

OSCILLATION CONSTANT FOR MODIFIED EULER TYPE HALF-LINEAR EQUATIONS

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ABSTRACT. Applying the modified half-linear Prüfer angle, we study oscillation properties of the half-linear differential equation

$$[r(t)t^{p-1}\Phi(x')] + \frac{s(t)}{t \log^p t} \Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x.$$

We show that this equation is conditionally oscillatory in a very general case. Moreover, we identify the critical oscillation constant (the borderline depending on the functions r and s which separates the oscillatory and non-oscillatory equations). Note that the used method is different from the standard method based on the half-linear Prüfer angle.

1. INTRODUCTION

For any $p > 1$, we analyze the oscillation behavior of the second-order half-linear differential equation

$$[R(t)\Phi(x')] + S(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x, \quad (1.1)$$

where R is a positive continuous function and S is a continuous function. In this paper, we focus on the so-called conditional oscillation. To recall the notion of the conditional oscillation in a simple form, we consider (1.1) with $S(t) = \gamma C(t)$, where C is a continuous function and $\gamma \in \mathbb{R}$, i.e.,

$$[R(t)\Phi(x')] + \gamma C(t)\Phi(x) = 0. \quad (1.2)$$

We say that (1.2) is conditionally oscillatory if there exists a positive constant Γ such that (1.2) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. Such a constant Γ is called the critical oscillation constant of (1.2). Of course, once we know its value, we are able to formulate oscillation results directly for (1.1). For this reason, some results of this kind are presented in different forms in the literature (see the below given results and references).

Now we collect the most relevant references of the treated topic. The reader can find the fundamental theory overview on half-linear equations in books [1, 5]. The conditional oscillation of half-linear differential equations is studied, e.g., in papers [4, 7, 11] (see also [3, 8, 24, 25]). Of course, the identification of oscillation

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constants is a subject of research also in the field of difference equations (see [26]) and in the field of dynamic equations on time scales (see [14]). In the linear case, we refer to [10, 12, 20, 28] (and also [16, 17, 18]).

Very general results about the conditional oscillation can be found in [13, 15, 27]. In [13, 27], there are treated equations with asymptotically almost periodic coefficients and coefficients having mean values (for the definitions of the asymptotic almost periodicity and the mean value, see Definitions 4.5 and 4.7 below). In addition, the results of [15] cover equations whose coefficients are unbounded. The main results of [15, 27] are reformulated in Theorems 1.1 and 1.2 below. Note that the main result of [27] is superior to the results of [13]. To mention the basic motivation explicitly in Theorems 1.1 and 1.2, we use the standard symbols $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}_a := [a, \infty)$ for any $a > 0$ in the rest of this article.

Theorem 1.1 ([27]). *Let $r : \mathbb{R}_1 \rightarrow \mathbb{R}^+$ be a continuous function, for which mean value $M(r^{1/(1-p)})$ exists and for which it holds*

$$0 < \inf_{t \in \mathbb{R}_1} r(t) \leq \sup_{t \in \mathbb{R}_1} r(t) < \infty,$$

and let $s : \mathbb{R}_1 \rightarrow \mathbb{R}$ be a continuous function having mean value $M(s)$. Let

$$\tilde{\Gamma} := \left(\frac{p-1}{p}\right)^p [M(r^{1/(1-p)})]^{1-p}.$$

Consider the equation

$$[r(t)\Phi(x')] + \frac{s(t)}{t^p}\Phi(x) = 0. \quad (1.3)$$

Equation (1.3) is oscillatory if $M(s) > \tilde{\Gamma}$, and non-oscillatory if $M(s) < \tilde{\Gamma}$.

Theorem 1.2 ([15]). *Let $r : \mathbb{R}_1 \rightarrow \mathbb{R}^+$ and $s : \mathbb{R}_1 \rightarrow \mathbb{R}$ be continuous functions such that there exist $\delta > 1$ and $\varepsilon \in (0, 1/2)$ for which*

$$\int_t^{t+1} r(\tau) d\tau < \delta t^\varepsilon, \quad \int_t^{t+1} |s(\tau)| d\tau < \delta t^\varepsilon, \quad t \in \mathbb{R}_1. \quad (1.4)$$

Consider the equation

$$[r^{1-p}(t)\Phi(x')] + \frac{s(t)}{t^p}\Phi(x) = 0. \quad (1.5)$$

(i) *If there exist $\alpha, R, S \in \mathbb{R}^+$ satisfying*

$$R^{p-1}S > \left(\frac{p-1}{p}\right)^p$$

and

$$\frac{1}{\alpha} \int_t^{t+\alpha} r(\tau) d\tau \geq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau \geq S$$

for all sufficiently large t , then (1.5) is oscillatory.

(ii) *If there exist $\alpha, R, S \in \mathbb{R}^+$ satisfying*

$$R^{p-1}S < \left(\frac{p-1}{p}\right)^p,$$

$$\frac{1}{\alpha} \int_t^{t+\alpha} r(\tau) d\tau \leq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau \leq S$$

for all sufficiently large t , then (1.5) is non-oscillatory.

In this paper, motivated by Theorems 1.1 and 1.2, our aim is to extend the family of conditionally oscillatory equations. More precisely, we identify the critical oscillation constant for the Euler type equations in the form

$$[r(t)t^{p-1}\Phi(x')] + \frac{s(t)}{t \log^p t} \Phi(x) = 0, \quad (1.6)$$

where $r : \mathbb{R}_e \rightarrow \mathbb{R}^+$ and $s : \mathbb{R}_e \rightarrow \mathbb{R}$ are continuous functions. We introduce a new modification of the half-linear Prüfer angle in combination with the Riccati transformation. We remark that the results proven in this paper are stronger than the ones known for (1.5) (compare (3.1) and (3.2) with (1.4)). To the best of our knowledge, the presented results are new even for (1.6) with periodic coefficients r, s .

The article is organized as follows. The notion of the Prüfer angle is mentioned in the next section. Auxiliary results are collected in Section 3. The content of Sections 2 and 3 gives the description of our method which is used in the proofs of the main results in Section 4. Section 4 is finished by corollaries and examples.

2. EQUATION FOR PRÜFER ANGLE

In this section, we derive the equation for the modified half-linear Prüfer angle which will be fundamental for our investigation. To do this, we have to begin with the well-known notion of the Riccati equation corresponding to the general half-linear equation

$$[R(t)\Phi(x')] + S(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x, \quad (2.1)$$

where $p > 1$ and $R : \mathbb{R}_e \rightarrow \mathbb{R}^+$ and $S : \mathbb{R}_e \rightarrow \mathbb{R}$ are continuous functions. The Riccati equation

$$w'(t) + S(t) + (p-1)R^{\frac{1}{1-p}}(t)|w(t)|^{\frac{p}{p-1}} = 0 \quad (2.2)$$

can be obtained applying the transformation

$$w(t) = R(t)\Phi\left(\frac{x'(t)}{x(t)}\right) \quad (2.3)$$

to (2.1), where x is a non-trivial solution of (2.1). Of course, the transformation is valid whenever $x(t) \neq 0$. For more details including the used calculations, see [5, Section 1.1.4].

To introduce the modified half-linear Prüfer transformation, we should shortly recall the half-linear trigonometric functions. The half-linear sine and cosine functions, denoted by \sin_p and \cos_p , are defined via the initial problem

$$[\Phi(x')] + (p-1)\Phi(x) = 0, \quad x(0) = 0, \quad x'(0) = 1. \quad (2.4)$$

More precisely, \sin_p is given as the odd extension of the solution of (2.4) with period

$$2\pi_p := \frac{4\pi}{p \sin \frac{\pi}{p}}.$$

Then, \cos_p is introduced as the derivative of \sin_p . The functions \sin_p and \cos_p have many properties similar to the common sine and cosine functions. In this paper, we will use only the Pythagorean identity, whose half-linear version is

$$|\sin_p x|^p + |\cos_p x|^p = 1, \quad x \in \mathbb{R}. \quad (2.5)$$

For more details concerning the half-linear trigonometric functions, we refer to [5, Section 1.1.2].

We introduce the modified Prüfer transformation in the form

$$x(t) = \rho(t) \sin_p \varphi(t), \quad R^{q-1}(t)x'(t) = \frac{\rho(t)}{\log t} \cos_p \varphi(t) \quad (2.6)$$

together with the substitution

$$v(t) = (\log t)^{p/q} w(t), \quad t \in \mathbb{R}_e, \quad (2.7)$$

where \log stands for the natural logarithm and q is the number conjugated with p , i.e.,

$$p + q = pq. \quad (2.8)$$

Note that substitutions similar to (2.7) can be used also in the Riccati equation (2.2). This approach leads to the so-called adapted (or weighted) Riccati equation. Nevertheless, we use this process only partially (see below) and we have to take into consideration the modified Prüfer transformation (2.6) as well.

Using (2.7), (2.3), (2.6), and (2.8) successively, we obtain

$$\begin{aligned} v(t) &= (\log t)^{p/q} w(t) = (\log t)^{p/q} R(t) \Phi\left(\frac{x'(t)}{x(t)}\right) \\ &= (\log t)^{p/q} R(t) \frac{\Phi(R^{1-q}(t) \cos_p \varphi(t))}{\Phi(\log t \sin_p \varphi(t))} \\ &= \Phi\left(\frac{\cos_p \varphi(t)}{\sin_p \varphi(t)}\right). \end{aligned} \quad (2.9)$$

One can easily verify that

$$\hat{v}(t) := \Phi\left(\frac{\cos_p t}{\sin_p t}\right) \quad (2.10)$$

solves the equation

$$\hat{v}'(t) + p - 1 + (p - 1)|\hat{v}(t)|^q = 0. \quad (2.11)$$

Equation (2.11) is the Riccati equation associated to the equation in (2.4). Hence, due to (2.5), (2.8), (2.10), and (2.11), we have

$$\begin{aligned} v'(t) &= [\hat{v}(\varphi(t))]' = [-p + 1 - (p - 1)|\hat{v}(\varphi(t))|^q] \varphi'(t) \\ &= (1 - p) \left[1 + \left| \frac{\cos_p \varphi(t)}{\sin_p \varphi(t)} \right|^p \right] \varphi'(t) \\ &= \frac{1 - p}{|\sin_p \varphi(t)|^p} \varphi'(t). \end{aligned} \quad (2.12)$$

On the other hand, considering (2.7) together with (2.2), we have

$$\begin{aligned} v'(t) &= \frac{p}{q} (\log t)^{\frac{p}{q}-1} \frac{w(t)}{t} + (\log t)^{p/q} w'(t) \\ &= \frac{p}{q} \cdot \frac{v(t)}{t \log t} + (\log t)^{p/q} [-S(t) - (p - 1)R^{1-q}(t)|w(t)|^q] \\ &= \frac{p}{q} \cdot \frac{v(t)}{t \log t} - (\log t)^{p/q} S(t) - (p - 1)R^{1-q}(t) \frac{|v(t)|^q}{\log t}. \end{aligned} \quad (2.13)$$

We combine (2.12) and (2.13). This leads to

$$\frac{1 - p}{|\sin_p \varphi(t)|^p} \varphi'(t) = \frac{p}{q} \cdot \frac{v(t)}{t \log t} - (\log t)^{p/q} S(t) - (p - 1)R^{1-q}(t) \frac{|v(t)|^q}{\log t}.$$

Taking into account (2.9), we obtain

$$(1-p)\varphi'(t) = \frac{p}{q} \cdot \frac{1}{t \log t} \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) \\ - (\log t)^{p/q} S(t) |\sin_p \varphi(t)|^p - (p-1)R^{1-q}(t) \frac{|\cos_p \varphi(t)|^p}{\log t}$$

which gives us the desired equation for the Prüfer angle as

$$\varphi'(t) = R^{1-q}(t) \frac{|\cos_p \varphi(t)|^p}{\log t} \\ - \frac{1}{t \log t} \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) + \frac{(\log t)^{p/q}}{p-1} S(t) |\sin_p \varphi(t)|^p. \quad (2.14)$$

In this article, we apply (2.14) to the study of the half-linear equations with the coefficients in the form

$$R(t) = r(t)t^{p-1}, \quad S(t) = \frac{s(t)}{t \log^p t}, \quad t \in \mathbb{R}_e. \quad (2.15)$$

Thus, we will analyze the equations

$$[r(t)t^{p-1}\Phi(x')] + \frac{s(t)}{t \log^p t} \Phi(x) = 0, \quad (2.16)$$

where $r : \mathbb{R}_e \rightarrow \mathbb{R}^+$ and $s : \mathbb{R}_e \rightarrow \mathbb{R}$ are continuous functions. We point out that the form of (2.14) for the coefficients in (2.15) is

$$\varphi'(t) = \frac{1}{t \log t} \left[r^{1-q}(t) |\cos_p \varphi(t)|^p \right. \\ \left. - \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) + s(t) \frac{|\sin_p \varphi(t)|^p}{p-1} \right], \quad (2.17)$$

which can be simply verified.

3. PRÜFER ANGLE OF AVERAGE FUNCTION

In this section, we assume that the coefficients $r : \mathbb{R}_e \rightarrow \mathbb{R}^+$ and $s : \mathbb{R}_e \rightarrow \mathbb{R}$ in (2.16) are such that

$$\lim_{t \rightarrow \infty} \frac{\int_t^{t+\alpha} r^{1-q}(\tau) d\tau}{\sqrt{t \log t}} = 0, \quad (3.1)$$

$$\lim_{t \rightarrow \infty} \frac{\int_t^{t+\alpha} |s(\tau)| d\tau}{\sqrt{t \log t}} = 0 \quad (3.2)$$

hold for some $\alpha \in \mathbb{R}^+$. For this number α , we define the function ψ which determines the average value of an arbitrarily given solution φ of (2.17) over intervals of the length α ; i.e., we put

$$\psi(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \varphi(\tau) d\tau, \quad t \in \mathbb{R}_e,$$

where φ is a solution of (2.17) on \mathbb{R}_e . We formulate and prove auxiliary results concerning the function ψ .

Lemma 3.1. *It holds*

$$\lim_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \psi(t)| = 0 \quad (3.3)$$

uniformly with respect to $s \in [t, t + \alpha]$.

Proof. For $s \in [t, t + \alpha]$, it is seen that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \psi(t)| \\ & \leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} \int_t^{t+\alpha} |\varphi'(\tau)| \, d\tau \\ & = \limsup_{t \rightarrow \infty} \sqrt{t \log t} \int_t^{t+\alpha} \left| \frac{1}{\tau \log \tau} \left[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \right. \right. \\ & \quad \left. \left. - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \right] \right| \, d\tau \\ & \leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} \int_t^{t+\alpha} \frac{1}{t \log t} \left[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \right. \\ & \quad \left. + |\Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)| + |s(\tau)| \frac{|\sin_p \varphi(\tau)|^p}{p-1} \right] \, d\tau. \end{aligned}$$

Since (see directly (2.5))

$$|\sin_p x|^p \leq 1, \quad |\cos_p x|^p \leq 1, \quad x \in \mathbb{R}, \quad (3.4)$$

and, consequently,

$$|\Phi(\cos_p x) \sin_p x| = |\cos_p x|^{p-1} |\sin_p x| \leq 1, \quad x \in \mathbb{R}, \quad (3.5)$$

we have

$$\begin{aligned} 0 & \leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \psi(t)| \\ & \leq \limsup_{t \rightarrow \infty} \frac{1}{\sqrt{t \log t}} \int_t^{t+\alpha} r^{1-q}(\tau) + 1 + \frac{|s(\tau)|}{p-1} \, d\tau, \quad s \in [t, t + \alpha]. \end{aligned}$$

Using (3.1) and (3.2), we obtain

$$0 \leq \liminf_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \psi(t)| \leq \limsup_{t \rightarrow \infty} \sqrt{t \log t} |\varphi(s) - \psi(t)| = 0$$

uniformly with respect to $s \in [t, t + \alpha]$. □

Lemma 3.2. *It holds*

$$\begin{aligned} \psi'(t) & = \frac{1}{t \log t} \left[\frac{|\cos_p \psi(t)|^p}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau) \, d\tau - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \right. \\ & \quad \left. + \frac{|\sin_p \psi(t)|^p}{(p-1)\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau + \Psi(t) \right] \end{aligned}$$

for all $t > e$, where $\Psi : \mathbb{R}_e \rightarrow \mathbb{R}$ is a continuous function such that $\lim_{t \rightarrow \infty} \Psi(t) = 0$.

Proof. For any $t > e$, we have

$$\psi'(t) = \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{\tau \log \tau} \left[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \right.$$

$$- \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Big] d\tau.$$

We can replace $\psi'(t)$ by

$$\frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t \log t} \left[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \right] d\tau,$$

because we can easily estimate (see (3.1), (3.2), (3.4), (3.5))

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{t \log t}{\alpha} \left| \int_t^{t+\alpha} \frac{1}{\tau \log \tau} \left[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \right. \right. \\ & \left. \left. + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \right] d\tau - \int_t^{t+\alpha} \frac{1}{t \log t} \left[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \right. \right. \\ & \left. \left. + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \right] d\tau \right| \\ & \leq \limsup_{t \rightarrow \infty} \frac{t \log t}{\alpha} \int_t^{t+\alpha} \left[\frac{1}{t \log t} - \frac{1}{\tau \log \tau} \right] \left[r^{1-q}(\tau) + 1 + |s(\tau)| \frac{1}{p-1} \right] d\tau \\ & \leq \limsup_{t \rightarrow \infty} \frac{t \log t}{\alpha} \int_t^{t+\alpha} \left[\frac{1}{t \log t} - \frac{1}{(t+\alpha) \log(t+\alpha)} \right] \\ & \quad \times \left[r^{1-q}(\tau) + 1 + |s(\tau)| \frac{1}{p-1} \right] d\tau \\ & \leq \limsup_{t \rightarrow \infty} \frac{t \log t}{\alpha} \cdot \frac{\alpha}{t(t+\alpha) \log t} \int_t^{t+\alpha} \left[r^{1-q}(\tau) + 1 + |s(\tau)| \frac{1}{p-1} \right] d\tau \\ & \leq \limsup_{t \rightarrow \infty} \frac{1}{t+\alpha} \int_t^{t+\alpha} \left[r^{1-q}(\tau) + 1 + |s(\tau)| \frac{1}{p-1} \right] d\tau = 0. \end{aligned}$$

Applying the uniform continuity of the half-linear trigonometric functions and (3.3) in Lemma 3.1, we see that

$$\lim_{t \rightarrow \infty} \left(\Phi(\cos_p \psi(t)) \sin_p \psi(t) - \frac{1}{\alpha} \int_t^{t+\alpha} \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) d\tau \right) = 0. \tag{3.6}$$

In addition, the half-linear trigonometric functions are continuously differentiable and periodic (see, e.g., [5, Section 1.1.2]). Hence, they have the Lipschitz property on \mathbb{R} and, consequently, there exists a constant $L \in \mathbb{R}^+$ for which

$$| |\cos_p x|^p - |\cos_p y|^p | \leq L|x - y|, \quad x, y \in \mathbb{R}, \tag{3.7}$$

$$| |\sin_p x|^p - |\sin_p y|^p | \leq L|x - y|, \quad x, y \in \mathbb{R}. \tag{3.8}$$

Therefore, from (3.1), Lemma 3.1, and (3.7), it follows

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \left| \frac{|\cos_p \psi(t)|^p}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau) d\tau - \frac{1}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p d\tau \right| \\ & \leq \frac{1}{\alpha} \limsup_{t \rightarrow \infty} \int_t^{t+\alpha} r^{1-q}(\tau) | |\cos_p \psi(t)|^p - |\cos_p \varphi(\tau)|^p | d\tau \\ & \leq \frac{1}{\alpha} \limsup_{t \rightarrow \infty} \frac{\sqrt{t \log t}}{\sqrt{t \log t}} \int_t^{t+\alpha} r^{1-q}(\tau) L |\psi(t) - \varphi(\tau)| d\tau = 0. \end{aligned} \tag{3.9}$$

Analogously, from (3.2), Lemma 3.1, and (3.8), we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \left| \frac{|\sin_p \psi(t)|^p}{(p-1)\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau - \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \, d\tau \right| \\ & \leq \frac{1}{(p-1)\alpha} \limsup_{t \rightarrow \infty} \int_t^{t+\alpha} |s(\tau)| \cdot \left| |\sin_p \psi(t)|^p - |\sin_p \varphi(\tau)|^p \right| \, d\tau \quad (3.10) \\ & \leq \frac{1}{(p-1)\alpha} \limsup_{t \rightarrow \infty} \frac{\sqrt{t \log t}}{\sqrt{t \log t}} \int_t^{t+\alpha} |s(\tau)| L |\psi(t) - \varphi(\tau)| \, d\tau = 0. \end{aligned}$$

Finally, the statement of the lemma directly comes from the combination of (3.6), (3.9), and (3.10). The continuity of Ψ is obvious. \square

4. OSCILLATION CONSTANT

At first, we recall known results concerning the studied equations with constant coefficients.

Theorem 4.1. *If $A, B \in \mathbb{R}^+$ satisfy $B/A > q^{-p}$, then the equation*

$$[At^{p-1}\Phi(x')] + \frac{B}{t \log^p t} \Phi(x) = 0$$

is oscillatory.

Proof. See, e.g., [5, Theorem 1.4.4] (or directly [6] and [7]). \square

Theorem 4.2. *If $C, D \in \mathbb{R}^+$ satisfy $D/C < q^{-p}$, then the equation*

$$[Ct^{p-1}\Phi(x')] + \frac{D}{t \log^p t} \Phi(x) = 0$$

is non-oscillatory.

Proof. See [5, Theorem 1.4.4] (or [6, 7]). \square

Now we can prove the announced results which identify the critical oscillation constant for the analyzed equations with very general coefficients.

Theorem 4.3. *Let $\alpha, R, S \in \mathbb{R}^+$ be such that (3.2) is valid and $R^{p-1}S > q^{-p}$. If there exists $T > e$ with the property that*

$$\frac{1}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau) \, d\tau \geq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau \geq S, \quad t \in \mathbb{R}_T,$$

then (2.16) is oscillatory.

Proof. We study the equation for the Prüfer angle corresponding to the average function ψ . It is well-known that the non-oscillation of solutions of (2.16) is equivalent to the boundedness from above of the Prüfer angle φ given by (2.17). We refer, e.g., to [3, 4, 19, 24]. It also suffices to consider directly (2.6) and (2.17) when $\sin_p \varphi(t) = 0$. Based on Lemma 3.1, we know that the boundedness (from above) of φ is equivalent to the boundedness (from above) of ψ . Hence, we will show that ψ is unbounded from above. At first, we assume that (3.1) is true.

Taking into account Lemma 3.2, we have

$$\psi'(t) = \frac{1}{t \log t} \left[\frac{|\cos_p \psi(t)|^p}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau) \, d\tau - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \right]$$

$$\begin{aligned} & + \frac{|\sin_p \psi(t)|^p}{(p-1)\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau + \Psi(t) \Big] \\ \geq & \frac{1}{t \log t} \left[R |\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \right. \\ & \left. + S \frac{|\sin_p \psi(t)|^p}{p-1} + \Psi(t) \right] \end{aligned}$$

for all $t \in \mathbb{R}_T$ and for some continuous function $\Psi : \mathbb{R}_T \rightarrow \mathbb{R}$ satisfying

$$\lim_{t \rightarrow \infty} \Psi(t) = 0. \tag{4.1}$$

Let $\varepsilon > 0$ be arbitrary. From (2.5) and (4.1), we have

$$\varepsilon \left(|\cos_p x|^p + \frac{|\sin_p x|^p}{p-1} \right) > |\Psi(t)| \tag{4.2}$$

for all $x \in \mathbb{R}$ and for all large $t \in \mathbb{R}_T$. Thus,

$$\begin{aligned} \psi'(t) & > \frac{1}{t \log t} \left[(R - \varepsilon) |\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \right. \\ & \left. + (S - \varepsilon) \frac{|\sin_p \psi(t)|^p}{p-1} \right] \end{aligned} \tag{4.3}$$

for all large $t \in \mathbb{R}_T$. Let ε be so small that $(R - \varepsilon)^{p-1}(S - \varepsilon) > q^{-p}$ and $R - \varepsilon > 0$. Using Theorem 4.1 for $A = (R - \varepsilon)^{\frac{1}{1-q}}$ and $B = S - \varepsilon$, we know that the equation

$$[(R - \varepsilon)^{\frac{1}{1-q}} t^{p-1} \Phi(x')] + \frac{S - \varepsilon}{t \log^p t} \Phi(x) = 0$$

is oscillatory; i.e., any solution $\hat{\varphi} : \mathbb{R}_T \rightarrow \mathbb{R}$ of the equation

$$\begin{aligned} \hat{\varphi}'(t) & = \frac{1}{t \log t} \left[(R - \varepsilon) |\cos_p \hat{\varphi}(t)|^p - \Phi(\cos_p \hat{\varphi}(t)) \sin_p \hat{\varphi}(t) \right. \\ & \left. + (S - \varepsilon) \frac{|\sin_p \hat{\varphi}(t)|^p}{p-1} \right] \end{aligned} \tag{4.4}$$

has the property that $\limsup_{t \rightarrow \infty} \hat{\varphi}(t) = \infty$. Indeed, one can simply compute that

$$B/A = (R - \varepsilon)^{p-1}(S - \varepsilon) > q^{-p}.$$

Considering (4.3) with (4.4) and the $2\pi_p$ -periodicity of \sin_p and \cos_p , we see that $\limsup_{t \rightarrow \infty} \hat{\varphi}(t) = \infty$ implies $\limsup_{t \rightarrow \infty} \psi(t) = \infty$.

To finish the proof, we have to consider the case when (3.1) is not valid. Evidently, there exists a continuous function $\tilde{r} : \mathbb{R}_e \rightarrow \mathbb{R}^+$ with the properties

$$r^{1-q}(t) \geq \tilde{r}^{1-q}(t), \quad \frac{1}{\alpha} \int_t^{t+\alpha} \tilde{r}^{1-q}(\tau) \, d\tau \geq R, \quad t \in \mathbb{R}_T,$$

and

$$\lim_{t \rightarrow \infty} \frac{\int_t^{t+\alpha} \tilde{r}^{1-q}(\tau) \, d\tau}{\sqrt{t \log t}} = 0. \tag{4.5}$$

We actually know that the equation

$$[\tilde{r}(t)t^{p-1}\Phi(x')] + \frac{s(t)}{t \log^p t} \Phi(x) = 0$$

is oscillatory (cf. (3.1) and (4.5)). Since $r(t) \leq \tilde{r}(t)$ for all $t \in \mathbb{R}_T$, the Sturmian half-linear comparison theorem (see, e.g., [5, Theorem 1.2.4]) gives the oscillation of (2.16). □

Theorem 4.4. *Let $\alpha, R, S \in \mathbb{R}^+$ be such that (3.2) is valid and $R^{p-1}S < q^{-p}$. If there exists $T > e$ with the property that*

$$\frac{1}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau) d\tau \leq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau \leq S, \quad t \in \mathbb{R}_T, \quad (4.6)$$

then (2.16) is non-oscillatory.

Proof. We proceed analogously as in the proof of Theorem 4.3. In this proof, we show that ψ is bounded from above. Note that (3.1) is valid (see the first inequality in (4.6)). From Lemma 3.2, we obtain

$$\psi'(t) \leq \frac{1}{t \log t} \left[R |\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t)) \sin_p \psi(t) + S \frac{|\sin_p \psi(t)|^p}{p-1} + \Psi(t) \right]$$

for any $t \in \mathbb{R}_T$ and for a continuous function $\Psi : \mathbb{R}_T \rightarrow \mathbb{R}$ satisfying (4.1). In addition, using (4.2), we obtain

$$\begin{aligned} \psi'(t) &< \frac{1}{t \log t} \left[(R + \varepsilon) |\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \right. \\ &\quad \left. + (S + \varepsilon) \frac{|\sin_p \psi(t)|^p}{p-1} \right] \end{aligned} \quad (4.7)$$

for any $\varepsilon > 0$ and all sufficiently large $t \in \mathbb{R}_T$. We choose ε in such a way that $(R + \varepsilon)^{p-1}(S + \varepsilon) < q^{-p}$. We put $C = (R + \varepsilon)^{\frac{1}{1-q}}$ and $D = S + \varepsilon$ in Theorem 4.2. Since

$$D/C = (R + \varepsilon)^{p-1}(S + \varepsilon) < q^{-p},$$

we know that the equation

$$[(R + \varepsilon)^{\frac{1}{1-q}} t^{p-1} \Phi(x')] + \frac{S + \varepsilon}{t \log^p t} \Phi(x) = 0$$

is non-oscillatory. This fact means that any solution $\check{\varphi} : \mathbb{R}_T \rightarrow \mathbb{R}$ of the equation

$$\begin{aligned} \check{\varphi}'(t) &= \frac{1}{t \log t} \left[(R + \varepsilon) |\cos_p \check{\varphi}(t)|^p - \Phi(\cos_p \check{\varphi}(t)) \sin_p \check{\varphi}(t) \right. \\ &\quad \left. + (S + \varepsilon) \frac{|\sin_p \check{\varphi}(t)|^p}{p-1} \right] \end{aligned} \quad (4.8)$$

has the property that $\limsup_{t \rightarrow \infty} \check{\varphi}(t) < \infty$. Finally, considering (4.7) together with (4.8) and considering the $2\pi_p$ -periodicity of the generalized trigonometric functions, we have the inequality $\limsup_{t \rightarrow \infty} \psi(t) < \infty$. Therefore, the statement of the theorem is proven. \square

Now we mention definitions which enable us to formulate the below given Corollaries 4.8 and 4.10 (motivated by the results of [13, 27] recalled in Introduction).

Definition 4.5. Let $f : \mathbb{R}_e \rightarrow \mathbb{R}$ be a continuous function. The mean value $M(f)$ of f is defined as

$$M(f) := \lim_{t \rightarrow \infty} \frac{1}{t} \int_a^{a+t} f(\tau) d\tau$$

if the limit is finite and if it exists uniformly with respect to $a \in \mathbb{R}_e$.

Definition 4.6. A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called almost periodic if, for all $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ of the real line contains at least one point s for which

$$|f(t+s) - f(t)| < \varepsilon, \quad t \in \mathbb{R}.$$

As a direct generalization of the almost periodicity, we consider the notion of the so-called asymptotic almost periodicity.

Definition 4.7. We say that a continuous function $f : \mathbb{R}_e \rightarrow \mathbb{R}$ is asymptotically almost periodic if f can be expressed in the form $f(t) = f_1(t) + f_2(t)$, $t \in \mathbb{R}_e$, where f_1 is almost periodic and f_2 has the property that $\lim_{t \rightarrow \infty} f_2(t) = 0$.

Concerning coefficients with mean values, we obtain a new result which reads as follows.

Corollary 4.8. *Let continuous functions $r : \mathbb{R}_e \rightarrow \mathbb{R}^+$ and $s : \mathbb{R}_e \rightarrow \mathbb{R}$ be such that the mean values $M(r^{1-q}) \in \mathbb{R}^+$, $M(s) \in \mathbb{R}$ exist and let (3.2) be valid for some $\alpha \in \mathbb{R}^+$. Let $[M(r^{1-q})]^{p-1}M(s) \neq q^{-p}$. Then, (2.16) is oscillatory if and only if $[M(r^{1-q})]^{p-1}M(s) > q^{-p}$.*

Proof. The corollary follows from Theorems 4.3 and 4.4. Let $\varepsilon > 0$ be arbitrary. The existence of $M(r^{1-q})$ and $M(s)$ implies the existence of $n \in \mathbb{N}$ such that

$$\left| \frac{1}{n\alpha} \int_t^{t+n\alpha} r^{1-q}(\tau) d\tau - M(r^{1-q}) \right| < \varepsilon, \quad \left| \frac{1}{n\alpha} \int_t^{t+n\alpha} s(\tau) d\tau - M(s) \right| < \varepsilon$$

for all $t \in \mathbb{R}_e$. If $[M(r^{1-q})]^{p-1}M(s) > q^{-p}$, then it suffices to choose ε so that

$$M(r^{1-q}) - \varepsilon > 0, \quad [M(r^{1-q}) - \varepsilon]^{p-1}[M(s) - \varepsilon] > q^{-p},$$

to put $R = M(r^{1-q}) - \varepsilon$ and $S = M(s) - \varepsilon$, and to replace α by $n\alpha$ in Theorem 4.3. Obviously, (3.2) is true also for $n\alpha$. If $[M(r^{1-q})]^{p-1}M(s) < q^{-p}$, then we choose ε so that

$$M(s) + \varepsilon > 0, \quad [M(r^{1-q}) + \varepsilon]^{p-1}[M(s) + \varepsilon] < q^{-p},$$

we consider $R = M(r^{1-q}) + \varepsilon$ and $S = M(s) + \varepsilon$, and we replace α by $n\alpha$ in Theorem 4.4. \square

Remark 4.9. We point out that the requirement about the validity of (3.2) for some $\alpha \in \mathbb{R}^+$ cannot be omitted in the statement of Corollary 4.8. Indeed, the existence of $M(s)$ does not imply (3.2). We remark that, from the same reason, Theorem 1.1 does not follow from Theorem 1.2.

For asymptotically almost periodic coefficients, we get a new result as well. Again, we formulate it explicitly.

Corollary 4.10. *Let functions $r^{1-q} : \mathbb{R}_e \rightarrow \mathbb{R}^+$ and $s : \mathbb{R}_e \rightarrow \mathbb{R}$ be asymptotically almost periodic and let $[M(r^{1-q})]^{p-1}M(s) \in \mathbb{R}^+ \setminus \{q^{-p}\}$. Then, (2.16) is oscillatory if and only if $[M(r^{1-q})]^{p-1}M(s) > q^{-p}$.*

Proof. Since any asymptotically almost periodic function has mean value and it is bounded (see, e.g., [2, 9]), this corollary is a consequence of Corollary 4.8. \square

Remark 4.11. Let us pay our attention to (1.3). We repeat that the result about (1.3), which corresponds to Corollary 4.8, is proven in [27] and that the one, which corresponds to Corollary 4.10, is proven in [13].

Remark 4.12. In Corollaries 4.8 and 4.10, the case $[M(r^{1-q})]^{p-1}M(s) = q^{-p}$ cannot be solved as oscillatory or non-oscillatory for general coefficients (which have mean values or which are asymptotically almost periodic). We conjecture that this case is not possible to solve even for general almost periodic coefficients. Our conjecture is based on constructions of almost periodic functions mentioned in [22] (see also [21, 23]).

At the end, we give some examples to illustrate the proven results.

Example 4.13. Let us consider constants $u > 0$, $v \in \mathbb{R}$, and $w \neq 0$ and the function $h : \mathbb{R}_1 \rightarrow \mathbb{R}$ given by the formula

$$h(t) := \begin{cases} v + nw2^n(t - n), & t \in [n, n + \frac{1}{2^n}), n \in \mathbb{N}; \\ v + nw2^n(n + \frac{2}{2^n} - t), & t \in [n + \frac{1}{2^n}, n + \frac{2}{2^n}), n \in \mathbb{N}; \\ v, & t \in [n + \frac{2}{2^n}, n + 1), n \in \mathbb{N}. \end{cases}$$

We analyze the equation

$$\left[\frac{(tx')^3}{u} \right]' + \frac{h(t)}{t \log^4 t} x^3 = 0. \quad (4.9)$$

Hence, we deal with (2.16), where $p = 4$ and

$$r(t) = \frac{1}{u}, \quad s(t) = h(t), \quad t \in \mathbb{R}_e.$$

One can verify that

$$M(h) = v, \quad M(r^{1-q}) = M(r^{-\frac{1}{3}}) = M(u^{\frac{1}{3}}) = u^{\frac{1}{3}} > 0.$$

Therefore, applying Corollary 4.8 (condition (3.2) is trivially valid for all $\alpha \in \mathbb{R}^+$), we know that (4.9) is oscillatory when $4^4 uv > 3^4$ and non-oscillatory when $4^4 uv < 3^4$. Note that the second coefficient h has mean value, but it is not asymptotically almost periodic (it suffices to consider that $\limsup_{t \rightarrow \infty} |h(t)| = \infty$).

Example 4.14. For $a, b, c > 0$, we consider the equation

$$\left[\left(\frac{t^2 x'}{1 + t[1 + \sin t \cos t]} \right)^5 \right]' + \frac{|\sin(at)| + \arctan[\sin(bt) + \cos(bt)]}{c(t + \sqrt{t}) \log^6 t} x^5 = 0 \quad (4.10)$$

which is in the form of (2.16) with

$$r(t) = \left(\frac{t}{1 + t[1 + \sin t \cos t]} \right)^5, \quad t \in \mathbb{R}_e, \\ s(t) = \frac{t(|\sin(at)| + \arctan[\sin(bt) + \cos(bt)])}{c(t + \sqrt{t})}, \quad t \in \mathbb{R}_e,$$

and $p = 6$. Since r^{1-q} and s are asymptotically almost periodic functions (see, e.g., [2, 9]), we can use directly Corollary 4.10. We have

$$M(r^{1-q}) = M(r^{-1/5}) = M\left(\frac{1}{t} + 1 + \sin t \cos t\right) = 1,$$

$$cM(s) = M(|\sin(at)| + \arctan[\sin(bt) + \cos(bt)]) = M(|\sin(at)|) = \frac{2}{\pi}.$$

Therefore, (4.10) is oscillatory for

$$c < \bar{\Gamma} := \left(\frac{6}{5}\right)^6 \frac{2}{\pi}$$

and non-oscillatory for $c > \bar{\Gamma}$.

Example 4.15. Let $p = 3/2$ and let $f, g : [-1, 1] \rightarrow \mathbb{R}^+$ be continuous functions. We find the oscillation constant for the equation

$$[\sqrt{tf(\sin t)} \Phi(x')] + \frac{g(\sin t)}{t \log^p t} \Phi(x) = 0. \quad (4.11)$$

Evidently, the functions $r(t) = \sqrt{f(\sin t)}$ and $s(t) = g(\sin t)$ are periodic and

$$M(r^{-2}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{f(\sin t)}, \quad M(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\sin t) dt.$$

We can apply, e.g., Corollary 4.10. If

$$\widehat{\Gamma} := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\sin t) dt \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{f(\sin t)}} > 3^{-3/2},$$

then (4.11) is oscillatory. If $\widehat{\Gamma} < 3^{-3/2}$, then (4.11) is non-oscillatory.

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