*Electronic Journal of Differential Equations*, Vol. 2009(2009), No. 120, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# LINKING METHOD FOR PERIODIC NON-AUTONOMOUS FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH SUPERQUADRATIC POTENTIALS

CHENGYUE LI, CHANGHUA SHI

Abstract. By means of the Schechter's Linking method, we study the existence of $2T$-periodic solutions of the non-autonomous fourth-order ordinary differential equation

$$u'''' - Au'' - Bu - V_u(t,u) = 0$$

where $A > 0$, $B > 0$, $V(t,u) \in \mathbb{C}^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is $2T$-periodic in $t$ and satisfies either $0 < \theta V(t,u) \le uV_u(t,u)$ with $\theta > 2$, or $uV_u(t,u) - 2V(t,u) \ge d_3|u|^r$ with $r \ge 1$.

## 1. Introduction

Pulse propagation through optical fibers involving a fourth-order negative dispersion term leads to a generalized nonlinear Schrodinger equation [1, 4]. After an appropriate scaling of the variables this equation takes the form

$$i\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial t^2} - \frac{\partial^4 w}{\partial t^4} + |w|^2 w = 0. \tag{1.1}$$

Considering harmonic spatial dependence $w(t,x) = u(t)e^{ikx}$ with $k < 0$, one obtains

$$u^{(4)} - u'' + ku - u^3 = 0. \tag{1.2}$$

Motivated by (1.2), we shall discuss the more general equation

$$u^{(4)} - Au'' - Bu - V_u(t,u) = 0, \tag{1.3}$$

where $A > 0$, $B > 0$, the potential $V(t,u) \in \mathbb{C}^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $V_u(t,u) = \partial V(t,u)/\partial u$.

Indeed, many other types of fourth-order differential equation models in physical, chemical or biological systems have been studied for recent years. We give some examples as follows:

(i) The equation $u^{(4)} - \gamma u'' - u + u^3 = 0$ serves as a model in studies of pattern formation and phase transitions near Lifshitz points. If $\gamma > 0$, it is the Extended Fisher-Kolmogorov equation proposed by Dee and Saarloos van in [6]. If $\gamma < 0$, it

2000 *Mathematics Subject Classification.* 58E05, 34C37, 70H05.
*Key words and phrases.* Periodic solutions; fourth-order differential equations; linking theorem; critical points.

Submitted June 18, 2009. Published September 27, 2009.
Supported by SRF for ROCS, SEM (2007-2008), and "211 Engineer" Project from the Ministry of Education in China.

is the Swift-Hohenberg equation which has been proposed by Swift and Hohenberg [12]. For the existence of its periodic solutions, we refer the readers to [8].

(ii) In the theory of shallow water waves driven by gravity and capillarity, the equation $u^{(4)} + pu'' + u - u^2 = 0$ has been studied with $p < 0$ [2], which was extensively considered by Buffoni [3].

(iii) Chen and McKenna [5] studied the equation $u^{(4)}+c^2u''+V'(u) = 0$ under the assumptions that $V \in \mathbb{C}^2(\mathbb{R})$ is a potential such that $V'(u) = (u+1)_+-1+g(u)$ with $|g''(u)| \le K$ for some $K > 0$ .This result was improved by Smets and Van den Berg [11] for almost every $c \in [-\sqrt{4\alpha}, \sqrt{4\alpha}]$, assuming that $\limsup_{u\to\infty} V(u)/|u|^2 = 0$.

(iv) Tersian and Chaparova [13] studied the equation $u^{(4)}+pu''+a(x)u-b(x)u^2-c(x)u^3 = 0$ where $a(x), b(x), c(x)$ are periodic, and $0 < a_1 \le a(x)$, $0 < c_1 \le c(x)$. They obtained the existence of periodic solutions of the equation for $p \neq 0$.

(v) Gyulov and Tersian [7] discussed the equation $u^{(4)} + au'' + bu + V_u(t, u) = 0$ where $V(t, u) \ge c|u|^p$ with $p > 2$, and obtained the existence and nonexistence of nontrival periodic solutions of the equation by Brezis-Nirenberg's linking Theorem and minimizing methods.

In the present paper, we shall study the existence of periodic solutions of the non-autonomous fourth-order equation (1.3). Our main results are as follows:

**Theorem 1.1.** *Let $A > 0$, $B > 0$. Assume that $V(t, u) \in C^1(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ satisfies the assumptions:*

- (V1) *$V(t, u) = V(t + 2T, u), V(t, u) = V(t, -u)$, for all $t \in \mathbb{R}, u \in \mathbb{R}$;*
- (V2) *$V(t, u) = o(|u|^2)$, as $u \to 0$ uniformly in $t \in \mathbb{R}$;*
- (V3) *There exists a constant $\theta > 2$ such that*

$$0 < \theta V(t, u) \le uV_u(t, u), \quad \forall t \in \mathbb{R}, u \in \mathbb{R} \setminus \{0\}.$$

*Then (1.3) has at least one nontrivial $2T$-periodic solution, provided that $\frac{T}{T_1} \notin \mathbb{N}$ with $T_1 = \pi\sqrt{2}/\sqrt{-A + \sqrt{A^2 + 4B}}$.*

**Theorem 1.2.** *Let $A > 0$, $B > 0$. Suppose that $V(t, u) \in C^1(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ satisfies that (V1), (V2) and the following conditions:*

- (V3') *$V(t, u)/|u|^2 \to \infty$, as $|u| \to \infty$ uniformly in $t \in \mathbb{R}$;*
- (V4) *There are constants $\mu, d_1, d_2 > 0$ such that $|V_u(t, u)| \le d_1|u|^\mu + d_2$, for all $t \in \mathbb{R}, u \in \mathbb{R}$;*
- (V5) *There are constants $h, d_3 > 0, r \ge max\{1, \mu\}$ such that*

$$uV_u(t, u) - 2V(t, u) \ge d_3|u|^r, \quad \forall t \in \mathbb{R},\ |u| > h.$$

*Then the conclusion of Theorem 1.1 holds.*

**Remark 1.3.** Hypothesis (V3) is so-called Ambrosetti-Rabinowitz superquadratic condition which implies that there exist constants $r_1 > 0$, $r_2 > 0$ such that

$$V(t, u) \ge r_1|u|^\mu - r_2, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}. \tag{1.4}$$

By direct computation we notice that, for example, $V(t, u) = u^2 \ln(1 + u^{2i}) \ln(1 + 2u^{2j})$ or $V(t, u) = u^2 \ln(1 + u^{2i})$ $(i, j \in \mathbb{N})$ satisfies (V3'), (V4), and (V5), but does not satisfy (1.4). Therefore, Theorems 1.1 and 1.2 study two types of superquadratic nonlinearities.

## 2. Preliminaries

To study the existence of $2T$-periodic solutions of (1.3), we first consider the solvability of the two-point boundary problem

$$\begin{gathered} u^{(4)} - Au'' - Bu - V_u(t,u) = 0, \quad 0 < t < T; \\ u(0) = u(T) = 0, u''(0) = u''(T) = 0. \end{gathered} \tag{2.1}$$

We shall obtain $2T$-periodic solutions of (1.3) which are antisymmetric with respect to $t = 0$ and $t = T$ taking the $2T$-periodic extension of the odd extension

$$\bar{u}(t) = \begin{cases} u(t), & 0 \le t \le T; \\ -u(-t), & -T \le t \le 0 \end{cases} \tag{2.2}$$

of the solution $u(t)$ for problem (2.1).

Assume $X = H^2([0,T]) \cap H_0^1([0,T])$ be a Hilbert space with the inner product

$$(u,v) = \int_0^T (u''(t)v''(t) + u'(t)v'(t) + u(t)v(t))dt, \tag{2.3}$$

which corresponds the norm

$$\|u\|_X = \Big( \int_0^T (|u''(t)|^2 + |u'(t)|^2 + |u(t)|^2)dt \Big)^{1/2}.$$

From the Poincare inequality

$$\int_0^T |u(t)|^2 dt \le \frac{T^2}{\pi^2} \int_0^T |u'(t)|^2 dt, \quad \int_0^T |u(t)|^2 dt \le \frac{T^4}{\pi^4} \int_0^T |u''(t)|^2 dt, \tag{2.4}$$

we know that $\|u\|_X$,

$$\|u\| = (\int_0^T (|u''(t)|^2 dt)^{1/2}, \tag{2.5}$$

$$\|u\|_* = (\int_0^T (|u''(t)|^2 + A|u'(t)|^2)dt)^{1/2}, \tag{2.6}$$

are equivalent norms in $X(T)$. In addition, an important fact in $X(T)$ is that the set of functions $\{sin\frac{k\pi t}{T}\}_{k=1}^{\infty}$is a complete orthogonal basis [7].

A function $u \in X(T)$ is said to be a weak solution of (2.1), if

$$\int_0^T (u''(t)v''(t) + Au'(t)v'(t) - Bu(t)v(t))dt - \int_0^T V_u(t,u)v dt = 0, \quad \forall v \in X(T).$$

Define the pertinent functional

$$I(u;T) = \int_0^T \frac{1}{2}(u''^2 + Au'^2 - Bu^2)dt - \int_0^T V(t,u)dt, \quad \forall u \in X(T). \tag{2.7}$$

Under the assumption of $V(t,u) \in \mathbb{C}^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we easily show that $I(u;T) \in C^1(X(T), R)$ and

$$I'(u;T)v = \int_0^T (u''v'' + Au'v' - Buv)dt - \int_0^T V_u(t,u)v dt = 0, \quad \forall u,v \in X(T). \tag{2.8}$$

So weak solutions of (2.1) are critical points of $I(u;T)$. In fact, by the standard way, weak solutions of (2.1) are exactly its classical solutions.

For $u \in X(T)$, using Fourier series, we have

$$u = \sum_{k=1}^{\infty} c_k \sin(\frac{k\pi t}{T}), \tag{2.9}$$

$$I(u;T) = \frac{T}{4}\sum_{k=1}^{\infty} c_k^2 P_k(T) - \int_0^T V(t,u)dt, \tag{2.10}$$

where $P_k(T) = P(\frac{k\pi}{T})$ with $P(\xi) = \xi^4 + A\xi^2 - B$, $\xi \in \mathbb{R}$. Clearly for every $T > 0$,

$$P_1(T) < P_2(T) < P_3(T) < \dots < P_n(T)) < \dots . \tag{2.11}$$

For every $n \in \mathbb{N}$, the equation $P_n(T) = 0$ has the unique solution

$$T_n = nT_1, \quad T_1 = \pi\sqrt{2}/\sqrt{-A + \sqrt{A^2 + 4B}}, \tag{2.12}$$

and $P_n(T) > 0$, if $T < nT_1$; $P_n(T) < 0$, if $T > nT_1$.

To prove Theorems 1.1 and 1.2, we shall use linking method due to Schechter. For that, we start recalling the definition of linking sets in the sense of homeomorphisms [9].

Let $E$ be a real Banach space and let $\Phi$ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0,1]$ to $E$ such that (i) $\Gamma(0) = I$, the identity map. (ii) For each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of $E$ into $E$ and $\Gamma^{-1}(t) \in \mathbb{C}(E \times [0,1], E)$. (iii) $\Gamma(1)E$ is a single point in $E$ and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$. (iv) For each $t_0 \in [0,1)$ and each bounded set $Y \subset E$, $\sup_{0 \le t \le t_0, u \in Y}\{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty$.

We say that $Y$ links $Z$ if $Y$ and $Z$ are subsets of $E$ such that $Y \cap Z = \phi$ and, for each $\Gamma \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)Y \cap Z \neq \phi$. Many examples of linking sets are presented in [9]. A typical one is as follows:

**Example.** [9, Example 3, P.38]. Let $M$ and $N$ be closed subspaces of Banach space $E$ such that $\dim N < \infty$ and $E = M \oplus N$. Let $w_0 \neq 0$ be an element of $M$, $0 < \rho < R$, and take

$$Y = \{v \in N : \|v\| \le R\} \cup \{v + \lambda w_0 : v \in N, \lambda \ge 0, \|v + \lambda w_0\| = R\},$$
$$Z = \partial B_\rho(0) \cap M.$$

Then $Y$ links $Z$.

It was shown in [9] that with the aid of linking method a deformation theorem was obtained and then, using standard minimax arguments, the following result was proved by Schechter:

**Theorem 2.1** (Linking Theorem 2.1.1 and Corollary 2.8.2 in [9]). *Assume that $E$ is a real Banach space, the functional $\varphi \in \mathbb{C}^1(\mathbb{E}, \mathbb{R})$. $Y$ and $Z$ are subsets of $E$ such that $Y$ is compact and $Y$ links $Z$, and satisfies that $a_0 := sup_Y \varphi \le b_0 := inf_Z \varphi$. If $a = inf_{\Gamma\in\Phi} sup_{0\le s\le 1, u\in Y}\varphi(\Gamma(s)u)$ is finite, then there is a sequence $(u_m) \subset E$ such that $\varphi(u_m) \to a \ge b_0$, $(1 + \|u_m\|)\varphi'(u_m) \to 0$. Furthermore, if $a = b_0$, then* $\operatorname{dist}(u_m, Z) \to 0$.

In addition, we also recall the limit case of Rabinowitz's Mountain Pass Lemma, which shall be employed in the section 3 and section 4.

**Theorem 2.2** ([14]). *Let $E$ be a real Banach space and $\varphi \in \mathbb{C}^1(\mathbb{E}, \mathbb{R})$ satisfying the $(PS)$ condition, $\varphi(0) = 0$. If $\varphi$ satisfies*

(a) *There is an open neighborhood* $Y$ *of the origin* $0$ *such that* $\varphi|_{\partial Y} \geq 0$*;*
(b) *There is* $e \notin \overline{Y}$ *such that* $\varphi(e) \leq 0$*,*

*then* $\varphi$ *possesses a critical value* $b \geq 0$ *at the level characterized by*

$$b = \sup_{Z\in W} \inf_{u\in\partial Z} \varphi(u),$$

*where* $W = \{Z \subset E : Z$ *is open* $0 \in Z$ *and* $e \notin \overline{Z}\}$*. Moreover, if* $b = 0$*, there is a critical point of* $\varphi$ *on* $\partial Y$*.*

## 3. Proof of Theorem 1.1

**Lemma 3.1.** *Under the assumptions of Theorem 1.1, the* $(PS)$ *condition holds for* $I(u;T)$*. Namely, if* $(u_m) \subset X(T)$ *satisfies that*

$$|I(u_m;T)| \leq M_1, \quad |I'(u_m;T)| \to 0, \tag{3.1}$$

*for some constant* $M_1 > 0$*,then there is a subsequence of* $(u_m)$ *converging to a limit* $u_0 \in X(T)$*.*

*Proof.* Choose $\theta^* \in (2, \theta)$. By (V3), (1.4) and (3.1), we have

$$\begin{aligned}
M_1 + \|u_m\| &\geq I(u_m;T) - \frac{1}{\theta^*} I'(u_m;T)u_m \\
&= \frac{1}{2}\int_0^T (u_m^{''2} + Au_m^{'2} - Bu_m^2)dt - \int_0^T V(t,u_m)dt \\
&\quad - \frac{1}{\theta^*}\Big(\int_0^T (u_m^{''2} + Au_m^{'2} - Bu_m^2)dt - \int_0^T V_u(t,u_m)u_m dt\Big) \\
&= (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T (u_m^{''2} + Au_m^{'2})dt - (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T Bu_m^2 dt \\
&\quad + \int_0^T (\frac{V_u(t,u_m)u_m}{\theta^*} - V(t,u_m))dt \\
&\leq (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T (u_m^{''2} + Au_m^{'2})dt - (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T Bu_m^2 dt \\
&\quad + \frac{\theta - \theta^*}{\theta^*}\int_0^T V(t,u_m)dt \\
&\leq (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T (u_m^{''2} + Au_m^{'2})dt - (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T Bu_m^2 dt \\
&\quad + r_1\frac{\theta - \theta^*}{\theta^*}\|u_m\|_{L^\theta}^\theta - Tr_2\frac{\theta - \theta^*}{\theta^*} \\
&\leq (\frac{1}{2} - \frac{1}{\theta^*})\int_0^T (u_m^{''2} + Au_m^{'2})dt - (\frac{1}{2} - \frac{1}{\theta^*})B\|u_m\|_{L^2}^2 \\
&\quad + r_3\|u_m\|_{L^2}^\theta - Tr_2\frac{\theta - \theta^*}{\theta^*}
\end{aligned} \tag{3.2}$$

with $r_3 > 0$. We claim that $\|u_m\|_{L^2}$ is bounded. Otherwise, $\|u_m\|_{L^2} \to \infty$, $\|u_m\| \to \infty$. Thus, since $\theta > 2$, for $m$ sufficiently large, we have

$$-(\frac{1}{2} - \frac{1}{\theta^*})B\|u_m\|_{L^2}^2 + r_3\|u_m\|_{L^2}^\theta - Tr_2\frac{\theta - \theta^*}{\theta^*} > 0. \tag{3.3}$$

Consequently, by (3.2) and (3.3), we deduce that

$$M_1 + \|u_m\| \geq (\frac{1}{2} - \frac{1}{\theta^*}) \int_0^T (u_m^{''2} + Au_m^{'2})dt \geq (\frac{1}{2} - \frac{1}{\theta^*})\|u_m\|^2,$$

which contradicts $\|u_m\| \to \infty$. So $\|u_m\|_{L^2}$ is bounded. Therefore, by (3.2), there exists $M_2 > 0$ such that

$$M_1 + \|u_m\| \geq (\frac{1}{2} - \frac{1}{\theta^*}) \int_0^T u_m^{''2} dt + M_2 = (\frac{1}{2} - \frac{1}{\theta^*})\|u_m\|^2 + M_2, \tag{3.4}$$

This inequality implies $\|u_m\|$ is bounded in $X(T)$. Then we can assume that, without loss of generation,

$$u_m \rightharpoonup u_0 \in X(T), \quad u_m \to u_0 \in \mathbb{C}([0,T]). \tag{3.5}$$

So, by (2.8) and (3.5), we have

$$\begin{aligned} &\|u_m - u_0\|^2 + A\int_0^T |u_m' - u_0'|^2 dt \\ &= B\int_0^T |u_m - u_0|^2 dt + (I'(u_m) - I'(u_0))(u_m - u_0) \\ &\quad + \int_0^T (V_u(t,u_m) - V_u(t,u_0))(u_m - u_0)dt \to 0, \end{aligned} \tag{3.6}$$

namely, $u_m \to u_0$ in $X(T)$. ☐

**Lemma 3.2.** *Under the assumptions of Theorem 1.1, if $T > T_1$ and $\frac{T}{T_1} \notin \mathbb{N}$, then the functional $I(u;T)$ possesses a nontrivial critical point in $X(T)$.*

*Proof.* There exists $n \in \mathbb{N}$ such that $nT_1 < T < (n+1)T_1$. Define

$$E_n = span\{sin\frac{\pi t}{T}, sin\frac{2\pi t}{T} \dots sin\frac{n\pi t}{T}\}, \tag{3.7}$$

$$Y = \{v \in E_n : \|v\| \leq R\} \cup \{v + \lambda e : v \in E_n, \lambda \geq 0, \|v + \lambda e\| = R\}, \tag{3.8}$$

$$Z = \partial B_\rho(0) \cap E_n^\perp. \tag{3.9}$$

where $e \in E_n^\perp$, $\|e\| = 1$ and $0 < \rho < R$. By the typical example in section 2, $Y$ links with $Z$. We shall verify for $R$ sufficiently large and $\rho$ sufficiently small, that the following inequality holds:

$$\sup_Y T(u;T) \leq 0 \leq \inf_Z I(u;T). \tag{3.10}$$

Firstly, for every $v \in E_n$, $v(t) = \sum_{k=1}^n c_k \sin(\frac{k\pi t}{T})$, since

$$P_1(T) < P_2(T) < P_3(T) < \dots P_n(T) < 0, \tag{3.11}$$

we have

$$I(v;T) = \frac{T}{4}\sum_{k=1}^n P_k(T)c_k^2 - \int_0^T V(t,v(t))dt \leq 0. \tag{3.12}$$

Secondly, we take $e = c_{n+1} \sin\frac{(n+1)\pi t}{T} \in E_n^\perp$ with $c_{n+1}$ such that $\|e\| = 1$, and let

$$w = u + \lambda e = \sum_{k=1}^n c_k \sin(\frac{k\pi t}{T}) + \lambda e, \lambda \geq 0.$$

Then $\|w\| = \|u\| + \|\lambda e\| = \|u\| + \lambda$. There exist $r_4, r_4'$ such that $r_4'\|w\|_{L^\theta} \le \|w\| \le r_4\|w\|_{L^\theta}$, for all $w \in E_{n+1}$. Therefore, by (1.4), we conclude that

$$\begin{aligned} I(w;T) &= \int_0^T \frac{1}{2}((w''^2 + Aw'^2 - Bw^2))dt - \int_0^T V(t,w)dt \\ &= \frac{T}{4}\sum_{k=1}^{n} P_k(T)c_k^2 + \frac{T}{4}P_{n+1}(T)\lambda^2 c_{n+1}^2 - \int_0^T V(t,w)dt \\ &\le \frac{T}{4}P_{n+1}(T)\lambda^2 c_{n+1}^2 - \int_0^T V(t,w)dt \\ &\le \frac{T}{4}P_{n+1}(T)c_{n+1}^2\|w\|^2 - r_1 r_4^{-\theta}\|w\|^\theta + r_2 T \le 0 \end{aligned} \tag{3.13}$$

for $\|w\| = R$ large enough. Finally, by (V2) for each $\varepsilon > 0$, there is $\delta \in (0,1)$ such that

$$|V(t,u)| \le \varepsilon|u|^2 \quad \text{if } |u| \le \delta \text{ and } t \in [0,T].$$

By the Sobolev embedding Theorem, there exists a constant $r_5 > 0$ such that

$$\|u\|_{C([0,T])} = \|u\|_{L^\infty[0,T]} \le r_5\|u\|, \quad \forall u \in X(T). \tag{3.14}$$

Let $0 < \rho < \min\{\delta/r_5, R\}$ and $\|u\| = \rho$, then $|u(t)| \le \delta$ for all $t \in [0,T]$. Therefore,

$$\int_0^T V(t,u(t))dt \le \varepsilon\|u\|_{L^2}^2.$$

Noticing $0 < p_{n+1}(T) < p_{n+2}(T)\dots$ for $u \in E_n^\perp \cap B_\rho(0)$, $u = \sum_{k=n+1}^\infty c_k \sin(\frac{k\pi t}{T})$, we have

$$\begin{aligned} I(u;T) &= \frac{T}{4}\sum_{k=n+1}^{\infty} P_k(T)c_k^2 - \int_0^T V(t,u(t))dt, \\ &\le P_{n+1}(T)\|u\|_{L^2}^2 - \varepsilon\int_0^T |u(t)|^2 dt, \\ &\le \frac{1}{2}P_{n+1}(T)\|u\|_{L^2}^2 \ge 0 \end{aligned} \tag{3.15}$$

if $0 < \varepsilon < \frac{1}{2}P_{n+1}(T)$. Then (3.12), (3.13) and (3.15) imply that (3.10) holds. Thus, by Theorem 2.1, there exists a sequence $(u_m) \subset X(T)$ satisfies that

$$I(u_m;T) \to d_0 \ge 0, \tag{3.16}$$

$$(1 + \|u_m\|)I'(u_m;T) \to 0. \tag{3.17}$$

By Lemma 3.1, we may assume that $u_m \to u_0 \in X(T)$. And, by (3.17), we can show that $u_0$ is a critical point of $I(u;T)$. If $d_0 > 0$, then $u_0 \ne 0$. If $d_0 = 0$, then $\operatorname{dist}(u_m, Z) \to 0$ by Theorem 2.1. Hence there is a sequence $(v_m) \subset Z$ such that $u_m - v_m \to 0$ in $X(T)$, so $v_m \to u_0$, thus $\|u_0\| = \lim_{m\to\infty}\|v_m\| = \rho \ne 0$. □

**Lemma 3.3.** *Under the assumptions of Theorem 1.1, if $0 < T < T_1$, then the functional $I(u;T)$ possesses a nontrivial critical point in $X(T)$.*

*Proof.* We shall use Theorem 2.2 to prove the existence of the critical point of $I(u;T)$. Under the condition $0 < T < T_1$, we have

$$0 < P_1(T) < P_2(T) < P_3(T) < \dots P_n(T) < \dots. \tag{3.18}$$

Similar to Lemma 3.2, for $0 < \varepsilon < \frac{1}{2}P_1(T)$, there exists $\delta \in (0,1)$ such that $V(t,u) \le \varepsilon |u|^2$ if $|u| \le \delta$ and $t \in [0,T]$. Then for every $u = \sum_{k=1}^{\infty} c_k \sin(\frac{k\pi t}{T}) \in X(T)$ such that $\|u\| = \rho < \delta/r_5$, where $r_5$ is defined in (3.14), we have

$$\begin{aligned} I(u;T) &= \frac{T}{4}\sum_{k=1}^{\infty} P_k(T)c_k^2 - \int_0^T (V(t,u(t))dt, \\ &\le P_1(T)\|u\|_{L^2}^2 - \varepsilon \int_0^T |u(t)|^2 dt, \\ &\le \frac{1}{2}P_1(T)\|u\|_{L^2}^2 \end{aligned} \tag{3.19}$$

which is non-negative. Next, for some $\overline{u} \in E\backslash\{0\}$ and all $\sigma > 0$, we have

$$\begin{aligned} I(\sigma\overline{u};T) &= \frac{\sigma^2}{2}\int_0^T (\overline{u}^{''2} + A\overline{u}^{'2} - B\overline{u}^2)dt - \int_0^T V(t,\sigma\overline{u})dt \\ &\le \frac{\sigma^2}{2}(\int_0^T (\overline{u}^{''2} + A\overline{u}^{'2} - B\overline{u}^2))dt - r_1\sigma^\theta \int_0^T |\overline{u}|^\theta dt + r_2 T. \end{aligned} \tag{3.20}$$

Then $I(\sigma\overline{u};T) \to -\infty$ as $\sigma \to \infty$. Hence, by Lemma 3.1 and Theorem 2.2, the functional $I(u;T)$ has at least one nontrivial critical point in $X(T)$. □

The proof of Theorem 1.1 follows from combining Lemmas 3.1, 3.2 and 3.3.

## 4. Proof of Theorem 1.1

**Lemma 4.1.** *Under the assumptions of Theorem 1.2, if $T > T_1$ and $\frac{T}{T_1} \notin \mathbb{N}$, then the functional $I(u;T)$ possesses a nontrivial critical point in $X(T)$.*

*Proof.* In the same way as (3.7)-(3.9), we define $E_n$, $Y$ and $Z$. Under the assumptions of Theorem 1.2, we can verify that (3.12), (3.13) and (3.15) still hold, whose proofs are similar to that of Lemma 3.1 with the exception of the inequality (3.13) resulting from (V3). In its place we proceed as follows

Still take $e = c_{n+1}\sin\big(\frac{(n+1)\pi t}{T}\big) \in E_n^{\perp}$ with $c_{n+1}$ such that $\|e\| = 1$, and let

$$w = u + \lambda e = \sum_{k=1}^{n} c_k \sin(\frac{k\pi t}{T}) + \lambda e, \lambda \ge 0.$$

Then $\|w\| = \|u\| + \|\lambda e\| = \|u\| + \lambda$. There exist $r_6, r_6' > 0$ such that $r_6'\|w\|_{L^2} \le \|w\| \le r_6\|w\|_{L^2}$, for all $w \in E_{n+1}$.

By (V3'), there exists $r_7 > 0$ such that

$$V(t;u) \ge (\frac{T}{4}P_{n+1}(T)c_{n+1}^2 r_6^2 + 1)|u|^2 - r_7, \quad \forall t \in \mathbb{R}, u \in \mathbb{R}. \tag{4.1}$$

Therefore,

$$\begin{aligned} I(w;T) &= \frac{1}{2}\int_0^T (w^{''2} + Aw^{'2} - Bw^2))dt - \int_0^T V(t,w)dt \\ &= \frac{T}{4}\sum_{k=1}^{n} c_k^2 P_k(T) + \frac{T}{4}\lambda^2 P_{n+1}(T)c_{n+1}^2 - \int_0^T V(t,w)dt \\ &\le \frac{T}{4}\lambda^2 P_{n+1}(T)c_{n+1}^2 - \int_0^T V(t,w)dt \end{aligned}$$

$$
\begin{aligned}
&\le \frac{T}{4}P_{n+1}(T)c_{n+1}^2 r_6^2\|w\|_{L^2}^2 - (\frac{T}{4}P_{n+1}(T)c_{n+1}^2 r_6^2 + 1)\|w\|_{L^2}^2 + r_7T \\
&= -\|w\|_{L^2}^2 + r_7T \\
&\le -r_6^{-2}\|w\|^2 + r_7T \to -\infty \quad (\text{as } \|w\| \to \infty).
\end{aligned}
$$

Hence, under the assumptions of Theorem 1.2, $Y$ links $Z$, so, by Theorem 2.1, there exists a sequence $(u_m) \subset X(T)$ such that (3.16) and (3.17) hold. We shall prove that $(u_m)$ is bounded in $X(T)$. If not, we may assume that $\|u_m\| \to \infty$. From $(V_5)$, we get

$$
\begin{aligned}
2I(u_m;T) - I'(u_m;T)u_m &= \int_0^T (u_m(t)V_u(t,u_m(t)) - 2V(t,u_m(t)))dt, \\
&\le d_3 \int_{|u_m(t)|\ge h} |u_m(t)|^r dt + d_4.
\end{aligned} \tag{4.2}
$$

with $d_4$ being a constant. By (3.16), (3.17) and (4.2), we obtain

$$
\frac{1}{\|u_m\|}\int_{|u_m(t)|\ge h} |u_m(t)|^r dt \to 0. \tag{4.3}
$$

On the other hand, in view of (V4), we have

$$
\begin{aligned}
&I'(u_m;T)u_m \\
&= \int_0^T (u_m''^2 + Au_m'^2 - Bu_m^2)dt - \int_0^T u_m(t)V_u(t,u_m)dt \\
&\le \int_0^T (u_m''^2 + Au_m'^2 - Bu_m^2)dt - d_1\int_0^T |u_m(t)|^{\mu+1}dt - d_2\int_0^T |u_m(t)|dt \\
&= \int_0^T (u_m''^2 + Au_m'^2 - Bu_m^2)dt - d_1(\int_{|u_m(t)|\ge h} |u_m(t)|^{\mu+1}dt \\
&\quad + \int_{|u_m(t)|\le h} |u_m(t)|^{\mu+1}dt) - d_2\int_0^T |u_m(t)|dt \\
&\le \int_0^T (u_m''^2 + Au_m'^2 - Bu_m^2)dt - d_1\|u_m\|_{L^\infty}\int_{|u_m(t)|\ge h} |u_m(t)|^{\mu}dt \\
&\quad - d_2\|u_m\|_{L^1} - d_5 \\
&\le \int_0^T (u_m''^2 + Au_m'^2 - Bu_m^2)dt - d_6\|u_m\|\int_{|u_m(t)|\ge h} |u_m(t)|^{\mu}dt \\
&\quad - d_7\|u_m\| - d_5 \\
&\le \int_0^T (u_m''^2 + Au_m'^2 - Bu_m^2)dt - d_6h^{\mu-r}\|u_m\|\int_{|u_m(t)|\ge h} |u_m(t)|^{r}dt \\
&\quad - d_7\|u_m\| - d_5,
\end{aligned} \tag{4.4}
$$

with $d_5, d_6, d_7$ being positive constants. The two sides of (4.4) are divided by $\|u_m\|^2$, by (2.6), we have

$$
\begin{aligned}
\frac{I'(u_m;T)u_m}{\|u_m\|^2} &\ge \frac{\|u_m\|_*^2}{\|u_m\|^2} - B\int_0^T (\frac{u_m}{\|u_m\|})^2 dt \\
&\quad - d_6h^{\mu-r}\frac{\int_{|u_m(t)|\ge h}|u_m(t)|^r dt}{\|u_m\|} - \frac{d_7\|u_m\| + d_5}{\|u_m\|^2}.
\end{aligned} \tag{4.5}
$$

Set $\widetilde{u_m}(t) = \frac{u_m(t)}{\|u_m\|}$, then $\widetilde{u_m}(t) = 1$. We may assume that $\widetilde{u_m}(t) \rightharpoonup \chi \in X(T)$ and $\widetilde{u_m}(t) \to \chi$ in $\mathbb{C}([0,T])$, and $\frac{\|u_m\|_*}{\|u_m\|} \to \tau > 0$. Letting $m \to \infty$ in (4.5), and by (4.2), we have $B\int_0^T (\chi(t))^2 dt \geq \tau^2 > 0$, which implies the measure of $\Omega := \{t \in [0,T] : \chi(t)) \neq 0\}$ is positive. For every $t \in \Omega$, we have $|u_m(t)| = \|u_m\||\widetilde{u_m}(t)| \to \infty$, so by (3.16), (3.17) and (V5), we have

$$
\begin{aligned}
2d_0 &\leftarrow 2I(u_m;T) - I'(u_m;T)u_m \\
&= \int_0^T (u_m(t)V_u(t,u_m(t)) - 2V(t,u_m(t)))dt, \\
&= \int_\Omega (u_m(t)V_u(t,u_m(t)) - 2V(t,u_m(t)))dt \\
&\quad + \int_{[0,T]\setminus\Omega} (u_m(t)V_u(t,u_m(t)) - 2V(t,u_m(t)))dt, \\
&\leq d_3 \int_\Omega |u_m(t)|^r dt + \text{a bounded term} \to \infty,
\end{aligned} \tag{4.6}
$$

which is a contradiction. Therefore, $(u_m)$ is bounded in $X(T)$. Referring to (3.5)-(3.6), we can show that $u_m$ converges to some critical point $u_0$ of $I(u;T)$ in $X(T)$. Following the proof of Lemma 3.2, we also have $u_0 \neq 0$. □

**Lemma 4.2.** *Under the assumptions of Theorem 1.2, if $0 < T < T_1$, then the functional $I(u;T)$ possesses a nontrivial critical point in $X(T)$.*

The proof of the above lemma is simple, so we omit it; see also Lemma 3.3.

The Proof of Theorem 1.2 follows from Lemma 4.1 and (4.2).

**Acknowledgements.** The authors would like to thank the anonymous referee for the valuable suggestions.

## References


[1] N. N. Akhmediev, A. V. Buryak, and M. Karlsson; *Radiationless optical solitons with oscillating tails*, Optics Comm., 110(1994), 540-544.

[2] B. Buffoni, M. D. Groves, and Toland J. F.; *A plethora of solitary gravity capillary water waves with nearly critical Bond and Froude numbers*, Philos. Trans. Roy. Soc. London Ser. A 354 (1996), no. 1707, 575-607.

[3] B. Buffoni; *Periodic and homoclinic orbits for Lorentz-Lagrangian systems via variational methods*, Nonlinear Anal. 26 (1996), no. 3, 443-462.

[4] A. V. Buryak, Y. Kivshar; *Solitons due to second harmonic generation*, Phys Lett A, 1995, 197:407-412.

[5] Y. Chen and P. J. McKenna; *Traveling waves in a nonlinearly suspended beam: theoretical results and numerical observations*, J. Differential Equations 136 (1997), no. 2, 325-355.

[6] G. T., Dee, W. Van Saarloos; *Bistable systems with propagating fronts leading to pattern formation*, Phys Rev Lett, 1988, 60: 2641-2644.

[7] T. Gyulov and S. Tersian; *Existence of trivial and nontrivial solutions of a fourth-order ordinary diff erential equation*, Electronic Journal of Differential Equations, Vol. 2004(2004), No. 41, pp.1-14.

[8] L. A. Peletier, W. C. Troy; *Spatial Patterns: Higher Order Models in Physics and Mechanics*, Boston: Birkhaser, 2001.

[9] M. Schechter; *Linking Methods in Critical Point Theory*, Birkhser, Boston, 1999.

[10] M. Schechter; *Periodic non-autonomous second-order dynamical systems*, Journal of Differential Equations 223 (2006) 290 -302.

[11] D. Smets and J. B. Van den Berg; *Homoclinic solutions for Swift-Hohenberg andsuspension bridge type equations*, J. Differential Equations 184 (2002), no. 1, 78-96.

[12] J. B. Swift, P. C. Hohenberg; *Hydrodynamic fluctuations at the convective instability*, Phys. Rev.A, 1977, 15: 319-328.

[13] S. Tersian and J. Chaparova; *Periodic and Homoclinic Solutions of Extended Fisher-Kolmogorov Equations*, J. Math. Anal. Applications 260, 490-506 (2001).

[14] P. Rabinowitz; *Minimax methods in critical point theory with applications for differential equations*, CBMS Regional Conference, 1984.

Chengyue Li
Department of Mathematics, Minzu University of China, Beijing 100081, China
*E-mail address*: cunlcy@163.com

Changhua Shi
Department of Mathematics, Minzu University of China, Beijing 100081, China
*E-mail address*: shichanghua88888@163.com