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LIE GROUP CLASSIFICATION AND EXACT SOLUTIONS OF THE GENERALIZED KOMPANEETS EQUATIONS

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Abstract. We study generalized Kompaneets equations (GKEs) with one functional parameter, and using the Lie-Ovsiannikov algorithm, we carried out the group classification. It is shown that the kernel algebra of the full groups of the GKEs is the one-dimensional Lie algebra. Using the direct method, we find the equivalence group. We obtain six non-equivalent (up to transformations from the equivalence group) GKEs that allow wider invariance algebras than the kernel one. We find a number of exact solutions of the non-linear GKE which has the maximal symmetry properties.

1. Introduction

In this article, we study the generalized Kompaneets equations (GKEs)

$$u_t = \frac{1}{x^2}[x^4(u_x + f(u))]_x, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{1.1}$$

where $u = u(t,x)$, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$; $f(u)$ is an arbitrary smooth function of the variable u.

Equation (1.1) with $f(u) = u^2 + u$, namely,

$$u_t = \frac{1}{x^2} \cdot [x^4(u_x + u^2 + u)]_x, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{1.2}$$

was obtained in 1950 by Kompaneets [19] (see also [35]). It describes the scattering of unpolarized, low energy photons on a dilute distribution of non-relativistic electrons when all the particles (both photons and electrons) are distributed isotropically in their momenta. Its possible applications (mainly, astrophysical) were investigated in detail in [9, 16, 35, 36] et al. It should also be noted that in the previous few decades much attention has been devoted to generalizing (1.2), for instance, on the relativistic case, etc. (see [4, 5, 11, 17, 27] and papers cited therein).

If $u \gg 1$ then one can put $f(u) = u^2$ (now induced scattering is only considered). The corresponding equation of the form (1.2) was studied, e.g., in [35]. If $u \ll 1$ then we get the linear equation with $f(u) = u$. The general solution of this equation was obtained by Kompaneets [19] using the Green function, whose properties have been

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investigated in [22, 23]. The Green function for the linear Kompaneets equation with $f(u) = 0$ was obtained in [36].

For the nonlinear Kompaneets equation (1.2), classical methods for solving linear partial differential equations (PDEs) such as the Green functions, the separation of variables, the integral transforms, are not available. Therefore, the construction of exact analytical solutions of the nonlinear Kompaneets equation (1.2) is an actual task of modern mathematical physics.

One of the most powerful methods for constructing exact solutions of nonlinear PDEs is the classical Lie method [2, 24, 26], and its various generalizations and modifications (see, e.g., [12]).

Group analysis of the Kompaneets equation (1.2) was held recently in [14]. It was shown that the maximal algebra of invariance (MAI) of this equation is the one-dimensional algebra $\langle \partial_t \rangle$, i.e. equation (1.2) only allows the one-parameter time translation group. This symmetry leads only to the well-known stationary solution found by Kompaneets [19].

At the same time, it has been shown [14] that for various limiting cases (e.g., for the prevailing induced scattering or the degenerate limiting case $u^2 \gg u$, $u^2 \gg u_x$) the corresponding equations of the form (1.1) allow extensions of the symmetry properties. This allow us to construct a series of new exact solutions, which were not known before.

We note also the recent paper [3], where the analysis of the nonlinear Kompaneets equation (1.2) in the case of prevailing induced scattering was held by using the Bluman–Cole method [1] (see also [12]) and a number of new exact solutions of this equation were built.

It is noteworthy that the class of equations (1.1) is the particular case of the class of GKEs of the form

$$u_t = \frac{1}{\beta(x)}[\alpha(x)(u_x + f(x,u))]_x, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

which was investigated by Wang et al. [33, 34, 10], using non-group-theoretical methods.

The results in [3, 14] indicate that in the class of GKEs (1.1) there are equations with nontrivial symmetry properties. This enables us to build exact analytical solutions of these equations using the method of symmetry reduction. So, it is naturally arised the problem of classification of symmetry properties of the differential equations of the form (1.1), i.e. the problem of group classification of the class of GKEs (1.1). It can be formulated as follows: find the kernel $\mathfrak{g}^{\cap}$ of the MAIs of equations from class (1.1), i.e. the MAIs of equation (1.1) with an *arbitrary* function $f(u)$, and describe *all non-equivalent* equations that admit invariance algebras of dimension, higher than $\mathfrak{g}^{\cap}$.

Hereafter, we are going to work with the class of equations (1.1), written in the form

$$u_t = x^2 u_{xx} + x[xf'(u) + 4]u_x + 4xf(u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{1.3}$$

Taking into account the physical meaning of the function u, we assume that $u > 0$ in (1.3).

The purpose of this article is to carry out the group classification of the class of GKEs (1.3), and to build the exact invariant solutions of the equations admitting the highest symmetry properties.

The structure of this article is as follows. In Section 2, using the direct method, we find the complete group of equivalence transformations of class (1.3), up to which we carried out the group classification of one. In Section 3, using the Lie method, we get the system of the determining equations for the infinitesimal symmetries of equations from class (1.3). Analysis of its classifying part is made in Section 4. In Section 5, exact invariant solutions are found for equation (1.3) with $f(u) = u^{4/3}$, which is a representative of the equivalence class of nonlinear equations (1.3) with the three-dimensional MAI. In Section 6, we give some remarks discussing our main results and a problem for further investigation.

2. Group of equivalence transformations

Performing the group classification of classes of differential equations, it is important to know the local transformations of variables that alter the functional parameters contained in the studied class of equations, but keep the differential structure of one. Such transformations induce an equivalence relation on the set of the functional parameters. In other words, isomorphic are the symmetry groups of two differential equations, which correspond to two different, but equivalent parameters.

Traditionally, finding a group of equivalence transformations, one use the Lie-Ovsyannikov infinitesimal method (see, for example, [15, 20, 26]). However, this method only allows to find all *continuous* equivalence transformations, while for finding a complete group (pseudogroup) of equivalence transformations (including both continuous ones and *discrete* ones) it should be used the *direct method* [18].

We start the construction of the group of equivalence transformations of the class of GKEs (1.3) from the previous study of the set of *admissible* transformations (other names, allowed or form-preserving transformations) of this class of equations. In other words, we look for all non-degenerate point transformations of variables

$$\bar{t} = T(t,x,u), \quad \bar{x} = X(t,x,u), \quad \bar{u} = U(t,x,u), \quad \frac{\partial(T,X,U)}{\partial(t,x,u)} \neq 0,$$

that map a fixed equation of the form (1.3) to an equation of the same form:

$$\bar{u}_{\bar{t}} = \bar{x}^2 \bar{u}_{\bar{x}\bar{x}} + \bar{x}[\bar{x}\bar{f}'(\bar{u}) + 4]\bar{u}_{\bar{x}} + 4\bar{x}\bar{f}(\bar{u}). \tag{2.1}$$

Without loss of generality, we can restrict ourselves by consideration of point transformations of the form

$$\bar{t} = T(t),\ \bar{x} = X(t,x),\ \bar{u} = U(t,x,u),$$

where T, X, and U are arbitrary smooth functions of their variables with $T_t X_x U_u \neq 0$ (see [18, 32]). Under these transformations, the partial derivatives are transformed as follows:

$$\begin{aligned} u_t &= \frac{1}{U_u}(T_t \bar{u}_{\bar{t}} + X_t \bar{u}_{\bar{x}} - U_t), \quad u_x = \frac{1}{U_u}(X_x \bar{u}_{\bar{x}} - U_x), \\ u_{xx} &= \frac{1}{U_u}\Big[X_x^2 \bar{u}_{\bar{x}\bar{x}} + \Big(X_{xx} - 2X_x \frac{U_{xu}}{U_u} + 2X_x \frac{U_x U_{uu}}{U_u^2}\Big)\bar{u}_{\bar{x}} \\ &\quad - X_x^2 \frac{U_{uu}}{U_u^2} \bar{u}_{\bar{x}}^2 - U_{xx} + 2\frac{U_x U_{xu}}{U_u} - \frac{U_x^2 U_{uu}}{U_u^2}\Big]. \end{aligned}$$

Substituting the last formulas in (1.3) and taking into account equality (2.1), we obtain the equation

$$\begin{aligned}&\bar u_{\bar x\bar x}(x^2X_x^2-T_tX^2)-\bar u_{\bar x}^2x^2X_x^2\frac{U_{uu}}{U_u^2}+\bar u_{\bar x}\Big[x^2\Big(X_{xx}-2X_x\frac{U_{xu}}{U_u}+2X_x\frac{U_xU_{uu}}{U_u^2}\Big)\\&+x(xf_u+4)X_x-\big(\bar f_{\bar u}X+4\big)T_tX-X_t\Big]-x^2\left(U_{xx}-2\frac{U_xU_{xu}}{U_u}+\frac{U_x^2U_{uu}}{U_u^2}\right)\\&-x(xf_u+4)U_x+4xfU_u-4\bar fT_tX+U_t=0.\end{aligned}$$

Splitting it in $\bar u_{\bar x}$, and $\bar u_{\bar x\bar x}$, we have:

$$\begin{gathered}\bar u_{\bar x\bar x}:x^2X_x^2-T_tX^2=0,\quad \bar u_{\bar x}^2:U_{uu}=0,\\ \bar u_{\bar x}:x^2\Big(X_{xx}-2X_x\frac{U_{xu}}{U_u}\Big)+x(xf_u+4)X_x-\big(\bar f_{\bar u}X+4\big)T_tX-X_t=0,\\ 1:x^2\Big(U_{xx}-2U_x\frac{U_{xu}}{U_u}\Big)+x(xf_u+4)U_x-4xfU_u+4\bar fT_tX-U_t=0\end{gathered}\tag{2.2}$$

(equality $U_{uu}=0$ have been taken into account in the last two equations immediately).

From the second equation of system (2.2) we obtain

$$U=C_1(t,x)u+C_2(t,x),$$

and from the first equation we obtain

$$T_t=\frac{x^2X_x^2}{X^2}.\tag{2.3}$$

Substituting (2.3) in the last equation of system (2.2), we have

$$\bar f=\frac{X}{4x^2X_x^2}\Big[x^2\Big(2\frac{U_xU_{xu}}{U_u}-U_{xx}\Big)-x(xf_u+4)U_x+4xfU_u+U_t\Big].\tag{2.4}$$

Differentiated equation (2.4) with respect to u, we find that

$$\bar f_{\bar u}=\frac{X}{4x^2X_x^2U_u}\Big[x^2\Big(2\frac{U_{xu}^2}{U_u}-U_{xxu}-f_{uu}U_x-f_uU_{xu}\Big)-4x\left(U_{xu}-f_uU_u\right)+U_{tu}\Big].\tag{2.5}$$

Now we substitute (2.3) and (2.5) into the third equation of system (2.2). After simple transformations, we arrive at the equation

$$\begin{aligned}&x^2\Big(X_{xx}-2X_x\frac{U_{xu}}{U_u}-\frac{4X_x^2}{X}\Big)-\frac{X}{4U_u}\Big(x^2\Big(2\frac{U_{xu}^2}{U_u}-U_{xxu}\Big)-4xU_{xu}+U_{tu}\Big)\\&+4xX_x-X_t+x^2\Big(X_x-\frac{X}{x}+\frac{XU_{xu}}{4U_u}\Big)f_u+\frac{x^2XU_x}{4U_u}f_{uu}=0.\end{aligned}$$

Further analysis of the obtained equation is not possible without additional assumptions on the function $f(u)$. Thereby, now we find the group of *equivalence* transformations of the class of equations (1.3).

The equivalence transformations of the class of equations (1.3) are picking out the set of all admissible transformations by the additional condition that they map *every* equation of the form (1.3) to an equation of the same form. In this case, the functional parameter f varies, and thus, the last equation can be split by the derivatives of f.

Solving the obtained system of equations, we have:

$$X=B(t)x,\quad U=\frac{C}{B^4(t)}u+C_2(t),\quad C\in\mathbb{R}.$$

Then from the first equation of system (2.2), it follows that $T = t + A$, where $A \in \mathbb{R}$.

If $B(t)$ is not constant, then substituting the expressions for T, X, and U in (2.4), we obtain

$$\bar{f} = \frac{C}{B^5(t)} f - \frac{C}{xB^6(t)} u + \frac{C_2'(t)}{4xB(t)}.$$

Multiplying this equality by $4xB(t)$ and differentiating the resulting equation by x, we obtain the equality

$$\bar{f} = \frac{C}{B^5(t)} f,$$

which implies that $C_2'(t) = 0$, and $C = 0$. But then $U_u = 0$, that is impossible.

Thus, $B(t) = \text{const}$. Then from (2.4), we obtain

$$\bar{f} = \frac{C}{B^5} f + \frac{C_2'(t)}{4xB}.$$

Here, as above, we can show that $C_2'(t) = 0$. Denoting now $\frac{C}{B^4}$ in C_1, we arrive at the following assertion.

Theorem 2.1. *The group of equivalence transformations $G^{\sim}$ of the class of GKEs* (1.3) *consists of the following transformations:*

$$\bar{t} = t + A, \quad \bar{x} = Bx, \quad \bar{u} = C_1 u + C_2, \quad \bar{f} = \frac{C_1}{B} f, \tag{2.6}$$

where A, B, C_1, C_2 are arbitrary real constants with $BC_1 \neq 0$.

Remark 2.2. From Theorem 2.1, it is directly followed that any transformation $\mathcal{T}$ from the group $G^{\sim}$ of equivalence transformations of the class of differential equations (1.3) can be represented as the composition

$$\mathcal{T} = \mathcal{T}(A)\mathcal{T}(B)\mathcal{T}(C_1)\mathcal{T}(C_2),$$

where each of the transformations

$$\mathcal{T}(A) : (t, x, u, f) \mapsto (t + A, x, u, f),$$
$$\mathcal{T}(B) : (t, x, u, f) \mapsto (t, Bx, u, B^{-1} f), \ B \neq 0,$$
$$\mathcal{T}(C_1) : (t, x, u, f) \mapsto (t, x, C_1 u, C_1 f), \ C_1 \neq 0,$$
$$\mathcal{T}(C_2) : (t, x, u, f) \mapsto (t, x, u + C_2, f)$$

belongs to the one-parameter family of equivalence transformations.

3. The kernel of MAIs

According to the classical Lie algorithm [24, 26], we look for the infinitesimal operators generating the invariance algebra of equation (1.3) in the class of differential operators of the first order

$$X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \tag{3.1}$$

where τ, ξ, and η are arbitrary smooth functions of their variables.

The condition of invariance of equation (1.3) with respect to operator (3.1) is as follows:

$$X^{(2)} \left\{ u_t - x^2 u_{xx} - x(x f_u + 4) u_x - 4xf \right\} \Big|_{(1.3)} = 0,$$

or, in detail,

$$\begin{aligned}&\eta^t-2xu_{xx}\xi-x^2\eta^{xx}-2(xf_u+2)u_x\xi\\&-x^2f_{uu}u_x\eta-x(xf_u+4)\eta^x-4f\xi-4xf_u\eta\Big|_{(1.3)}=0.\end{aligned}\tag{3.2}$$

We used the following notation:

$$X^{(2)}=X+\eta^t\partial_{u_t}+\eta^x\partial_{u_x}+\eta^{tt}\partial_{u_{tt}}+\eta^{tx}\partial_{u_{tx}}+\eta^{xx}\partial_{u_{xx}}$$

is the second prolongation of the operator X;

$$\begin{gathered}\eta^i=\mathrm{D}_i(\eta)-u_t\mathrm{D}_i(\tau)-u_x\mathrm{D}_i(\xi),\quad i\in\{t,x\},\\ \eta^{ij}=\mathrm{D}_j(\eta^i)-u_{ti}\mathrm{D}_j(\tau)-u_{ix}\mathrm{D}_j(\xi),\quad i,j\in\{t,x\},\end{gathered}$$

where D_i, $i\in\{t,x\}$, is the operator of the total differentiation with respect to i; condition $|_{(3)}$ in (3.2) means replacing u_t to $x^2u_{xx}+x(xf_u+4)u_x+4xf$.

Substituting the expressions for η^t, η^x, and η^{xx} in equation (3.2) and splitting the obtained equality with respect to the various derivatives of u, we get the following system of determining equations:

$$\begin{gathered}u_xu_{tx}:\tau_u=0,\\ u_{tx}:\tau_x=0,\\ u_xu_{xx}:\xi_u=0,\\ u_x^2:\eta_{uu}=0,\\ u_{xx}:x(2\xi_x-\tau_t)-2\xi=0,\\ u_x:x(xf_u+4)(\tau_t-\xi_x)+2(xf_u+2)\xi+\xi_t+x^2(f_{uu}\eta+2\eta_{xu}-\xi_{xx})=0,\\ 1:4xf(\tau_t-\eta_u)+4f\xi+4xf_u\eta-\eta_t+x(xf_u+4)\eta_x+x^2\eta_{xx}=0.\end{gathered}\tag{3.3}$$

The last two equations including the functional parameter f form the *classifying part* of the system. Using the remaining equations, we find that

$$\tau=\tau(t),\quad \xi=x\Big(\frac{1}{2}\tau'(t)\ln x+\gamma(t)\Big),\quad \eta=\alpha(t,x)u+\beta(t,x),$$

where α,β,γ, and τ are smooth functions of their variables.

If f is an arbitrary function, we can further split system (3.3) with respect to derivatives of f and find the *kernel* $\mathfrak{g}^\cap$ of MAIs of the equations of the form (1.3) (i.e., those operators that are allowed by arbitrary equation from class (1.3)). Splitting yields: $\xi=\eta=0$, $\tau_t=0$. From the equations, it directly follows the next statement.

Theorem 3.1. *The kernel of MAIs of GKEs* (1.3) *is the one-dimensional Lie algebra* $\mathfrak{g}^\cap=\langle\partial_t\rangle$.

Extensions of the kernel can exist only in the cases when the equations of the classifying part are satisfied not only for an arbitrary function f. The analysis of these equations will be made in the next section.

4. Extensions of the kernel of MAIs

Rewrite the classifying part of system (3.3) as follows

$$\begin{aligned}&3\tau_t+\tau_{tt}\ln x+4x\alpha_x+2\gamma_t+x[\tau_t(1+\ln x)+2\gamma]f_u\\&+2x\beta f_{uu}+2x\alpha uf_{uu}=0,\\&4\beta_x+x\beta_{xx}-x^{-1}\beta_t+\left(4\alpha_x+x\alpha_{xx}-x^{-1}\alpha_t\right)u\\&+2[\tau_t(2+\ln x)+2(\gamma-\alpha)]f+(4\beta+x\beta_x)f_u+(4\alpha+x\alpha_x)uf_u=0.\end{aligned} \tag{4.1}$$

For the analysis of system (4.1) we apply the method of structural constants. To do this, we first show that this system is equivalent to the system of two *ordinary* differential equations on the function $f(u)$:

$$a+bf_u+cf_{uu}+duf_{uu}=0, \tag{4.2}$$

$$a^*+b^*u+c^*f+d^*f_u+e^*uf_u=0, \tag{4.3}$$

with the constant coefficients a,b,c,d,a^*,b^*,c^*,d^*, and e^*.

Indeed, since f depends only on u, (4.1) satisfies only if all the coefficients in these equations are equal to zero, or proportional (with constant coefficients) of some function $\lambda=\lambda(t,x)\not\equiv 0$:

(1) $3\tau_t+\tau_{tt}\ln x+4x\alpha_x+2\gamma_t=a\lambda$, $x[\tau_t(1+\ln x)+2\gamma]=b\lambda$, $2x\beta=c\lambda$, $2x\alpha=d\lambda$;
(2) $4\beta_x+x\beta_{xx}-x^{-1}\beta_t=a^*\lambda$, $4\alpha_x+x\alpha_{xx}-x^{-1}\alpha_t=b^*\lambda$, $2[\tau_t(2+\ln x)+2(\gamma-\alpha)]=c^*\lambda$, $4\beta+x\beta_x=d^*\lambda$, $4\alpha+x\alpha_x=e^*\lambda$.

It is easy to verify that if all the coefficients in (4.2) and (4.3) are simultaneously equal to zero, it corresponds to an arbitrary function f. Therefore, extensions of the kernel of MAIs are only possible for the function f, which satisfy the overdetermined system of classifying equations of the form (4.2) and (4.3) with constant coefficients.

The analysis of this system allows to obtain the following result.

Theorem 4.1. *The GKE of the form* (1.3) *may allow the invariance algebra of dimension, higher than* $\mathfrak{g}^\cap$*, when the function* f *belongs to one of the following classes (non-equivalent up to the transformations from the group* $G^\sim$*):*

(1) $f(u)=e^u+ku$ $(k\neq 0)$;
(2) $f(u)=e^u+n$;
(3) $f(u)=u\ln u+ku+n$;
(4) $f(u)=\ln u+ku+n$;
(5) $f(u)=u^m+ku+n$ $(m\neq 0,1,2)$;
(6) $f(u)=u^2+n$;
(7) $f(u)=u$;
(8) $f(u)=1$;
(9) $f(u)=0$,

where k,m,n *are arbitrary real constants.*

Proof. Let us consider equation (4.2). The analysis of it for the purpose of constructing the general solution strongly depends on the constant d.

If $d=0$, the corresponding equation has the general solutions of the following forms:

(1) $f(u)=ku^2+lu+n$ if $b=0$;
(2) $f(u)=ke^{mu}+lu+n$ (where $km\neq 0$) if $b\neq 0$,

where k, l, m, n are arbitrary real constants that satisfy the specified conditions.

Now let $d \neq 0$. For the purpose of analysis of equation (4.2), we use the fact that the equivalence relations in the class (1.3) are transferred to the system of classifying equations (4.1), and hence to equation (4.2) and (4.3). Applying the transformations from the group $G^{\sim}$ to equation (4.2), we find that in the new variables, the structure of this equation is preserved, but its coefficients change as follows:

$$a \mapsto \frac{a}{B}, \quad b \mapsto b, \quad c \mapsto cC_1 - dC_2, \quad d \mapsto d.$$

Now it is easy to see that picking in the correct way the value of C_1, the coefficient c in equation (4.2) can be reduced to zero.

Solving the resulting equation (with $c = 0$), we obtain the following expressions for its general solution (depending on the coefficients b and d):

(1) $f(u) = ku \ln u + lu + n$ if $b = 0$;
(2) $f(u) = k \ln u + lu + n$ (where $k \neq 0$) if $b = d$;
(3) $f(u) = ku^m + lu + n$ (where $k \neq 0$, $m \neq 0, 1$) in other cases,

where k, m, n are arbitrary real constants that satisfy the specified conditions.

Now, collecting all the possible cases for the function $f(u)$ in such a way that they are not mutually disjoint, and taking into account the non-used transformations from the group $G^{\sim}$, we get the nine non-equivalent (up to the transformations from the group $G^{\sim}$) classes listed in the formulation of the theorem. If a fixed function $f(u)$ belongs to some of these classes, then the extension of the kernel of MAIs of the corresponding GKE of the form (1.3) may be exist.

Analysis of equation (4.3) can be made similarly and gives the same result as in the case of equation (4.2). □

Substituting the resulting expressions for the function $f(u)$ in system (4.1) and holding the corresponding calculations, we arrive at such statement (detailed proof is in [29]).

Theorem 4.2. *All possible MAIs of the GKEs* (1.3) *with some fixed function* $f(u)$ *are described in Table* 1. *Any other equation of the form* (1.3) *with nontrivial Lie symmetry maps to one of the equations given in Table* 1 *by means of the equivalence transformations of the form* (2.6).

5. Exact solutions of GKE (1.3) with $f(u) = u^{4/3}$

In the previous section, it was shown that the equation

$$u_t = x^2 u_{xx} + 4x\Big(\frac{1}{3} x\, u^{1/3} + 1\Big) u_x + 4xu^{4/3}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{5.1}$$

(corresponding to Case 3 of Table 1) has the highest symmetry properties among the *non-linear* GKEs of the form (1.3), namely, this equation admits as MAI the three-dimensional Lie algebra of infinitesimal symmetries. It is natural to expect that this fact will allow us to build a number of non-trivial invariant exact solutions of equation (5.1). We could obtain a comprehensive list of the non-equivalent invariant solutions of this equation by operators from the admitted Lie algebra constructing an optimal system of subalgebras of this algebra (see, e.g., [24, Section 3.3]).

However, equation (5.1) can be reduced to the equation

$$u_t = u_{xx} + \frac{4}{3}\, u^{1/3} u_x \tag{5.2}$$

Table 1. The group classification of GKEs (1.3)

N	$f(u)$	Basis of MAI
1	e^u	$\partial_t,\, x\partial_x - \partial_u$
2	$u^k\ (k \neq 0, 1, \frac{4}{3})$	$\partial_t,\, x\partial_x - \frac{1}{k-1}u\partial_u$
3	$u^{4/3}$	$\partial_t,\, x\partial_x - 3u\partial_u,\, 2t\partial_t + (3t + \ln x)x\partial_x - 3(1 + 3t + \ln x)u\partial_u$
4	u	$\partial_t,\, u\partial_u,\, \varphi(t, x)\partial_u$[a]
5	1	$\partial_t,\, x\partial_x + u\partial_u,\, (x + u)\partial_u,$ $2t\partial_t + (\ln x - 3t)x\partial_x - (\ln x - 3t)x\partial_u,$ $4t^2\partial_t + 4tx \ln x\partial_x - [((\ln x + 3t)^2 + 2t)(x + u) + 4tx \ln x]\partial_u,$ $2tx\partial_x - [(\ln x + 3t)(x + u) + 2tx]\partial_u,\, \psi(t, x)\partial_u$[b]
6	0	$\partial_t,\, x\partial_x,\, u\partial_u,\, 2t\partial_t + (\ln x - 3t)x\partial_x,\, 2tx\partial_x - (\ln x + 3t)u\partial_u,$ $4t^2\partial_t + 4tx \ln x\partial_x - [(\ln x + 3t)^2 + 2t]u\partial_u,\, \psi(t, x)\partial_u$[b]

[a]The function $\varphi(t, x)$ is an arbitrary smooth solution of $u_t = x^2u_{xx} + x(x + 4)u_x + 4xu$.
[b]The function $\psi(t, x)$ is an arbitrary smooth solution of $u_t = x^2u_{xx} + 4xu_x$.

by the local transformation of variables

$$\bar{t} = t, \quad \bar{x} = \ln x - 3t, \quad \bar{u} = x^3 u. \tag{5.3}$$

Equation (5.2) is a well-known non-linear diffusion-convection equation (see, e.g., [6, 25], and also [31, Subs. 5.1.5, No. 9]).

Using the results of [6], we immediately obtain that the basis of MAI of equation (5.2) can be chosen as follows:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t + \frac{1}{2}x\partial_x - \frac{3}{2}u\partial_u,$$

and the optimal system of one-dimensional subalgebras of the MAI consists of the following ones:

$$\langle X_2\rangle, \quad \langle X_3\rangle, \quad \langle X_1 + cX_2 \mid c \in \mathbb{R}\rangle.$$

A symmetry of the type $\langle X_1 + cX_2 \mid c \in \mathbb{R} \setminus \{0\}\rangle$ generates a travelling wave solution. Solutions of this type can be found in [31] (see Subs. 5.1.5, No. 9.2):

$$u(t, x) = \Big(ce^{-\frac{\lambda}{3}(x+\lambda t)} + \frac{1}{\lambda}\Big)^{-3};$$

$$u(t, x) = \frac{\lambda^3}{8}\big\{1 + \tanh[\frac{\lambda}{6}(x + \lambda t) + c]\big\}^3.$$

The operator $\langle X_1\rangle$ gives the ansatz $u = \varphi(x)$ and reduces equation (5.2) to the equation

$$\varphi'' + \frac{4}{3}\varphi^{1/3}\varphi' = 0.$$

We obtained a generalized Emden-Fowler equation, for which the solution in the parametric form is known [30, Subs. 2.5.2, No. 1.1] We rewrite it as a function $x(\varphi)$:

$$x = \int \frac{d\varphi}{c - \varphi^{4/3}}.$$

After integrating, we have such solutions (depending on the value of c):

$$\varphi(x) = \frac{27}{(x + c_0)^3}, \quad \text{if } c = 0;$$

$$x=\frac{3}{4}c_1\Big(\ln|\frac{c_1\sqrt[3]{\varphi}+1}{c_1\sqrt[3]{\varphi}-1}|-2\arctan(c_1\sqrt[3]{\varphi})\Big)+c_2,\quad\text{if } c>0;$$
$$x=\frac{3}{4}c_3\Big(\ln\frac{2c_3^2\sqrt[3]{\varphi^2}+2c_3\sqrt[3]{\varphi}+1}{2c_3^2\sqrt[3]{\varphi^2}-2c_3\sqrt[3]{\varphi}+1}-2\arctan\frac{2c_3\sqrt[3]{\varphi}}{1-2c_3^2\sqrt[3]{\varphi^2}}\Big)+c_4,\quad\text{if } c<0.$$

Further, the operator $\langle X_3\rangle$ gives the ansatz [6, p. 153]

$$u=\frac{\varphi(y)}{\sqrt{t^3}},\quad y=\frac{x}{\sqrt{t}}$$

and reduces equation (5.1) to

$$\varphi''+\Big(\frac{4}{3}\varphi^{1/3}+\frac{1}{2}y\Big)\varphi'+\frac{3}{2}\varphi=0. \tag{5.4}$$

We could not find the general solution of equation (5.4), however, it is easy to see that one has a particular solution $\varphi(y)=27y^{-3}$. Then we have the following particular solution of equation (5.2),

$$u(x)=27x^{-3}.$$

It is a self-similar solution mentioned in [31] (see Subs. 5.1.5, No. 9.3).

Finally, it is easy to see that the operator $\langle X_2\rangle$ generate only trivial solution $u=c$.

Now, using transformation (5.3), we obtain such solutions of (5.1):

(1) $u(x)=cx^{-3}$;
(2) $u(t,x)=27x^{-3}(\ln x-3t+c)^{-3}$;
(3) $u(t,x)=x^{-3}\big\{c[xe^{(\lambda-3)t}]^{-\frac{\lambda}{3}}+\frac{1}{\lambda}\big\}^{-3}$;
(4) $u(t,x)=\frac{\lambda^3}{8}x^{-3}\big\{1+\tanh[\frac{\lambda}{6}(\ln x+(\lambda-3)t)+c]\big\}^3$,

Also there are two solutions in implicit form (here $u(t,x)=x^{-3}\psi(y)$, $y=\ln x-3t$):

(5) $x=\lambda\big\{\big|\frac{c\sqrt[3]{\psi}+1}{c\sqrt[3]{\psi}-1}\big|^{c/4}\exp[t-\frac{c}{2}\arctan(c\sqrt[3]{\psi})]\big\}^3$, $\lambda>0$;
(6) $x=\lambda\big\{\big(\frac{2c^2\sqrt[3]{\psi^2}+2c\sqrt[3]{\psi}+1}{2c^2\sqrt[3]{\psi^2}-2c\sqrt[3]{\psi}+1}\big)^{c/4}\exp\big(t-\frac{c}{2}\arctan\frac{2c\sqrt[3]{\psi}}{1-2c^2\sqrt[3]{\psi^2}}\big)\big\}^3$, $\lambda>0$.

The obtained solutions can potentially be used for solving some physical problems. As an illustrative example, we present their usage for solving a boundary value problem (BVP).

Consider the set of exact solutions of equation (5.1) in an explicit form, namely, solutions 1)–4). Let S be the class of all positive continuous in $\mathbb{R}_+\times\mathbb{R}_+$ solutions $u(t,x)$ from them. Solutions 2) does not belong to this class, because they are discontinuous. Solutions 1), 3), 4) belong to S, if $c>0$, $c>0$ and $\lambda>0$, $\lambda>0$, respectively.

Proposition 5.1. *In the class S, the boundary-value problem*

$$\begin{gathered}u_t=\frac{1}{x^2}\cdot[x^4(u_x+u^{4/3})]_x,\quad(t,x)\in\mathbb{R}_+\times\mathbb{R}_+,\\ x^4(u_x+u^{4/3})\to0,\quad\textit{as } x\to+0,\\ x^4(u_x+u^{4/3})\to k,\quad\textit{as } x\to+\infty,\\ u(0,x)=F(x),\quad\textit{where }\int_0^{+\infty}F(x)dx=1\end{gathered} \tag{5.5}$$

(where k is a constant, not $\pm\infty$):

(1) *has no solutions when* $k < -8\frac{139}{256}$*;*
(2) *has a unique solution when* $k = -8\frac{139}{256}$ *or* $k \geq -8$*; this solution reads as*

$$u(t,x) = \lambda^3 x^{-3}\Big\{\big[3(\lambda-3)(\lambda-6)\frac{\pi}{\sin\frac{6\pi}{\lambda}}\big]^{\lambda/6}[xe^{(\lambda-3)t}]^{-\lambda/3} + 1\Big\}^{-3}, \quad (5.6)$$

where λ *is a real root of the equation*

$$\lambda^4 - 3\lambda^3 = k; \quad (5.7)$$

(3) *has two solutions when* $-8\frac{139}{256} < k < -8$*; these solutions read as* (5.6)*, where* $\lambda = \lambda_1$ *and* $\lambda = \lambda_2$ *are two real roots of equation* (5.7)*.*

Proof. First of all, note that solutions (1) do not satisfy the initial condition. Thus, only solutions (3) and (4) can be ones of BVP (5.5).

Further, using the programm of symbolic calculations *Mathematica*, we obtain that solutions (3) and (4) satisfy the boundary conditions for all $\lambda > 0$ (moreover, $k = \lambda^4 - 3\lambda^3$), and the initial condition, only if $\lambda > 2$. In this case,

$$c = \frac{1}{\lambda}\big[3(\lambda-3)(\lambda-6)\frac{\pi}{\sin\frac{6\pi}{\lambda}}\big]^{\lambda/6} \quad (5.8)$$

for solution (3), and

$$c = -\frac{\lambda}{12}\log\big[3(\lambda-3)(\lambda-6)\frac{\pi}{\sin\frac{6\pi}{\lambda}}\big] \quad (5.9)$$

for solution (4) (the values of $c(3)$ and $c(6)$ can be found as the corresponding limits of these expressions).

If $\lambda > 2$, then equation (5.7) has two real solutions, when $-8\frac{139}{256} < k < -8$, and the unique real solution, when $k = -8\frac{139}{256}$ or $k \geq -8$; otherwise this equation has no solutions. Now, substituting (5.8) into solution (3), we obtain the solution of BVP (5.5) in the form (5.6), if one exists. Executing routine calculations, we can reduce solution (4) (for the same λ and c of the form (5.9)) to (5.6). □

Example 5.2. Let $k = 0$. Then $\lambda = 3$, $c = \sqrt{3/2}$, and $u(t,x) = 27(x + 3\sqrt{3/2})^{-3}$. So, in this (and only in this) case BVP (5.5) has only *stationary* solution.

Note that the similar BVPs with vanishing boundary condition at infinity were considered previously in [7, 8, 10, 33, 34] et al.

Example 5.3. Let $k = 64$. Then $\lambda = 4$, $c = \frac{1}{4}\sqrt[3]{36\,\pi^2}$, and

$$u(t,x) = 64\,x^{-3}\Big(\sqrt[3]{36\,\pi^2}[xe^t]^{-\frac{4}{3}} + 1\Big)^{-3}.$$

The graph of this solution is presented on Figure 1.

Example 5.4. Let $k = -8,2944$. Then $\lambda_1 \approx 2,0853$; $\lambda_2 = -2,4$. For λ_2, $c = \frac{5}{12}\sqrt[5]{41,9904\,\pi^2}$, and

$$u(t,x) = 13,824\,x^{-3}\Big(\sqrt[5]{41,9904\,\pi^2}[xe^{-\frac{3}{5}t}]^{-\frac{4}{5}} + 1\Big)^{-3}.$$

The graph of this solution is presented on Figure 2.

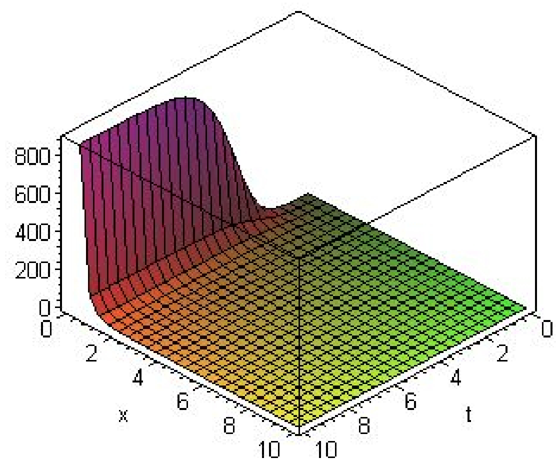

FIGURE 1. Solution of BVP (5.5), $k = 64$

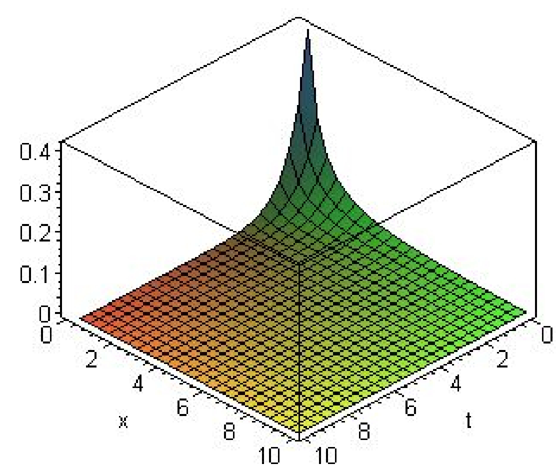

FIGURE 2. Solution of BVP (5.5), $k = -8,2944$

6. Discussion

Now we give some remarks discussing our main results.

Remark 6.1. It should be emphasized that Theorem 4.2 gives a comprehensive description of all non-equivalent GKEs of the form (1.3) up to the transformations of variables from the group $G^{\sim}$. However, it can be shown that the GKE (1.3) with $f(u) = 1$ may be mapped to the GKE (1.3) with $f(u) = 0$ by the local transformation of variables (which does not belong to the group $G^{\sim}$):

$$\bar{t} = t, \quad \overline{x} = x, \quad \overline{u} = u + x.$$

Note also that the GKEs (1.3) with $f(u) = u^k$ ($k \neq 0, 1, \frac{4}{3}$) and $f(u) = e^u$ admit the isomorphic MAIs (type $A_{2.1}$ by the Mubarakzyanov classification [21]). However, the direct analysis of (2.4) shows that among the admissible transformations in the class of GKEs of the form (1.3) there are no such point transformations of variables mapping these two equations into each other.

Remark 6.2. Lie classification of the linear parabolic second order differential equations with two independent variables is widely known (see, e.g., [13]). One of the main results of this classification is the fact that any equation of the form

$$P(t,x)u_t + Q(t,x)u_x + R(t,x)u_{xx} + S(t,x)u = 0, \quad P \neq 0,\ R \neq 0,$$

which admits a five-dimensional nontrivial Lie algebra of infinitesimal symmetries is reduced to the linear heat equation

$$v_\tau = v_{yy} \tag{6.1}$$

by the change of variables

$$\tau = \alpha(t), \quad y = \beta(t,x), \quad v = \gamma(t,x)u, \quad \alpha_t\beta_x \neq 0, \tag{6.2}$$

with some fixed functions α, β, and γ.

Therefore, Theorem 4.2 implies that equation (1.3) with $f(u) = \text{const}$ admitting the five-dimensional non-trivial algebra of infinitesimal symmetries is reduced to the linear heat equation by some change of variables of the form (6.2). Indeed, it has been shown [36] that the linear Kompaneets equation (1.3) with $f(u) = 0$ reduces to equation (6.1) by the change of variables

$$\tau = t, \quad y = 3t + \ln x, \quad v = u.$$

Remark 6.3. Since equation (5.2) is a PDE with two independent variables t and x, then it can be looked for the invariant solutions of ranks $\rho = 0$ or $\rho = 1$. We perform only the analysis of invariant solutions of rank $\rho = 1$. To find the invariant solutions of equation (5.2) of rank $\rho = 0$, it is necessary to use the two-dimensional subalgebras of the algebra $\langle X_1, X_2, X_3\rangle$. Using the results of [28], it is easy to see that there are three non-equivalent (up to conjugacy, which is determined by the actions of the group of inner automorphisms of the algebra $\langle X_1, X_2, X_3\rangle$) two-dimensional subalgebras of this algebra, namely, $\langle X_1, X_2\rangle$, $\langle X_1, X_3\rangle$, $\langle X_2, X_3\rangle$. However, their analysis shows that they do not lead to any new invariant solutions of equation (5.2) compared with ones obtained as a result of analysis of the one-dimensional subalgebras.

In the forthcoming article, we are going to consider from the group-theoretical point of view a class of the variable coefficients GKEs with two functional parameters.

Conclusions. In the present article, Lie group classification problem for the class of GKEs (1.3) was solved exhaustively. The main result of the paper is the classification list (see Table 1), which consists of the six non-equivalent cases (up to equivalence transformations obtained in Section 2). Among the corresponding non-linear equations from the class under study, the GKE with $f(u) = u^{4/3}$ has the maximal symmetry properties, namely, it admits a three-dimensional MAI. A number of exact solutions were constructed for this equation. As an illustrative example, a BVP for this equation was solved using the solutions obtained.

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