.1. *Let $\varphi:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be the continuous function, such that there exist the positive numbers $A<\mu_0<B$ satisfying*

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{(x,y)\in\overline{\Omega},s\in\mathbb{R}}0\le s\le\delta\Rightarrow|\varphi((x,y),s)-Bs|\le\varepsilon|s|; \tag{3.4}$$

$$\forall_{\varepsilon>0}\exists_{R>0}\forall_{(x,y)\in\overline{\Omega},s\in\mathbb{R}}s\ge R\Rightarrow|\varphi((x,y),s)-As|\le\varepsilon|s|. \tag{3.5}$$

Then

(a) *there exists the nonnegative solution u of* (3.1) *and for almost all $n\in\mathbb{N}$ there exists solution u_n of* (3.2)*, such that, there exists the subsequence $\{u_{\gamma(n)}\}$ satisfying $\lim_{n\to+\infty}\|u_{\gamma(n)}-u\|_0=0$, and*

(b) *there exists the nonnegative solution u of* (3.1) *and for almost all $n\in\mathbb{N}$ there exists solution v_n of* (3.3)*, such that, there exists the subsequence $\{v_{\gamma(n)}\}$ satisfying $\lim_{n\to+\infty}\|v_{\gamma(n)}-u\|_0=0$.*

Proof. Let us assume that $\varphi((x,y),s)=B|s|$, for $s<0$ and $(x,y)\in\overline{\Omega}$. Let us now define the continuous function $\psi:(0,+\infty)\times\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$, by

$$\psi(\lambda,(x,y),s)=\lambda\varphi((x,y),s).$$

We may now consider the nonlinear spectral problem

$$\begin{gathered}(Lu)(x,y)+\psi(\lambda,(x,y),u(x,y))=0 \quad \text{for } (x,y)\in\Omega\\ u(x,y)=0 \quad \text{for } (x,y)\in\partial\Omega.\end{gathered} \tag{3.6}$$

Function ψ satisfies (1.6) and (1.7), so Theorem 1.3 may be applied to problem (3.6). So, there exists the noncompact component C of $\mathcal{R}$ for the problem (3.6), such that $(\frac{\mu_0}{B},0)\in C$ and for $(\lambda,u)\in C$, the inequality holds $u\ge 0$.

Let us now observe that, because C is not compact, one of the three situations takes place

(i) there exists sequence $(\lambda_n,u_n)\in C$, such that $\lambda_n\to 0$;
(ii) there exists sequence $(\lambda_n,u_n)\in C$, such that $\|u_n\|_0\to+\infty$.
(iii) there exists sequence $(\lambda_n,u_n)\in C$, such that $\lambda_n\to+\infty$;

For all those situations, we may assume that $u_n\ne 0$. In the reasoning below let $\Phi:C_0(\overline{\Omega})\to C_0(\overline{\Omega})$ denote the Niemytzki operator associated with the function φ.

The first conclusion is that (i) implies (ii). Assume, contrary to the claim, that $\lambda_n \to 0$ and $\|u_n\|_0 \le M$, for a constant $M > 0$. So $u_n = \lambda_n T\Phi(u_n)$ implies $u_n \to 0$. Then,

$$v_n = \lambda_n BTv_n + \lambda_n T\frac{\Phi(u_n) - Bu_n}{\|u_n\|_0},$$

where $v_n = \frac{u_n}{\|u_n\|_0}$. We can see that $\{Tv_n\}$ contains convergent subsequence. Because of our assumption the sequence $\{\frac{\Phi(u_n)-Bu_n}{\|u_n\|_0}\}$ converges to zero. Letting $n \to +\infty$ we have $v_n \to 0$, so we came to the contradiction, with $\|v_n\|_0 = 1$.

So at least one of (ii) and (iii) holds true. That is why in the component C, there exist the point (λ, u), such that the sum $\|u\|_0 + |\lambda|$ is arbitrary large. We are now going to show that there exists $\lambda_1 > 1$ and $R_1 > 0$, such that

$$\forall_{(\lambda,u)\in C} \|u\|_0 \ge R_1 \Rightarrow \lambda \in (\lambda_1, +\infty).$$

Assume contrary to our claim that, there exists the sequence $(\lambda_n, u_n) \in C$, such that $\|u_n\|_0 \to +\infty$ and $\lambda_n \to \lambda_0 \in [0, 1]$. Then, for $v_n = \frac{u_n}{\|u_n\|_0}$, the relation holds

$$v_n = \lambda_n T\frac{\Phi(u_n) - Au_n}{\|u_n\|_0} + \lambda_n ATv_n.$$

Let us now observe that $\frac{\Phi(u_n)-Au_n}{\|u_n\|_0} \to 0$. We will show, that for any positive $\eta > 0$, the inequality $|\frac{\Phi(u_n)(x,y)-Au_n(x,y)}{\|u_n\|_0}| < \eta$ holds for almost all $n \in \mathbb{N}$. First, let us choose $R > 0$, such that for $s > R$ the relation holds

$$\Big|\frac{\varphi((x, y), s) - As}{s}\Big| < \eta.$$

Because the function $|\varphi((x, y), s) - As|$ is bounded on $\overline{\Omega} \times [0, R]$ we can see that for $n \in \mathbb{N}$ large enough the implication holds

$$u_n(x, y) \in [0, R] \Rightarrow \Big|\frac{\Phi(u_n)(x, y) - Au_n(x, y)}{\|u_n\|_0}\Big| < \eta.$$

Moreover, for $\|u_n\|_0 > R$ the implication holds

$$u_n(x, y) > R \Rightarrow \frac{|\Phi(u_n)(x, y) - Au_n(x, y)|}{|u_n(x, y)|} \le \Big|\frac{\Phi(u_n)(x, y) - Au_n(x, y)}{\|u_n\|_0}\Big| < \eta$$

what proves $|\frac{\Phi(u_n)(x,y)-Au_n(x,y)}{\|u_n\|_0}| < \eta$, for any $(x, y) \in \overline{\Omega}$. Hence $\frac{1}{\|u_n\|_0}T(\Phi(u_n) - Au_n) \to 0$. Moreover, there exists the convergent subsequence of $\{Tv_n\}$. So we may assume that $v_n \to v_0$ and then we have

$$v_0 = \lambda_0 ATv_0$$

for $v_0 \ne 0$ and $v_0 \ge 0$. This implies $\lambda_0 = \frac{\mu_0}{A} > 1$, what contradicts our assumption.

By Theorem 1.4 there exists the sequence of connected sets $C_n^i \subset (0, +\infty) \times C_0(\overline{\Omega})$, such that for $n \in \mathbb{N}$ large enough $C_n^i \cap ((0, 1) \times \{0\}) \ne \emptyset$ and for any $R > 0$ the relation holds $C_n^i \cap \partial([\frac{\mu_0}{2B}, 2] \times \overline{B(0, R)} \ne \emptyset$. Let us observe that for $R > 0$ large enough, there exists $(\lambda_n, u_n) \in C_n^i$ such that $\lambda_n \in (1, 2]$.

So, from the connectedness of the sets C and C_n^i, we may conclude that there exist pairs $(1, u) \in C$, $(1, u_n) \in C_n^1$ and $(1, v_n) \in C_n^2$, and because of Theorem 1.5 the point $(1, u)$ may be selected in such way that there exists infinitely many points $(1, u_n) \in C_n^1$ or $(1, v_n) \in C_n^2$ being arbitrarily close to $(1, u)$. $\square$

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