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MONOTONICITY AND UNIQUENESS OF TRAVELING WAVES IN BISTABLE SYSTEMS WITH DELAY

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Abstract. This article establishes the monotonicity, uniqueness and Liapunov stability of traveling waves for bistable systems with delay. We use an elementary super-subsolution comparison method and a moving plane technique. Also an example is given to illustrate our results.

1. Introduction

In this article, we are concerned with the monotonicity, Liapunov stability and uniqueness of traveling wave solutions of the bistable reaction-diffusion systems with delay

$$\begin{aligned}\frac{\partial u_i(x,t)}{\partial t} &= D_i\frac{\partial^2 u_i(x,t)}{\partial x^2} + F^i\Big(u_1(x,t-\tau),\dots,u_{i-1}(x,t-\tau),u_i(x,t),\\ &\qquad u_{i+1}(x,t-\tau),\dots,u_n(x,t-\tau)\Big),\quad (x,t)\in\mathbb{R}\times(0,\infty),\ u_i\in\mathbb{R},\\ &\qquad u_i(x,s)=u_{0i}(x,s),\quad x\in\mathbb{R},\ s\in[-\tau,0].\end{aligned}\tag{1.1}$$

In the previous paper [23, 24], based on the assumption of monotone traveling waves, we established the globally exponential asymptotic stability of traveling waves of system (1.1) by using the squeezing technique developed by Chen [7]. Generally, the comparison principle can not be used if there is no monotonicity of traveling waves for the investigated systems. Thus, the monotonicity of traveling waves is important and necessary in this method. Here, we give the full detail proofs that the traveling waves of (1.1) are monotone, which could be as a kind of continuity for our work in [23]. Moreover, we also investigated Liapunov stability and uniqueness of traveling waves of system (1.1) by using an elementary super-subsolution comparison method and a moving plane technique.

It is well known that traveling wave solutions of reaction-diffusion systems have been applied to several subjects, such as ecology, chemistry, biology, and so on. See for example [7, 11, 12, 13, 15, 16, 17, 18, 20, 21, 24]. There are many works for the existence, uniqueness and stability of traveling waves in this field. For example, by using squeezing technique, Chen [7] established the global asymptotic exponential stability of traveling waves for nonlocal evolution equations, and then proved the

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uniqueness of traveling waves by a moving plane technique. He also established the existence of traveling waves of this problem based on the asymptotic behavior of solutions obtained from the stability, combining with the monotonicity and uniqueness of traveling waves. For the uniqueness of traveling waves, see [1, 2, 3, 4, 5, 6, 9, 27]. In the classical paper, Diekmann and Kaper [9] studied a particular monostable nonlocal model. Recently, Chen and Guo [5] obtained a complete uniqueness result for a generalized discrete version of the nonlocal monostable equations. The above uniqueness results are based on the method of a moving plane, see also [2]. Later, Carr and Chmaj [3] extended the method in [9]. In another paper, Chen and Guo [5] established the uniqueness result of traveling waves by constructing a Laplace transform representation of a solution and using the powerful Tauberian-Ikerhara's Theorem developed in [9, 10, 19]. For the delayed scalar equations, Schaff [13] studied the existence of monotone traveling wave solutions and uniqueness of wave speeds by a phase plane method. Smith and Zhao [15] established globally exponential stability and uniqueness of monotone traveling waves. For one-dimension systems, Volpert et al. [17, 18] obtained the existence of traveling waves by topological methods. Recently, Tsai [16] studied globally exponential stability of traveling waves in monotone bistable systems with partial diffusion coefficients being zero.

However, there are few results relatively about traveling waves of reaction-diffusion systems with delay, one can be referred to [12, 20, 21, 24, 25, 26]. Wu and Zou [21] obtained the existence of traveling waves in quasi-monotone and non-quasi-monotone reaction-diffusion systems with delay via the monotone iteraction method. Ou and Wu [12] established existence results of non-monotone traveling waves in monostable and bistable cases without quasi-monotonicity for a nonlocal reaction-diffusion system with delay. Recently, we established the globally exponential asymptotic stability results of traveling waves of the bistable system (1.1) with delay in [23]. Motivated by Chen [7], we studied the monotonicity, Liapunov stability and uniqueness of traveling waves for system (1.1) in this article.

The rest of this article is organized as follows. In Section 3, we give and prove the monotonicity result of traveling wave solutions of (1.1). And then based on the stability result of traveling wave solutions in [23], we establish and prove the Liapunov stability and uniqueness (up to translation) of traveling wave solutions in Section 4. Before doing these, we introduce some assumptions and notations in Section 2. Finally, as application, we give an example in the last Section.

2. Preliminaries

Our main results in this paper depend strongly on the construction of super-sub solutions of (1.1) and the comparison principle, we first state some assumptions and definitions of super-sub solutions of (1.1) as follows.

The following assumptions are made and the standard notation $\mathbb{R}^n$ is used, see [24].

- (H1) Positive diffusion coefficients: $D_i > 0$ for $i = 1, 2, \dots, n$;
- (H2) Monotone system: the reaction term

$$\mathbf{F}(\mathbf{u}) = (\mathbf{F}^1(\mathbf{u}), \dots, \mathbf{F}^n(\mathbf{u})),$$

is defined on a bounded domain $\Omega \subset \mathbb{R}^n$ and class C^1 in $\mathbf{u} = (u_1, u_2, \dots, u_n)$. We also require that $\mathbf{F}$ satisfy

$$\frac{\partial F^i}{\partial u_j}(\mathbf{u}) \geq 0 \quad \text{for } \mathbf{u} \in \Omega \text{ and } 1 \leq i \neq j \leq n;$$

(H3) Bistable nonlinearity: $\mathbf{F}$ has two stable equilibrium points $\mathbf{0} \ll \mathbf{1}$; i.e. $\mathbf{F}(\mathbf{0}) = \mathbf{F}(\mathbf{1}) = mathbf0$ and all the eigenvalues of $\mathbf{F}'(\mathbf{0})$ and $\mathbf{F}'(\mathbf{1})$ lie in the open left-half complex plane. We also assume that the matrixes $\mathbf{F}'(\mathbf{0})$ and $\mathbf{F}'(\mathbf{1})$ are irreducible.

By (H2)-(H3) and Perron-Frobenius Theorem (see [14, page61, Remark 3.1]), we know that there exists a small enough vector $\mathbf{d_0} > 0$ such that $F^i(-\mathbf{d_0}) \gg 0$ and $F^i(1 + \mathbf{d_0}) \ll 0$. Namely, $\mathbf{v}^+ = \mathbf{1} + \mathbf{d}_0$ and $\mathbf{v}^- = -\mathbf{d}_0$ are an ordered pair of super- and subsolutions of (1.1) on $[0, \infty)$. Here $-\mathbf{d}_0 = (-d_0, -d_0, \dots, -d_0)$ and $\mathbf{1} + \mathbf{d}_0 = (1 + d_0, 1 + d_0, \dots, 1 + d_0)$.

Let $C = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm, where $X = BUC(\mathbb{R}, \mathbb{R}^n)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$ with the the usual supremum norm. Let $X^+ = \{\mathbf{\Lambda} \in X; \mathbf{\Lambda}(x) \geq 0, x \in \mathbb{R}\}$ and $C^+ = \{\mathbf{\Lambda} \in C; \mathbf{\Lambda}(s) \in X^+, s \in [-\tau, 0]\}$. We can see that X^+ is a closed cone of X and C^+ is a positive cone of C. We can similarly define $X_0 = BUC(\mathbb{R}, \mathbb{R})$ and X_0^+. For convenience, we identify an element $\mathbf{\Lambda} \in C$ as a function from $\mathbb{R} \times [-\tau, 0]$ into $\mathbb{R}^n$ defined by $\mathbf{\Lambda}(x, s) = (\mathbf{\Lambda}(s))(x)$, and $\mathbf{\Lambda} = (\Lambda_i)_{i=1}^n$, $i = 1, 2, \dots, n$. For any continuous vector function $\mathbf{\Gamma}(\cdot) : [-\tau, b] \to X$, $b > 0$, we define $\mathbf{\Gamma}_t \in C, t \in [0, b)$ by $\mathbf{\Gamma}_t(s) = \mathbf{\Gamma}(t + s)$, $s \in [-\tau, 0]$. It is then easy to see that $t \mapsto \mathbf{\Gamma}_t$ is a continuous vector function from $[0, b)$ to C. For any $\mathbf{\Lambda} \in [-d_0, 1 + d_0]_C^n = \{\mathbf{\Lambda} \in C; \mathbf{\Lambda}_i(x, s) \in [-d_0, 1 + d_0], x \in \mathbb{R}, s \in [-\tau, 0], i = 1, 2, \dots, n\}$, define

$$\begin{aligned} f^i(\mathbf{\Lambda}(s))(x) = F^i\big(&\Lambda_1(x, -\tau), \dots, \Lambda_{i-1}(x, -\tau), \Lambda_i(x, 0), \\ &\Lambda_{i+1}(x, -\tau), \dots, \Lambda_n(x, -\tau)\big), \end{aligned}$$

where $x \in \mathbb{R}$; therefore,

$$\begin{aligned} f^i(\mathbf{\Lambda}_t(s))(x) = F^i\big(&\Lambda_1(x, t - \tau), \dots, \Lambda_{i-1}(x, t - \tau), \Lambda_i(x, t), \\ &\Lambda_{i+1}(x, t - \tau), \dots, \Lambda_n(x, t - \tau)\big), \end{aligned}$$

where

$$[-d_0, 1 + d_0]_C^n = \underbrace{[-d_0, 1 + d_0]_C \times [-d_0, 1 + d_0]_C \times \dots \times [-d_0, 1 + d_0]_C}_{n \text{ times}}.$$

By the global Lipschitz continuity of $F^i(\cdot)$ (because $\mathbf{F} \in C^1$ in $\mathbf{u}$) on $[-d_0, 1 + d_0]^n$, we can verify that $\mathbf{f}(\mathbf{\Lambda}) = (f^1(\mathbf{\Lambda}), \dots, f^n(\mathbf{\Lambda})) \in X$ and globally Lipschitz continuous.

We are interested in traveling wave solutions $\mathbf{U}(\cdot)$ of (1.1) connecting the two equilibria $\mathbf{0}$ and $\mathbf{1}$. More precisely, functions $\mathbf{U}(\xi) = (U_1(\xi), U_1(\xi), \dots, U_n(\xi)) \in C^2(\mathbb{R})$ are said to be a traveling wave solution of (1.1), if for some $c \in \mathbb{R}, \mathbf{u}(x, t) = \mathbf{U}(x - ct) = \mathbf{U}(\xi)$ is a solution of (1.1) with the property that

$$\mathbf{U}(-\infty) = \mathbf{0}, \quad \text{and} \quad U(+\infty) = \mathbf{1}. \tag{2.1}$$

Here c is the so-called wave speed associated with the profile of the traveling wave $\mathbf{U}$. Without generality, we always assume $c > 0$ throughout this paper. Therefore,

$\mathbf{U}(\xi)$ satisfies the following ordinary functional differential system

$$D_i\ddot{U}_i+c\dot{U}_i+F^i(U_1(\xi+c\tau),\dots,U_{i-1}(\xi+c\tau),U_i(\xi),U_{i+1}(\xi+c\tau),\dots,U_n(\xi+c\tau))=0, \tag{2.2}$$

where $\xi = x - ct \in \mathbb{R}$, for $i = 1, 2, \dots, n$, where "$\cdot$" denotes $\frac{d}{d\xi}$.

Definition 2.1. A continuous function $\mathbf{v} = (v_1, v_2, \dots, v_n) : [-\tau, b] \to X$, $b > 0$, is called a supersolution (subsolution) of (1.1) on $[0, b)$ if

$$v_i(t) \geq (\leq) T_i(t-s)v_i(s) + \int_s^t T_i(t-r)f^i(\mathbf{v}_r)dr \tag{2.3}$$

for $0 \leq s < t < b$ and $i = 1, 2 \dots, n$. If $\mathbf{v}$ is both a supersolution and a subsolution on $[0, b)$, then we call it a mild solution of (1.1).

We note that $\mathbf{T}(t) = (T_i(t))_1^n$ is a strongly continuous analytic semigroup on X generated by the X-realization $\mathbf{D}\Delta_X$ of $\mathbf{D}\Delta$ with the help of [8, Theorem 1.5]. Moreover, by the explicit expression of solutions of the heat equation

$$\frac{\partial u_i}{\partial t} = D_i \Delta u_i, \quad x \in \mathbb{R},\ t > 0,\ i = 1, 2, \dots, n,$$
$$u_i(x, 0) = u_{0,i}(x), \quad x \in \mathbb{R},$$

we have

$$T_i(t)u_{0,i}(x) = \frac{1}{\sqrt{4\pi D_i}} \int_{-\infty}^{+\infty} \exp\Big(-\frac{(x-y)^2}{4D_i t}\Big)u_{0,i}(y)dy,$$

for $x \in \mathbb{R}$, $t > 0$, $u_{0,i}(\cdot) \in BUC(\mathbb{R}, \mathbb{R})$.

We have another definition that is equivalent to the one above.

Definition 2.2. Assume that there is a $\mathbf{v} = (v_i)_1^n \in BUC(\mathbb{R} \times [-\tau, b), \mathbb{R}^n)$, $b > 0$ such that v_i is C^2 in $x \in \mathbb{R}, C^1$ in $t \in [0, b)$ for $i = 1, 2 \dots, n$, and v_i satisfies

$$\begin{aligned}\frac{\partial v_i}{\partial t} \geq (\leq) D_i \Delta v_i + F^i\big(&v_1(x, t-\tau), \dots, v_{i-1}(x, t-\tau), v_i(x, t), v_{i+1}(x, t-\tau),\\ &\dots, v_n(x, t-\tau)\big), \quad x \in \mathbb{R},\ t \in (0, b),\ i = 1, 2 \dots, n,\end{aligned} \tag{2.4}$$

and that $|\frac{\partial v_i}{\partial t} - D_i \Delta v_i - F^i(v_1(x, t-\tau), \dots, v_{i-1}(x, t-\tau), v_i(x, t), v_{i+1}(x, t-\tau), \dots,$ $v_n(x, t-\tau))|$ is bounded on $\mathbb{R} \times [0, b)$, and that $-d_0 \leq v_i(x, t) \leq 1 + d_0$ for $(x, t) \in \mathbb{R} \times [0, b)$, $i = 1, 2 \dots, n$. Then $\mathbf{v}$ is a supersolution (subsolution) of (1.1) on $[0, b)$.

By the positivity of the linear semigroup $\mathbf{T}(t) : X \to X$, it easily follows that (2.3) holds. Therefore, Definition 2.2 is equivalent to Definition 2.1.

Now, we give the comparison principle for (1.1), proved in [23, 24].

Theorem 2.3. *Assume* (H1)–(H3) *hold. Then for any* $\mathbf{u}_0 \in [-d_0, 1 + d_0]_C^n$*, Equation* (1.1) *has a unique mild solution* $\mathbf{u}(x, t, \mathbf{u}_0)$ *on* $[0, \infty)$ *and it is a classical solution to* (1.1) *for* $(x, t) \in \mathbb{R} \times (\tau, \infty)$*. Furthermore, for any pair of supersolution* $\mathbf{u}_1(x, t)$ *and subsolution* $\mathbf{u}_2(x, t)$ *of* (1.1) *on* $[0, \infty)$ *with* $\mathbf{u}_1(x, t), \mathbf{u}_2(x, t) \in [-d_0, 1 + d_0]^n, x \in \mathbb{R}, t \in [-\tau, \infty)$ *and* $\mathbf{u}_1(x, s) \geq \mathbf{u}_2(x, s), x \in \mathbb{R}, s \in [-\tau, 0]$*, then there holds* $\mathbf{u}_1(x, t) \geq \mathbf{u}_2(x, t), x \in \mathbb{R}, t \geq 0$*, where* $\mathbf{u}_1 = (u_{1,i})_{i=1}^n, \mathbf{u}_2 = (u_{2,i})_{i=1}^n$*. At the same time, there exists*

$$u_{1,i}(x, t) - u_{2,i}(x, t) \geq \theta_i(J, t - t_0) \int_z^{z+1} \big(u_{1,i}(y, t_0) - u_{2,i}(y, t_0)\big)dy$$

for any $J \geq 0, x \in \mathbb{R}, z \in \mathbb{R}$ with $|x-z| \leq J$ and $t > t_0 \geq 0$, where

$$\theta_i(J,t) = \frac{1}{\sqrt{4\pi D_i t}} \exp\Big\{-L_i t - \frac{(J+1)^2}{4D_i t}\Big\}, \quad J \geq 0,\ t > 0,$$
$$L_i = \max\{|\partial_j F^i(\mathbf{u})| : \mathbf{u} \in [-d_0, 1+d_0]^n\}, \quad i,j = 1,2,\dots,n.$$

Before giving some lemmas about the construction of super- and subsolution of (1.1), we need some preparations: First, we use the traveling wave solution $\mathbf{U}$ and a positive bounded vector function $\mathbf{p}$ to construct super- and subsolution of (1.1). By the hypotheses (H2)-(H3) and Perron-Frobenius Theorem, we know that the principal eigenvalues (the eigenvalue with the maximal real part) of $\mathbf{F}'(\mathbf{0})$ and $\mathbf{F}'(\mathbf{1})$ are negative and the corresponding eigenvectors are positive. Therefore, there exist irreducible constant matrixes $M^{\pm} = (\alpha_{ij}^{\pm})$ such that $\frac{\partial F^i}{\partial u_j}(\mathbf{0}) < \alpha_{ij}^-$ and $\frac{\partial F^i}{\partial u_j}(\mathbf{1}) < \alpha_{ij}^+$ for $i,j = 1,2,\dots,n$, and that the principal eigenvalues of $M^{\pm}$ are negative. Then we can choose positive vectors $\mathbf{p}^{\pm} = (p_1^{\pm}, p_2^{\pm}, \dots, p_n^{\pm})$ and $p_i^- < p_i^+$, $i = 1,2,\dots,n$, such that $\mathbf{p}^{\pm}$ are positive eigenvectors corresponding to the principal eigenvalues of $M^{\pm}$. Define

$$\nu(s) = \frac{1}{2}\Big(1 + \tanh\frac{s}{2}\Big)$$

and let the positive vector function $\mathbf{p}(\xi) = (p_1(\xi), p_2(\xi), \dots, p_n(\xi))$ defined by

$$p_i(\xi) = \nu(\xi)p_i^+ + (1-\nu(\xi))p_i^-, \quad i = 1,2,\dots,n.$$

It is easy to check $\mathbf{p}(\xi)$ satisfies the following conditions:

$$p_i(\cdot) \in \big[\min\{p_i^-, p_i^+\}, \max\{p_i^-, p_i^+\}\big] = [p_i^-, p_i^+] \quad \text{on } \mathbb{R},$$
$$\min_{1 \leq j \leq n} \inf_{\xi \in \mathbb{R}} p_j(\xi) > 0, \quad p_i'(\xi) > 0,\ \xi \in \mathbb{R},$$
$$p_i(\xi) \to p_i^{\pm} \text{ and } p_i'(\xi) \to 0 \quad \text{as } \xi \to \pm\infty \text{ for } i = 1,2,\dots,n.$$

Thus the needed pair of super- and sub-solution of (1.1) can be constructed in the following lemma (see [23, 24]).

Lemma 2.4. *There exist positive constants β_0, σ_0 and $\tilde{d_0} \in (0, \frac{1}{2})$ such that for any $\delta \in (0, \tilde{d_0}]$ and every $\xi_0 \in \mathbb{R}$, the following functions $\mathbf{w}^{\pm}$ defined by*

$$\mathbf{w}^{\pm}(x,t) = \mathbf{U}(x - ct + \xi_0 \pm \sigma_0\delta(1 - e^{-\beta_0 t})) \pm \delta\mathbf{p}(x-ct)e^{-\beta_0 t} \tag{2.5}$$

are a super-solution and a sub-solution respectively of (1.1), here

$$\mathbf{w}^{\pm} = (w_1^{\pm}, w_2^{\pm}, \dots, w_1^{\pm}).$$

In the sequel, we still keep the notation $\beta_0, \sigma_0, \tilde{d_0}$ and $\mathbf{p}(\cdot)$.

3. Monotonicity of traveling waves

In this section, we establish the monotonicity of a traveling wave solution $\mathbf{U}(\cdot)$ of (1.1). We show that $\mathbf{U}(\cdot)$ is strictly monotone in the following theorem.

Theorem 3.1. *If $\mathbf{u}(x,t) = \mathbf{U}(x-ct)$ is a traveling wave solution of (1.1) satisfying $\lim_{x\to+\infty}\mathbf{U}(x) = \mathbf{1}$ and $\lim_{x\to-\infty}\mathbf{U}(x) = \mathbf{0}$ with $\mathbf{0} \leq \mathbf{U}(\cdot) \leq \mathbf{1}$, then $\mathbf{U}$ is strictly increasing and $\mathbf{U}'(x) > \mathbf{0}$ for almost all $x \in \mathbb{R}$.*

Proof. As mentioned before, let

$$p_{\min} = \min\{p_i^-, p_i^+\}, \quad \mathbf{p}^- = \big(p_1^-, \dots, p_i^-, \dots, p_n^-\big), \quad i = 1, 2, \dots, n,$$
$$p_{\max} = \max\{p_i^-, p_i^+\}, \quad \mathbf{p}^+ = \big(p_1^+, \dots, p_i^+, \dots, p_n^+\big), \quad i = 1, 2, \dots, n,$$

and let β_0 be a positive constant satisfying

$$\beta_0 \leq \frac{3}{4}\gamma e^{\beta_0 \tau}, \tag{3.1}$$

where $\gamma > 0$. Let $\zeta_0(x)$ be a smooth function such that

$$\zeta_0(x) = 0 \text{ for } x \leq -2, \quad \zeta_0(x) = 1 \text{ for } x \geq 2,$$
$$0 \leq \zeta_0'(x) \leq 1 \text{ and } |\zeta_0''(x)| \leq 1 \text{ for all } x \in \mathbb{R}.$$

Define

$$b_i(x,t) = [\big(1 - \zeta_0(x)\big)p_i^- + \zeta_0(x)p_i^+]e^{-\beta_0 t}. \tag{3.2}$$

Then, we divide the proof into three steps.

Step 1. It is obvious that $\mathbf{1}$ and $\mathbf{0}$ are super-solution and sub-solution of (1.1), respectively. By the comparison principle, we have

$$\mathbf{0} < \mathbf{U}(x) < \mathbf{1} \tag{3.3}$$

for all $x \in \mathbb{R}$.

Step 2. We claim that, for some z_* large enough,

$$\mathbf{0} < \mathbf{U}(x - z) < \mathbf{U}(x) \tag{3.4}$$

for all $x \in \mathbb{R}$ and $z \in \mathbb{R}$ with $z \geq z_*$.

Let b_i be defined as in (3.2) and let

$$v_i^\alpha(x, t - \tau) = U_i(x + c\tau) + \alpha b_i(x, t - \tau), \quad i = 1, 2, \dots, n, \tag{3.5}$$

where the constant τ is the delay in (1.1). If $\tau = 0$ in (3.5), then

$$v_i^\alpha(x,t) = U_i(x) + \alpha b_i(x,t), \quad i = 1, 2, \dots, n,$$

where $U_i(\cdot)$ is the traveling wave solutions of (2.2). Hence $v_{it}^\alpha = \alpha b_{it}$, $v_{ix}^\alpha = U_{ix}(x) + \alpha b_{ix}$, $v_{ixx}^\alpha = U_{ixx}(x) + \alpha b_{ixx}$.

Now, we claim that there exists $\xi_* \gg 1$ and $\alpha_* > 0$ such that, for all $0 < \alpha < \alpha_*$,

$$Lv_i^\alpha(x,t) = v_{it}^\alpha - D_i v_{ixx}^\alpha - c v_{ix}^\alpha - F^i\big(v_1^\alpha(x, t-\tau), \dots, v_i^\alpha(x,t), \dots, v_n^\alpha(x, t-\tau)\big) \geq 0 \tag{3.6}$$

for all $x \in \mathbb{R}$ with $|x| > \xi_*$ and all $t \in \mathbb{R}^+$. In fact,

$$\begin{aligned} Lv_i^\alpha(x,t) = {} & F^i\big(U_1(x + c\tau), \dots, U_i(x), \dots, U_n(x + c\tau)\big) \\ & - F^i\big(v_1^\alpha(x, t-\tau), \dots, v_i^\alpha(x,t), \dots, v_n^\alpha(x, t-\tau)\big) \\ & + \alpha(b_{it} - D_i b_{ixx} - c b_{ix}). \end{aligned} \tag{3.7}$$

Since $\lim_{x \to +\infty} \mathbf{U}(x) = \mathbf{1}$ and $\lim_{x \to -\infty} \mathbf{U}(x) = \mathbf{0}$, we have

$$\begin{aligned} & F^i\big(U_1(x + c\tau), \dots, U_i(x), \dots, U_n(x + c\tau)\big) - F^i\big(v_1^\alpha(x, t-\tau), \dots, \\ & v_i^\alpha(x,t), \dots, v_n^\alpha(x, t-\tau)\big) \\ & = F^i\big(U_1(x + c\tau), \dots, U_i(x), \dots, U_n(x + c\tau)\big) - F^i\Big(U_1(x + c\tau) + \alpha b_1(x, t-\tau), \dots, \\ & \quad U_i(x) + \alpha b_i(x,t), \dots, U_n(x + c\tau) + \alpha b_n(x, t-\tau)\Big) \end{aligned}$$

$$
\begin{aligned}
&= -\alpha \sum_{1\le i\ne j\le n} \frac{\partial F^i}{\partial u_j}\big(U_1(x+c\tau)+\theta_1\alpha b_1(x,t-\tau),\dots,U_i(x)+\theta_i\alpha b_i(x,t),\dots,\\
&\quad U_n(x+c\tau)+\theta_n\alpha b_n(x,t-\tau)\big)\cdot b_j(x,t-\tau)\\
&\quad -\alpha\frac{\partial F^i}{\partial u_i}\big(U_1(x+c\tau)+\theta_1\alpha b_1(x,t-\tau),\dots,U_i(x)+\theta_i\alpha b_i(x,t),\dots,\\
&\quad U_n(x+c\tau)+\theta_n\alpha b_n(x,t-\tau)\big)\cdot b_i(x,t)\\
&\ge -\alpha \sum_{1\le i\ne j\le n} \alpha^{\pm}_{ij} b_j(x,t-\tau) - \alpha\alpha^{\pm}_{ii} b_i(x,t)\\
&\ge -\alpha \sum_{1\le i\ne j\le n} \alpha^{\pm}_{ij} b_j(x,t-\tau) - \alpha\alpha^{\pm}_{ii} b_i(x,t-\tau)\\
&\quad \big(\text{because } \alpha^{\pm}_{ii}\ge 0 \text{ and } b_i(x,t)\le b_i(x,t-\tau)\big)\\
&= -\alpha\sum_{j=1}^n \alpha^{\pm}_{ij} b_j(x,t-\tau)\\
&\ge \alpha\gamma b_i(x,t-\tau)\\
&= \alpha\gamma[\big(1-\zeta_0(x)\big)p_i^- + \zeta_0(x)p_i^+]e^{-\beta_0(t-\tau)} \to \alpha\gamma e^{\beta_0\tau} p_i^+ e^{-\beta_0 t}
\end{aligned}
$$

uniformly in t as $x\to\infty$ and $\alpha\to 0$. Therefore, there exist $\xi_1>2$ and $\alpha_0>0$ such that

$$
\begin{aligned}
&F^i\big(U_1(x+c\tau),\dots,U_i(x),\dots,U_n(x+c\tau)\big)\\
&-F^i\big(v_1^\alpha(x,t-\tau),\dots,v_i^\alpha(x,t),\dots,v_n^\alpha(x,t-\tau)\big)\\
&>\frac{3}{4}\alpha\gamma e^{\beta_0\tau}p_i^+e^{-\beta_0 t}
\end{aligned}
\tag{3.8}
$$

for all $t\in\mathbb{R}^+$, $x\in\mathbb{R}$ with $x>\xi_1$ and $0<\alpha<\alpha_0$.

For $x>2$, $b_{ix}=0$ and $b_{ixx}=0$, therefore, from (3.7)-(3.8),

$$
\begin{aligned}
&Lv_i^\alpha(x,t)\\
&= F^i\big(U_1(x+c\tau),\dots,U_i(x),\dots,U_n(x+c\tau)\big)\\
&\quad -F^i\big(v_1^\alpha(x,t-\tau),\dots,v_i^\alpha(x,t),\dots,v_n^\alpha(x,t-\tau)\big)+\alpha(b_{it}-D_ib_{ixx}-cb_{ix})\\
&> \frac{3}{4}\alpha\gamma e^{\beta_0\tau}p_i^+e^{-\beta_0 t}+\alpha(-\beta_0)e^{-\beta_0 t}p_i^+\\
&= \alpha e^{-\beta_0 t}p_i^+\big(\frac{3}{4}\gamma e^{\beta_0\tau}-\beta_0\big)\ge 0
\end{aligned}
$$

(by (3.1)). Choose $\alpha_*=\alpha_0$ and $\xi_*=\xi_1$, then, when $0<\alpha<\alpha_*$ and $x>\xi_*$, we have

$$
Lv_i^\alpha(x,t)\ge 0.
$$

This proves the claim (3.6) for x near $+\infty$. Similarly we can prove the claim for x near $-\infty$.

We assume that (3.6) holds. Since $\lim_{x\to+\infty}\mathbf{U}(x)=\mathbf{1}$ and $\lim_{x\to-\infty}\mathbf{U}(x)=\mathbf{0}$, there exists $z_*>0$ such that

$$
\mathbf{U}(x-z)\le\begin{cases}\mathbf{U}(x), & x\in[-\xi_*,\xi_*]\\ \mathbf{U}(x)+\alpha_*\mathbf{p}^-, & x\bar{\in}[-\xi_*,\xi_*]\end{cases}
$$

for all $z \in \mathbb{R}$ with $z \geq z_*$, where $\mathbf{p}^- = \big(p_1^-, \dots, p_i^-, \dots, p_n^-\big)$, $i = 1, 2, \dots, n$. We claim that (3.4) holds with this choice of z_*. In fact, $v_i^{\alpha_*}(x,t)$ in (3.5) satisfies

$$v_i^{\alpha_*}(x,t) \geq U_i(x) \geq U_i(x-z)$$

for all $x \in [-\xi_*, \xi_*]$, $t \in \mathbb{R}$ and

$$v_i^{\alpha_*}(x,0) \geq U_i(x) + \alpha_* p_i^- \geq U_i(x-z)$$

for all $x \in \mathbb{R}$. Applying the comparison principle to $v_i^{\alpha_*}(x,t) - U_i(x-z)$, we have

$$U_i(x-z) \leq U_i(x) + \alpha_* b_i(x,t) \tag{3.9}$$

for all $x \in \mathbb{R}, t \in \mathbb{R}$. Let $t \to +\infty$, we get (3.4).

Step 3. We prove that (3.4) holds for all $z \geq 0$. Let

$$z_0 = \inf\{\tilde{z} \geq 0 : U_i(x-z) \leq U_i(x),\ \forall z \geq \tilde{z},\ x \in \mathbb{R}\}, \tag{3.10}$$

we prove that

$$U_i(x-z_0) = U_i(x) \tag{3.11}$$

for all $x \in \mathbb{R}$. Otherwise, by the comparison principle we have

$$U_i(x-z_0) < U_i(x) \tag{3.12}$$

for all $x \in \mathbb{R}$. There exists ε_* such that

$$\mathbf{U}(x-z) \leq \begin{cases} \mathbf{U}(x), & \text{for } x \in [-\xi_*, \xi_*] \\ \mathbf{U}(x) + \alpha_* \mathbf{p}^-, & \text{for } x \bar{\in} [-\xi_*, \xi_*] \end{cases}$$

for all $z \in \mathbb{R}$ and $z \geq z_0 - \varepsilon_*$. By a similar argument to that in **Step 2**, one can show that (3.4) holds for all $z \geq z_0 - \varepsilon_*$, which contradicts the choice of z_0.

Since $\lim_{x\to+\infty} \mathbf{U}(x) = \mathbf{1}$ and $\lim_{x\to-\infty} \mathbf{U}(x) = \mathbf{0}$, we deduce from (3.11) that $z_0 = 0$. Therefore, (3.4) holds for all $z \geq 0$. Hence $\mathbf{U}'(x) \geq \mathbf{0}$ for almost all $x \in \mathbb{R}$. By the comparison principle, we have $\mathbf{U}'(x) > \mathbf{0}$ for almost all $x \in \mathbb{R}$. □

4. Liapunov stability and uniqueness of traveling waves

In this section, based on the monotonicity result obtained in Section 3 and on the globally asymptotic exponential stability with phase shift of monotone traveling wave solutions of (1.1) in [24], we establish the Liapunov stability and uniqueness up to translation of traveling wave solutions combining suitably constructed super-sub solutions comparison and a moving plane method.

Theorem 4.1 ([24]). *Suppose* (H1)–(H3) *hold and* (1.1) *has a monotone traveling wave solution* $\mathbf{U}(x-ct) = (U_1(x-ct), U_2(x-ct), \dots, U_n(x-ct))$. *Let* $\mathbf{u} = (u_1, u_2, \dots, u_n)$ *be the solution of* (1.1) *with the initial data* $\mathbf{u}(x,s) = \mathbf{u}_0(x,s)$, $x \in \mathbb{R}$, $s \in [-\tau, 0]$. *For any* $u_{0,i} \in [0,1]_C$, $i = 1, 2, \dots, n$ *and* $\mathbf{u}_0 = (u_{0,i})_{i=1}^n$, *if*

$$\varepsilon(\mathbf{u}_0(x,s)) := \sup\Big\{\limsup_{x\to-\infty} \|\mathbf{u}_0(x,s) - \mathbf{0}\|, \limsup_{x\to+\infty} \|\mathbf{u}_0(x,s) - \mathbf{1}\|\Big\}, \quad s \in [-\tau, 0]$$

is small enough, then $\mathbf{U}(x-ct)$ *is globally exponential stable with phase in the sense that there exists a positive constant* $k > 0$ *such that the solution* $\mathbf{u}(x,t,\mathbf{u}_0)$ *of* (1.1) *satisfies*

$$|u_i(x,t,\mathbf{u}_0) - U_i(x-ct+\xi)| \leq Ke^{-kt}, \quad x \in \mathbb{R},\ t \geq 0,$$

for some $K = K(\mathbf{u}_0)$ *and* $\xi = \xi(\mathbf{u}_0)$.

As a direct consequence of Theorem 4.1, we have the other main result in this paper.

Theorem 4.2. *Every monotone traveling wave solution of* (1.1) *is Liapunov stable. If* (1.1) *has a monotone traveling wave solution* $\mathbf{U}(x-ct) = (U_1(x-ct), U_2(x-ct), \dots, U_n(x-ct))$*, then the traveling wave solutions of* (1.1) *are unique up to a translation in the sense that for any traveling wave solution* $\bar{\mathbf{U}}(x-\bar{c}t) = (\bar{U}_1(x-\bar{c}t), \bar{U}_2(x-\bar{c}t), \dots, \bar{U}_n(x-\bar{c}t))$*, with* $0 \le \bar{U}_i(\xi) \le 1, \xi \in \mathbb{R}, i = 1, 2, \dots, n$*, we have* $\bar{c} = c$ *and* $\bar{U}_i(\cdot) = U_i(\xi_0 + \cdot)$ *for some* $\xi_0 = \xi_0(\bar{\mathbf{U}}) \in \mathbb{R}, i = 1, 2, \dots, n$.

Proof. Let $\mathbf{U}(x-ct) = (U_1(x-ct), U_2(x-ct), \dots, U_n(x-ct))$ be a monotone traveling wave solutions of (1.1). By the uniform continuity of $U_i(\cdot)$ on $\mathbb{R}$, it follows that for any $\varepsilon > 0$, there exists a $\delta_3 = \delta_3(\varepsilon) > 0$ such that for all $|y| \le \delta_3$, there is

$$|U_i(x-ct+y) - U_i(x-ct)| < \frac{\varepsilon}{2}, \quad x \in \mathbb{R},\ t \ge 0. \tag{4.1}$$

We then choose a $\delta = \delta(\varepsilon) > 0$ such that $\delta < \min\{\frac{\varepsilon}{2p_i^+}, \frac{\delta_3 e^{-\beta_0\tau}}{\sigma_0}, \tilde{d}_0\}$, where β_0, σ_0 and $\tilde{d}_0$ are as in Lemma 2.4. For any $\mathbf{u}_0 \in C([-\tau, 0], X)$ with $|u_{0,i}(x,s) - U_i(x-cs)| < \delta$ for $s \in [-\tau, 0]$ and $x \in \mathbb{R}$, $i = 1, 2, \dots, n$, we have

$$\begin{aligned} &U_i(x-cs+\sigma_0\delta(1-e^{\beta_0\tau}) - \sigma_0\delta(1-e^{-\beta_0 s})) - \delta p_i(x-cs)e^{-\beta_0 s} \\ &\le u_{0,i}(x,s) \\ &\le U_i(x-cs+\sigma_0\delta(e^{\beta_0\tau}-1) + \sigma_0\delta(1-e^{-\beta_0 s})) + \delta p_i(x-cs)e^{-\beta_0 s} \end{aligned} \tag{4.2}$$

By Lemma 2.4 and Theorem 2.3, it follows that

$$\begin{aligned} &U_i(x-ct+\sigma_0\delta(1-e^{\beta_0\tau}) - \sigma_0\delta(1-e^{-\beta_0 t})) - \delta p_i(x-ct)e^{-\beta_0 t} \\ &\le u_i(x,t,\mathbf{u}_0) \\ &\le U_i(x-ct+\sigma_0\delta(e^{\beta_0\tau}-1) + \sigma_0\delta(1-e^{-\beta_0 t})) + \delta p_i(x-ct)e^{-\beta_0 t} \end{aligned}$$

for $x \in \mathbb{R}$, $t \ge 0$, $i = 1, 2, \dots, n$. By the fact that $p_i(\cdot) \in [p_i^-, p_i^+]$ on $\mathbb{R}$, we have

$$\begin{aligned} &U_i(x-ct+\sigma_0\delta(1-e^{\beta_0\tau}) - \sigma_0\delta(1-e^{-\beta_0 t})) - \delta p_i^+ e^{-\beta_0 t} \\ &\le u_i(x,t,\mathbf{u}_0) \\ &\le U_i(x-ct+\sigma_0\delta(e^{\beta_0\tau}-1) + \sigma_0\delta(1-e^{-\beta_0 t})) + \delta p_i^+ e^{-\beta_0 t}. \end{aligned} \tag{4.3}$$

By the choice of $\delta = \delta(\varepsilon)$, we have that for all $t \ge 0$,

$$\begin{aligned} |\sigma_0\delta(1-e^{\beta_0\tau}) - \sigma_0\delta(1-e^{-\beta_0 t})| &\le \sigma_0\delta(e^{\beta_0\tau}-1) + \sigma_0\delta(1-e^{-\beta_0 t}) \\ &\le \sigma_0\delta e^{\beta_0\tau} < \delta_3(\varepsilon), \end{aligned}$$

and

$$\begin{aligned} |\sigma_0\delta(e^{\beta_0\tau}-1) + \sigma_0\delta(1-e^{-\beta_0 t})| &\le \sigma_0\delta(e^{\beta_0\tau}-1) + \sigma_0\delta(1-e^{-\beta_0 t}) \\ &\le \sigma_0\delta e^{\beta_0\tau} < \delta_3(\varepsilon). \end{aligned}$$

Then by (4.1) and (4.3), it follows that $U_i(x-ct) - \varepsilon \le u_i(x,t,\mathbf{u}_0) \le U_i(x-ct) + \varepsilon$, for $x \in \mathbb{R}$, $t \ge 0$, $i = 1, 2, \dots, n$. That is to say, $|u_i(x,t,\mathbf{u}_0) - U_i(x-ct)| < \varepsilon$, for $x \in \mathbb{R}$, $t \ge 0$, $i = 1, 2, \dots, n$. Therefore, $U_i(x-ct)$, $i = 1, 2, \dots, n$. is Liapunov stable; i.e. $\mathbf{U}(x-ct)$ is Liapunov stable.

We are ready to prove the uniqueness of traveling wave solutions. Let $\mathbf{U}(x-ct) = (U_1(x-ct), U_2(x-ct), \dots, U_n(x-ct))$, be the given monotone traveling wave solution of (1.1), and let $\bar{\mathbf{U}}(x-\bar{c}t) = (\bar{U}_1(x-\bar{c}t), \bar{U}_2(x-\bar{c}t), \dots, \bar{U}_n(x-\bar{c}t))$, with $0 \le \bar{U}_i(\xi) \le 1$, $\xi \in \mathbb{R}$, $i = 1, 2, \dots, n$, be any traveling wave solution of

(1.1) with $0 \le \bar{U}_i \le 1$ on $\mathbb{R}$, $i = 1, 2, \dots, n$. Since $\lim_{x\to\infty} \bar{U}_i(x - \bar{c}s) = 1$ and $\lim_{x\to-\infty} \bar{U}_i(x - \bar{c}s) = 0$ uniformly for $s \in [-\tau, 0]$, $i = 1, 2, \dots, n$, thus there exists

$$\varepsilon(\bar{U}_i(x, s)) := \sup\Big\{\limsup_{x\to-\infty} \|\bar{U}_i(x - \bar{c}s) - 0\|, \limsup_{x\to+\infty} \|\bar{U}_i(x - \bar{c}s) - 1\|\Big\}, \quad s \in [-\tau, 0] \tag{4.4}$$

is small enough. Then, by Theorem 4.1, there exist $\bar{K} = \bar{K}(\bar{\mathbf{U}}) > 0$, and $\xi_0 = \xi_0(\bar{\mathbf{U}}) \in \mathbb{R}$ such that

$$|\bar{U}_i(x - \bar{c}t) - U_i(x - ct + \xi_0)| \le \bar{K}e^{-kt}, \quad x \in \mathbb{R},\ t \ge 0,\ i = 1, 2, \dots, n. \tag{4.5}$$

Let $\bar{\xi} \in \mathbb{R}$ such that $0 < \bar{U}_i(\bar{\xi}) < 1$, and define $L(\bar{\xi}) := \{(x, t)|x \in \mathbb{R},\ t \ge 0,\ x - \bar{c}t = \bar{\xi}\}$. Then, by (4.5), we have

$$U_i(\bar{\xi} + \xi_0 + (\bar{c} - c)t) - \bar{K}e^{-kt} \le \bar{U}_i(\bar{\xi}) \le U_i(\bar{\xi} + \xi_0 + (\bar{c} - c)t) + \bar{K}e^{-kt}, \tag{4.6}$$

for all $(x, t) \in L(\bar{\xi})$, $i = 1, 2, \dots, n$. Since $U_i(+\infty) = 1$ and $U_i(-\infty) = 0$, $i = 1, 2, \dots, n$, letting $t \to \infty$ in (4.6), we obtain that $\bar{c} \le c$ from the left inequality and that $\bar{c} \ge c$ from the right inequality. Therefore $\bar{c} = c$. For any $\xi \in \mathbb{R}$, again by (4.5), we then have

$$|\bar{U}_i(\xi) - U_i(\xi + \xi_0)| \le \bar{K}e^{-kt} \tag{4.7}$$

for all $(x, t) \in L(\bar{\xi})$, $i = 1, 2, \dots, n$. Naturally, letting $t \to \infty$ in (4.7), we get $\bar{U}_i(\xi) = U_i(\xi + \xi_0)$ for all $\xi \in \mathbb{R}$, $i = 1, 2, \dots, n$. That is $\bar{U}_i(\cdot) = U_i(\xi_0 + \cdot)$, $i = 1, 2, \dots, n$. Therefore, the traveling wave solutions of (1.1) are unique up to a translation. This completes the proof. □

5. Applications

As an application, we consider the epidemic model with delay

$$\begin{aligned}
\frac{\partial}{\partial t} u_1(t, x) &= d\frac{\partial^2}{\partial x^2} u_1(t, x) - a_{11}u_1(t, x) + a_{12}u_2(t - \tau, x),\\
\frac{\partial}{\partial t} u_2(t, x) &= \tilde{d}\frac{\partial^2}{\partial x^2} u_2(t, x) - a_{22}u_2(t, x) + g(u_1(t - \tau, x)),
\end{aligned} \tag{5.1}$$

where g satisfies the following conditions:

- (A1) $g \in C^2(I)$, where I is an open interval in $\mathbb{R}$. $g(0) = 0$, $g'(0) \ge 0$, $g'(z) > 0$, for all $z > 0$, $\lim_{z\to\infty} g(z) = 1$, and there exists a $\varsigma > 0$ such that $g''(z) > 0$ for $z \in (0, \varsigma)$ and $g''(z) < 0$ for $z > \varsigma$.
- (A2) $g'(0) < \gamma_1 = \frac{a_{11}a_{22}}{a_{12}} < \gamma_1^*$, where the equation $g(z) = \gamma_1 z$ has one and only one root when $\gamma_1 = \gamma_1^*$.

Obviously, system (5.1) has three non-negative equilibria $E^- = (0, 0)$, $E^0 = (a, \frac{a_{11}a}{a_{12}})$ and $E^+ = (e_1^+, e_2^+) = (b, \frac{a_{11}b}{a_{12}})$, ($E^+$ may not be the point $(1, 1)$), where a and b satisfy $0 < a < b$ are the two positive roots of the equation $g(x) = \frac{a_{11}a_{22}}{a_{12}}x$. In this case, E^0 is a saddle point, E^- and E^+ are both stable nodes. Therefore we investigate the bistable waves, see [22].

Moreover, it is easy to verify assumptions (H1)–(H3) hold. As a result, we obtain the following theorems.

Theorem 5.1 (Stability). *Suppose* (A1)–(A2) *hold and* (5.1) *has a monotone traveling wave solution* $\mathbf{U}(x - ct) = (U_1(x - ct), U_2(x - ct))$. *Let* $\mathbf{u} = (u_1, u_2)$ *be the* *solution of* (5.1) *with the initial data* $\mathbf{u}(x,s) = \mathbf{u}_0(x,s)$, $x \in \mathbb{R}$, $s \in [-\tau, 0]$. *For any* $u_{0,i} \in [0,1]_C$, $i = 1,2$ *and* $u_0 = (u_{0,i})_{i=1}^n$, *if*

$$\varepsilon(\mathbf{u}_0(x,s)) := \sup\big\{\limsup_{x\to-\infty} \|\mathbf{u}_0(x,s) - E^-\|, \limsup_{x\to+\infty} \|\mathbf{u}_0(x,s) - E^+\|\big\}, \quad s \in [-\tau, 0]$$

is small enough, then $\mathbf{U}(x-ct)$ *is globally exponential stable in the sense that there exists a positive constant* $k > 0$ *such that the solution* $\mathbf{u}(x,t,\mathbf{u}_0)$ *of* (5.1) *satisfies*

$$|u_i(x,t,\mathbf{u}_0) - U_i(x - ct + \xi)| \le Ke^{-kt}, \quad x \in \mathbb{R},\ t \ge 0,\ i = 1,2$$

for some $K = K(\mathbf{u}_0)$ *and* $\xi = \xi(\mathbf{u}_0)$.

Theorem 5.2 (Monotonicity). *If* $\mathbf{u}(x,t) = \mathbf{U}(x-ct) = (U_1(\xi), U_2(\xi))$ *is a traveling wave solution of* (5.1) *satisfying* $\lim_{\xi\to+\infty} \mathbf{U}(\xi) = E^+$ *and* $\lim_{\xi\to-\infty} \mathbf{U}(\xi) = E^-$ *with* $E^- \le \mathbf{U}(\cdot) \le E^+$, *then* $\mathbf{U}$ *is strictly increasing and* $\mathbf{U}'(\xi) > 0$ *for almost all* $\xi \in \mathbb{R}$.

Next we have Liapunov stability and uniqueness.

Theorem 5.3. *Every monotone traveling wave solution of* (5.1) *is Liapunov stable. If* (5.1) *has a monotone traveling wave solution* $\mathbf{U}(x-ct) = (U_1(x-ct), U_2(x-ct))$, *then the traveling wave solutions of* (1.1) *are unique up to a translation in the sense that for any traveling wave solution* $\bar{\mathbf{U}}(x-\bar{c}t) = (\bar{U}_1(x-\bar{c}t), \bar{U}_2(x-\bar{c}t))$, *with* $0 \le \bar{U}_i(\xi) \le 1, \xi \in \mathbb{R}$, $i = 1,2$, *we have* $\bar{c} = c$ *and* $\bar{U}_i(\cdot) = U_i(\xi_0 + \cdot)$ *for some* $\xi_0 = \xi_0(\bar{\mathbf{U}}) \in \mathbb{R}$, $i = 1,2$.

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