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MILD SOLUTIONS FOR NON-AUTONOMOUS IMPULSIVE SEMI-LINEAR DIFFERENTIAL EQUATIONS WITH ITERATED DEVIATING ARGUMENTS

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Abstract. In this work, we consider an impulsive non-autonomous semi-linear equation with iterated deviating arguments in a Banach space. We establish the existence and uniqueness of a mild solution. Also we present an example that illustrates our main result.

1. Introduction

In the previous decades, impulsive differential equations have received much attention of researchers mainly because its demonstrated applications in widespread fields of science and engineering. Differential equation systems which are characterized by the occurrence of an abrupt change in the state of the system are known as impulsive differential equations. These changes occur at certain time instants over a period of negligible duration. Such process is investigated in various fields such as biology, physics, control theory, population dynamics, economics, chemical technology, medicine and so on. Impulsive differential equations are an appropriate model to hereditary phenomena for which a delay argument arises in the modelling equations. For more details for impulsive differential equation, we refer to the monographs [2, 20] and papers [3, 4, 5, 6, 19, 21, 22, 27] and references given therein.

In this article, we investigate the existence and uniqueness of solution for impulsive differential equation with iterated deviating arguments in a complex Banach space $(E, \|\cdot\|)$. We study the differential equation

$$\begin{aligned} \frac{d}{dt}[u(t)+G(t,u(a(t)))] &= -A(t)[u(t)+G(t,u(a(t)))] \\ &\quad + F(t,u(t),u(h_1(t,u(t)))), \quad t>0 \end{aligned} \tag{1.1}$$

geq1

$$u(0)=u_0, \quad u_0\in E, \tag{1.2}$$

$$\Delta u(t_i)=I_i(u(t_i)), \quad i=1,\dots,\delta\in\mathbb{N}, \tag{1.3}$$

geq2

where $h_1(t,u(t))=b_1(t,u(b_2(t,\dots,u(b_\delta(t,u(t)))\dots)))$ and $-A(t):D(A(t))\subseteq E\to E$, $t\geq 0$ is a closed densely defined linear operator. The functions F, b_i, G,

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$I_i : E \to E$ $(i = 1, \dots, \delta)$ are appropriate functions to be mentioned later. Here, $0 = t_0 < t_1 < \dots < t_\delta < t_{\delta+1} = T$ are fixed numbers, $0 < T < \infty$, $\Delta u|_{t=t_i} = u(t_i^+) - u(t_i^-)$ and $u(t_i^-) = \lim_{\varepsilon \to 0-} u(t_i + \varepsilon)$ and $u(t_i^+) = \lim_{\varepsilon \to 0+} u(t_i + \varepsilon)$ denote the left and right limits of $u(t)$ at $t = t_i$, respectively. In (1.1), $-A(t)$ is assumed to be the infinitesimal generator of an analytic semigroup of bounded linear operators on a Banach space E.

A Differential equation with a deviated argument is a valuable tool for the modeling of many phenomena in several fields of science and engineering such as theory of automatic control, biological systems, problems of long-term planning in economics, theory of self-oscillating systems, study of problems related to combustion in rocket motion and so on. The existence and uniqueness of the solution to the differential equations with deviated argument have been discussed by many authors (see [11]-[14]). For a detailed discussion of differential equations with iterated deviating arguments, we refer to monograph [7] and papers [11, 12, 14, 17, 19, 21, 23, 25, 28, 29] and references given therein.

The existence and uniqueness of the solution for the following problem with a deviated argument has been established by Gal [11],

$$u'(t) = Au(t) + F(t, u(t), u(h(u(t), t))), \quad t > 0, \tag{1.4}$$

$$u(0) = u_0, \quad u_0 \in E, \tag{1.5}$$

in a Banach space $(E, \|\cdot\|)$. Where $-A$ generates an analytic semigroup of bounded linear operators on E and the function $F : [0, \infty) \times E_\alpha \times E_{\alpha-1} \to E$, $h : E_\alpha \times [0, \infty) \to [0, \infty)$ are Hölder continuous with exponent $\mu_1 \in (0, 1]$ and $\mu_2 \in (0, 1]$ respectively. For $0 < \alpha \le 1$, E_α denotes the domain of $(-A)^\alpha$ which is a Banach space with the norm $\|u\|_\alpha = \|(-A)^\alpha u\|$, $u \in D((-A)^\alpha)$.

In [14], authors considered the following problem in a Banach space $(E, \|\cdot\|)$,

$$u'(t) + A(t)u(t) = F(t, u(t), u(h(u(t), t))), \quad t > 0, \tag{1.6}$$ req1

$$u(0) = u_0, \quad u_0 \in E, \tag{1.7}$$

where A is a closed, densely defined linear operator with domain $D(A) \subset E$. In (1.6), $-A$ generates an analytic semigroup of bounded linear operators on Banach space E. The function $F : \mathbb{R} \times E_\alpha \times E_{\alpha-1} \to E$, $h : E_\alpha \times \mathbb{R}_+ \to [0, \infty)$ are appropriated functions. The authors have established the existence of the solution for (1.6) by using Banach fixed point theorem.

The rest of this article is organized as follows: Section 2 provides some basic definitions, lemmas and theorems, assumptions as these are useful for proving our results. Section 3 focuses on the existence of a mild solution to problem (1.1)-(1.3). Section 4 present an example to illustrate the theory.

2. Preliminaries

In this section, we provide basic definitions, preliminaries, lemmas and assumptions which are useful for proving main result in later section.

Throughout the work, we assume that $(E, \|\cdot\|)$ is a complex Banach space. The notation $C([0, T], E)$ stands for the space of E-valued continuous functions on $[0, T]$ with the norm $\|z\| = \sup\{\|z(\tau)\|, \tau \in [0, T]\}$ and $L^1([0, T], E)$ denotes the space of E-valued Bochner integrable functions on $[0, T]$ endowed with the norm $\|\mathcal{F}\|_{L^1} = \int_0^T \|\mathcal{F}(t)\| dt$, $\mathcal{F} \in L^1([0, T], E)$. We denote by $C^\beta([0, T]; E)$ the space of all uniformly Hölder continuous functions from $[0,T]$ into E with exponent $\beta > 0$. It is easy to verify that $C^\beta([0,T];E)$ is a Banach space with the norm

$$\|y\|_{C^\beta([0,T];E)} = \sup_{0\le t\le T}\|y(t)\| + \sup_{0\le t,s\le T,\ t\ne s}\frac{\|y(t)-y(s)\|}{|t-s|^\beta}. \tag{2.1}$$

Let $\{A(t) : 0 \le t \le T\}$, $T \in [0,\infty)$ be a family of closed linear operators on the Banach space E. We impose following restrictions as [8]:

- (P1) The domain $D(A)$ of $\{A(t) : t \in [0,T]\}$ is dense in E and $D(A)$ is independent of t.
- (P2) For each $0 \le t \le T$ and $\operatorname{Re}\lambda \le 0$, the resolvent $R(\lambda; A(t))$ exists and there exists a positive constant K (independent of t and λ) such that
$$\|R(\lambda;A(t))\| \le K/(|\lambda|+1), \quad \operatorname{Re}\lambda \le 0,\ t\in[0,T].$$
- (P3) For each fixed $\xi \in [0,T]$, there are constants $K > 0$ and $0 < \mu \le 1$ such that
$$\|[A(\tau)-A(s)]A^{-1}(\xi)\| \le K|\tau-s|^\mu, \quad \text{for all } \tau,s\in[0,T] \tag{2.2}$$
where μ and K are independent of τ, s and ξ.

The assumptions (P1)–(P3) allow the existence of a unique linear evolution system (linear evolution operator) $U(t,s)$, $0 \le s \le t \le T$ which is generated by the family $\{A(t) : t \in [0,T]\}$ and there exists a family of bounded linear operators $\{\Phi(t,s) : 0 \le s \le t \le T\}$ such that $\|\Phi(t,s)\| \le \frac{K}{|t-s|^{1-\mu}}$. We also have that $U(t,s)$ can be written as

$$U(t,s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(\tau)}\Phi(\tau,s)d\tau. \tag{2.3}$$

Assumption (P2) guarantees that $-A(s)$, $s \in [0,T]$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \ge 0\}$ in $B(E)$, where the symbol $B(E)$ stands for the Banach algebra of all bounded linear operators on E.

The assumptions (P1)–(P3) allow the existence of a unique fundamental solution $\{U(t,s) : 0 \le s \le t \le T\}$ for the homogenous Cauchy problem such that

- (i) $U(t,s) \in B(E)$ and the mapping $(t,s) \to U(t,s)z$ is continuous for $z \in E$, i.e., $U(t,s)$ is strongly continuous in t,s for all $0 \le s \le t \le T$.
- (ii) For each $z \in E$, $U(t,s)z \in D(A)$, for all $0 \le s \le t \le T$.
- (iii) $U(t,\tau)U(\tau,s) = U(t,s)$ for all $0 \le s \le \tau \le t \le T$.
- (iv) For each $0 \le s < t \le T$, the derivative $\frac{\partial U(t,s)}{\partial t}$ exists in the strong operator topology and an element of $B(E)$, and strongly continuous in t, where $s < t \le T$.
- (v) $U(t,t) = I$.
- (vi) $\frac{\partial U(t,s)}{\partial t} + A(t)U(t,s) = 0$ for all $0 \le s < t \le T$.

We have also the following inequalities:

$$\|e^{-tA(\tau)}\| \le Ke^{-dt},\ \ t \ge 0; \tag{2.4}$$

$$\|A(\tau)e^{-tA(\tau)}\| \le \frac{Ke^{-dt}}{t},\ \ t > 0, \tag{2.5}$$

$$\|A(t)U(t,\tau)\| \le K|t-\tau|^{-1},\ \ 0 \le \tau \le t \le T. \tag{2.6}$$

for all $\tau \in [0,T]$. Where d is a positive constant. For $\alpha > 0$, we may define the negative fractional powers $A(t)^{-\alpha}$ as

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-sA(t)} ds. \tag{2.7}$$

Then, the operator $A(t)^{-\alpha}$ is a bounded linear and one to one operator on E. Therefore, it implies that there exists an inverse of the operator $A(t)^{-\alpha}$. We can define $A(t)^{\alpha} \equiv [A(t)^{-\alpha}]^{-1}$ which is the positive fractional powers of $A(t)$. The operator $A(t)^{\alpha} \equiv [A(t)^{-\alpha}]^{-1}$ is a closed densely defined linear operator with domain $D(A(t)^{\alpha}) \subset E$ and for $\alpha < \beta$, we get $D(A(t)^{\beta}) \subset D(A(t)^{\alpha})$. Let $E_\alpha = D(A(0)^{\alpha})$ be a Banach space with the norm $\|y\|_\alpha = \|A(0)^{\alpha} y\|$. For $0 < \omega_1 \leq \omega_2$, we have that the embedding $E_{\omega_2} \hookrightarrow E_{\omega_1}$ is continuous and dense. For each $\alpha > 0$, we may define $E_{-\alpha} = (E_\alpha)^*$, which is the dual space of E_α. The dual space is a Banach space with natural norm $\|y\|_{-\alpha} = \|A(0)^{-\alpha} y\|$.

In particular, by the assumption (P3), we conclude a constant $K > 0$, such that

$$\|A(t)A(s)^{-1}\| \leq K, \quad \text{for all } 0 \leq s,t \leq T. \tag{2.8}$$

For $0 < \alpha \leq 1$, let U_α and $U_{\alpha-1}$ be open sets in E_α and $E_{\alpha-1}$, respectively. For every $v' \in U_\alpha$ and $v'' \in U_{\alpha-1}$, there exist balls such that $B_\alpha(v', r') \subset U_\alpha$ and $B_{\alpha-1}(v'', r'') \subset U_{\alpha-1}$, for some positive numbers r' and r''. Let F, a, h and I_i $(i = 1, \dots, \delta)$ be the continuous functions satisfying following conditions:

(P4) The nonlinear map $F : [0,T] \times U_\alpha \times U_{\alpha-1} \to E$ is a Hölder continuous and there exist positive constants $L_F \equiv L_F(t, v', v'', r', r'')$ and $0 < \mu_1 \leq 1$ such that

$$\begin{aligned} &\|F(t, z_1, w_1) - F(s, z_2, w_2)\| \\ &\leq L_F(|t-s|^{\mu_1} + \|z_1 - z_2\|_\alpha + \|w_1 - w_2\|_{\alpha-1}), \end{aligned} \tag{2.9}$$

Feq1

for all $(z_1, w_1), (z_2, w_2) \in B_\alpha \times B_{\alpha-1}$ and $s, t \in [0,T]$.

(P5) The functions $b_i : [0,\infty) \times U_{\alpha-1} \to [0,\infty)$, $(i = 1, \dots, \delta)$ are continuous functions and there are positive constants $L_{b_i} \equiv L_{b_i}(t, v', r')$ and $0 < \mu_2 \leq 1$ such that

$$|b_i(t,z) - b_i(s,w)| \leq L_{b_i}(|t-s|^{\mu_2} + \|z - w\|_{\alpha-1}), \tag{2.10}$$

aeq1

for all $(t,z),\ (s,w) \in [0,T] \times B_\alpha$.

(P6) For $0 \leq \alpha < \beta < 1$, $G : [0,T] \times U_{\alpha-1} \to E_\beta$ is a continuous map and there exists a positive constant $L_G = L_G(t, v'', r'', \beta)$ such that

$$\|A^\beta G(t_1, z_1) - A^\beta G(t_2, z_2)\| \leq L_G[|t_1 - t_2| + \|z_1 - z_2\|_{\alpha-1}], \tag{2.11}$$

$$4L_G\|A(0)^{\alpha-\beta-1}\| < 1, \tag{2.12}$$

for each $(t_1, z_1),\ (t_2, z_2) \in [0,T] \times B_{\alpha-1}$.

(P7) The function $a : [0,T] \to [0,T]$ is a continuous function and satisfies the following conditions:

(i) $a(t) \leq t$ for all $t \in [0,T]$.

(ii) There exist a constant $L_a > 0$ such that

$$|a(t_1) - a(t_2)| \leq L_a|t_1 - t_2|, \tag{2.13}$$

for all $t_1, t_2 \in [0,T]$ and $L_a\|A^{-1}\| < 1$.

(P8) $I_i : U_\alpha \to U_\alpha$ $(i = 1, \dots, \delta)$ are continuous functions and there exist positive constants $L_i \equiv L_i(t, v', r')$ such that

$$\|I_i(z) - I_i(w)\|_\alpha \le L_i \|z - w\|_\alpha, \quad i = 1, \dots, \delta, \tag{2.14}$$

$$\|I_i(z)\| \le C_i, \;\; i = 1, \dots, \delta, \tag{2.15}$$

for all $z, w \in B_\alpha$, where C_i are positive constants.

Now, we turn to the Cauchy problem which is illustrated as follows,

$$u'(t) = -A(t)u(t) + f(t), \tag{2.16}$$

$$u(t_0) = u_0, \quad t \ge 0. \tag{2.17}$$

cheq1 cheq2

thm1 **Theorem 2.1** ([26]). *Assume that* (P1)–(P3) *hold. If f is a Hölder continuous function from $[t_0, T]$ into E with exponent β. Then, there exists a unique solution of the problem* (2.16)-(2.17) *given by*

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^{t} U(t, s) f(s) ds, \quad \forall t_0 \le t \le T. \tag{2.18}$$

Indeed, $u : [t_0, T] \to E$ is strongly continuously differentiable solution on $(t_0, T]$.

We also have following results.

lem1 **Lemma 2.2** ([8]). *Suppose that* (P1)–(P3) *are satisfied. If $0 \le \gamma \le 1$, $0 \le \beta \le \alpha < 1 + \mu$, $0 < \alpha - \gamma \le 1$, then for any $0 \le \tau < t < t + \Delta t \le t_0$, $0 \le \zeta \le T$,*

$$\|A^\gamma(\zeta)[U(t + \Delta t, \tau) - U(t, \tau)]A^{-\beta}(\tau)\| \le K(\gamma, \beta, \alpha)(\Delta t)^{\alpha - \gamma}|t - \tau|^{\beta - \alpha}. \tag{2.19}$$

lem2 **Lemma 2.3** ([8]). *Suppose that* (P1)–(P3) *are satisfied and let $0 \le \gamma < 1$. Then for any $0 \le \tau \le t \le t + \Delta t \le t_0$ and for any continuous function $f(s)$,*

$$\begin{aligned} &\|A^\gamma(\zeta)[\int_t^{t+\Delta t} U(t + \Delta t, s) f(s) ds - \int_\tau^t U(t, s) f(s) ds]\| \\ &\le K(\gamma)(\Delta t)^{1-\gamma}(|\log(\Delta t)| + 1) \max_{\tau \le s \le t + \Delta t} \|f(s)\|. \end{aligned} \tag{2.20}$$

For more details, we refer to the monographs [8, 26].

3. Existence result

In this section, the existence of mild solution for the problem (1.1)–(1.3) is established by using fixed point theorem. Let $(E, \|\cdot\|)$ be a complex Banach space. The symbol $\mathcal{C}_\alpha^{T_0}$ denotes the Banach space of all E_α-valued continuous functions on $J = [0, T_0]$, $0 < T_0 < T < \infty$ endowed with the sup-norm $\sup_{t \in J} \|z(t)\|$, $z \in C(J; E_\alpha)$.

We choose T_0 sufficiently small, $0 < T_0 < T$ such that

$$\|(U(t, 0) - I)(u_0 + G(0, u_0))\|_\alpha + K(\alpha) \sum_{0 < t_i < t} C_i \le \frac{r}{6}, \quad \forall t \in [0, T_0], \tag{3.1}$$

$$\|G(t, u(a(t))) - G(0, u_0)\|_\alpha \le \frac{r}{6}, \quad t \in [0, T_0], \tag{3.2}$$

$$K(\alpha) N \frac{{T_0}^{1-\alpha}}{1 - \alpha} \le \frac{2r}{3}, \quad \forall t \in [0, T_0]. \tag{3.3}$$

We define

$$
\begin{aligned}
Y &= PC([0,T_0];E_\alpha) = PC(E_\alpha)\\
&= \big\{u : J \to E_\alpha : u \in C((t_i,t_{i+1}],E_\alpha),\ i = 1,\dots,\delta\\
&\quad \text{and } u(t_i^+), u(t_i^-) = u(t_i) \text{ exist}\big\}.
\end{aligned}
\tag{3.4}
$$

Clearly, the space Y is a Banach space with the supremum norm

$$
\|u\|_{PC,\alpha} = \max\{\sup_{t\in J}\|u(t+0)\|_\alpha,\ \sup_{t\in J}\|u(t-0)\|_\alpha\}. \tag{3.5}
$$

Consider

$$
\begin{aligned}
Y_{\alpha-1} = PC_{\mathcal{L}}(J;E_{\alpha-1}) &= \big\{u \in Y : \|u(t)-u(s)\|_{\alpha-1} \le \mathcal{L}|t-s|,\\
&\text{for all } t,s \in (t_i,t_{i+1}],\ i = 0,1,\dots,\delta\big\},
\end{aligned}
\tag{3.6}
$$

where $\mathcal{L} > 0$ is a constant to be defined later. It is easy to see that $Y_{\alpha-1}$ is a Banach space under the supremum norm of $\mathcal{C}_\alpha^{T_0} = C(J,E_\alpha)$.

Before expressing and demonstrating the main result, we present the definition of the mild solution to the problem (1.1)-(1.3).

def3.1 **Definition 3.1.** A piecewise continuous function $u(\cdot) : [0,T_0] \to E$ is called a mild solution for the problem (1.1)-(1.3) if $u(0) = u_0$ and $u(\cdot)$ satisfies the integral equation

$$
\begin{aligned}
u(t) &= U(t,0)[u_0 + G(0,u_0)] - G(t,u(a(t)))\\
&\quad + \int_0^t U(t,s)F(s,u(s),u(h_1(s,u(s))))ds + \sum_{0<t_i<t} U(t,t_i)I_i(u(t_i^-)).
\end{aligned}
\tag{3.7}
$$

Let $0 < \eta < \beta - \alpha$ be the fixed constants. For $0 < \alpha \le 1$, let

$$
\begin{aligned}
\mathcal{S}_\alpha = \big\{y \in Y \cap Y_{\alpha-1} &: y(0) = u_0,\ \sup_{t\in J}\|y(t)-u_0\|_\alpha \le r,\\
&\|y(t_1)-y(t_2)\|_\alpha \le P|t_1-t_2|^\eta \text{ for all } t_1,t_2 \in J\big\},
\end{aligned}
\tag{3.8}
$$

where P and r are positive constants to be defined later. Thus, $\mathcal{S}_\alpha$ is a non-empty closed and bounded subset of $Y_{\alpha-1}$. Next, we prove the following theorem for the existence of a mild solution to the problem (1.1). We adopt the ideas of Friedman [8] and Gal [11] to prove the theorem.

thm3.1 **Theorem 3.2.** *Let $u_0 \in E_\beta$, where $0 < \alpha < \beta \le 1$. Suppose that assumptions* (P1)–(P8) *are satisfied and*

$$
\|A(0)^{\alpha-\beta}\|L_G + K(\alpha)L_F(2+\mathcal{L}L_b)\frac{T_0^{1-\alpha}}{(1-\alpha)} + K(\alpha)\sum_{i=1}^{\delta} L_i < 1. \tag{3.9}
$$

thmeq1

Then, there exists a unique solution $u(t) \in \mathcal{S}_\alpha$ for the problem (1.1)-(1.3) *on* $[0,T_0]$.

Proof. Let us assume that $u_0 \in E_\beta$. We define a map $Q : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$ by

$$
\begin{aligned}
Qu(t) &= U(t,0)(u_0 + G(0,u_0)) - G(t,u(a(t)))\\
&\quad + \int_0^t U(t,s)F(s,u(s),u(h_1(u(s),s)))ds\\
&\quad + \sum_{0<t_i<t} U(t,t_i)I_i(u(t_i^-)), \quad u \in S_\alpha,\ t \in [0,T_0].
\end{aligned}
\tag{3.10}
$$

We firstly claim that $Q(\mathcal{S}_\alpha) \subset \mathcal{S}_\alpha$. Clearly, it can easily be shown that $Qu \in Y$. Now, we want to show that $Qu \in Y_{\alpha-1}$. Indeed, if $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$, then we have

$$
\begin{aligned}
\|Qu(\tau_2) - Qu(\tau_1)\|_{\alpha-1} &\le \|[U(\tau_2,0) - U(\tau_1,0)](u_0 + G(0,u_0))\|_{\alpha-1} \\
&\quad + \|G(\tau_2, u(a(\tau_2))) - G(\tau_1, u(a(\tau_1)))\|_{\alpha-1} \\
&\quad + \Big\| \Big[\int_0^{\tau_2} U(\tau_2,s) F(s,u(s),u(h_1(u(s),s)))ds \\
&\quad - \int_0^{\tau_1} U(\tau_1,s) F(s,u(s),u(h_1(u(s),s)))ds \Big] \Big\|_{\alpha-1} \\
&\quad + \sum_{0<t_i<t} \|[U(\tau_2,t_i) - U(\tau_1,t_i)] I_i(u(t_i^-))\|_{\alpha-1},
\end{aligned}
\tag{3.11}
$$

eqcts1

From Lemma 2.2 for the first term on the right hand side of (3.11), we obtain

$$
\|[U(\tau_2,0) - U(\tau_1,0)](u_0 + G(0,u_0))\|_{\alpha-1} \le K_1 \|(u_0, G(0,u_0))\|_\alpha (\tau_2 - \tau_1), \tag{3.12}
$$

fteq1

where K_1 is some positive constant. By the assumptions $(P6)$ and $(P7)$, it follows that

$$
\|G(\tau_2, u(a(\tau_2))) - G(\tau_1, u(a(\tau_1)))\|_{\alpha-1} \le K_2 |\tau_2 - \tau_1|, \tag{3.13}
$$

where $K_2 = \|A(0)^{\alpha-\beta-1}\| L_G (1 + \mathcal{L} L_a)$ is a positive constant. By using Lemma 2.3 [8, Lemma 14.4], third term on the right hand side of the inequality (3.11) can be calculated as

$$
\begin{aligned}
&\Big\| \Big[\int_0^{\tau_2} U(\tau_2,s) F(s,u(s),u(h_1(u(s),s)))ds \\
&- \int_0^{\tau_1} U(\tau_1,s) F(s,u(s),u(h_1(u(s),s)))ds \Big] \Big\|_{\alpha-1} \\
&\le K_3 N \big(|\log(\tau_2 - \tau_1)| + 1 \big) (\tau_2 - \tau_1),
\end{aligned}
\tag{3.14}
$$

steq1

where $N = \sup_{0 \le s \le T} \|F(s,u(s),u(h_1(u(s),s)))\|$ and K_3 are positive constants depending on α. By Lemma 2.2, we conclude the last term of the right hand side of (3.11),

$$
\|[U(\tau_2,t_i) - U(\tau_1,t_i)] I_i(u(t_i^-))\| \le K_4 C_i (\tau_2 - \tau_1), \tag{3.15}
$$

tteq1

where K_4 is some positive constant and

$$
\sum_{0<t_i<t} \|I_i(u(t_i^-))\| \le \sum_{0<t_i<t} C_i, \quad i = 1, \dots, \delta.
$$

From equations (3.11)-(3.14) and (3.15), we obtain

$$
\|Qu(\tau_2) - Qu(\tau_1)\|_{\alpha-1} \le \mathcal{L} |\tau_2 - \tau_1|, \tag{3.16}
$$

where $\mathcal{L}$ is a constant such that

$$
\begin{aligned}
\mathcal{L} = \max \Big\{ &K_1(u_0, G(0,u_0)), \frac{\|A(0)^{\alpha-\beta-1}\| L_G}{1 - L_a L_G \|A(0)^{\alpha-\beta-1}\|}, \\
&K_3 N (\log |(\tau_2 - \tau_1)| + 1), \sum_{0<t_i<t} K_4 C_i \},
\end{aligned}
$$

which depends on K_1, K_2, K_3, N, T_0. Thus, we get $Qu \in Y_\alpha$.

Next, we show that $\sup_{t\in J}\|(Qu)(t)-u_0\|_\alpha \le r$ for $t\in[0,T_0]$. Since $u_0\in E_\alpha$. For $u\in\mathcal{S}_\alpha$, we have

$$\begin{aligned}\|Qu(t)-u_0\|_\alpha &\le \|(U(t,0)-I)(u_0+G(0,u_0))\|_\alpha + \|G(t,u(a(t)))-G(0,u_0)\|_\alpha\\ &\quad+\int_0^t \|U(t,s)F(s,u(s),u(h_1(u(s),s)))\|_\alpha ds\\ &\quad+\sum_{0<t_i<t}\|U(t,t_i)I_i(u(t_i^-))\|_\alpha,\end{aligned}\tag{3.17}$$

beq12

Since $u_0\in E_\alpha$ and $u_0+G(0,u_0)\in E_\alpha$ and for $t\in[0,T_0]$, we have following inequalities

$$\|(U(t,0)-I)(u_0+G(0,u_0))\|_\alpha + K(\alpha)\sum_{0<t_i<t} C_i \le \frac{r}{6},\quad \forall t\in[0,T_0],\tag{3.18}$$

beq2

$$\|G(t,u(a(t)))-G(0,u_0)\|_\alpha \le L_G[T_0+r]\le \frac{r}{6},\quad t\in[0,T_0],\tag{3.19}$$

beq01

$$K(\alpha)N\frac{{T_0}^{1-\alpha}}{1-\alpha}\le\frac{2r}{3},\quad \forall t\in[0,T_0].\tag{3.20}$$

beq4

We estimate the third term on the right hand side of equation (3.17) as [see [8, (14.13)page 160 and Line 12 page 163]]

$$\begin{aligned}\Big\|\int_0^t U(t,s)F(s,u(s),u(h_1(u(s),s)))ds\Big\|_\alpha &\le K(\alpha)N\int_0^t (t-s)^{-\alpha}ds\\ &\le K(\alpha)N\frac{{T_0}^{1-\alpha}}{1-\alpha}.\end{aligned}\tag{3.21}$$

beq3

Thus, from (3.17), (3.18),(3.20) and (3.21), we conclude that

$$\sup_{t\in J}\|(Qu)(t)-u_0\|_\alpha\le r,\quad t\in[0,T_0],\tag{3.22}$$

Now, we show that $\|Qu(t+h)-Qu(t)\|_\alpha\le Ph^\eta$ for $0<\eta<1$ and some positive constant P. If $0\le t\le t+h\le T_0$, then for $0\le\alpha<\beta\le 1$, we have

$$\begin{aligned}&\|Qu(t+h)-Qu(t)\|_\alpha\\ &\le \|[U(t+h,0)-U(t,0)](u_0+G(0,u_0)\|_\alpha + \|G(t+h,u(a(t)))-G(t,u(a(t)))\|_\alpha\\ &\quad+\Big\|\Big[\int_0^{t+h} U(t+h,s)F(s,u(s),u(h_1(u(s),s)))ds\\ &\quad-\int_0^t U(t,s)F(s,u(s),u(h_1(u(s),s)))ds\Big]\Big\|_\alpha\\ &\quad+\sum_{0<t_i<t}\|[U(t+h,t_i)-U(t,t_i)]I_i(u(t_i^-))\|_\alpha,\end{aligned}\tag{3.23}$$

kp

From Lemmas 2.2 and 2.3, [8, Lemmas 14.1, 14.4], we obtain the following results

$$\|[U(t+h,0)-U(t,0)]u_0\|_\alpha\le K(\alpha)\|u_0+G(0,u_0))\|_\beta h^{\beta-\alpha},\tag{3.24}$$

peq1

$$\|[U(t+h,t_i)-U(t,t_i)]I_i(u(t_i^-))\|_\alpha\le K(\alpha)h^{\beta-\alpha}\|I_i(u(t_i^-))\|_\beta,\tag{3.25}$$

peq2

$$\|G(t+h,u(a(t+h)))-G(t,u(a(t)))\|_\alpha\le\|A(0)^{\alpha-\beta}\|L_G(1+L_a\mathcal{L})h\tag{3.26}$$

peq21

$$\begin{aligned}
&\| \int_0^{t+h} U(t+h,s)F(s,u(s),u(h_1(u(s),s)))ds \\
&- \int_0^t U(t,s)F(s,u(s),u(h_1(u(s),s)))ds\|_\alpha \\
&\le K(\alpha)Nh^{1-\alpha}(1+|\log(h)|).
\end{aligned} \tag{3.27}$$

peq3

Using (3.24)-(3.27) in (3.23), we obtain

$$\begin{aligned}
&\|Qu(t+h)-Qu(t)\|_\alpha \\
&\le h^\eta[K\|u_0+G(0,u_0)\|{T_0}^{\beta-\alpha-\eta} + \|A(0)^{\alpha-\beta}\|L_G(1+L_a\mathcal{L})h^{1-\gamma} \\
&\quad + K(\alpha)N{T_0}^\varsigma h^{1-\alpha-\eta-\varsigma}(1+|\log(h)|) + K(\alpha)h^{\beta-\alpha-\eta}\|I_i(u(t_i^-))\|_\beta],
\end{aligned} \tag{3.28}$$

where $\varsigma > 0$ is a positive constant and $\varsigma < 1-\alpha-\eta$. Thus, for $t \in [0,T_0]$,

$$\|Qu(t+h)-Qu(t)\| \le Ph^\eta, \tag{3.29}$$

for $P > 0$ defined as

$$\begin{aligned}
P &= K\|u_0+G(0,u_0)\|{T_0}^{\beta-\alpha-\eta} + \|A(0)^{\alpha-\beta}\|L_G(1+L_a\mathcal{L})h^{1-\gamma} \\
&\quad + K(\alpha)N{T_0}^\varsigma h^{1-\alpha-\eta-\varsigma}(1+|\log(h)|) + K(\alpha)h^{\beta-\alpha-\eta}\|I_i(u(t_i^-))\|_\beta.
\end{aligned} \tag{3.30}$$

Hence $Q : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$. Now, it remains to show that Q is a contraction map. For $z_1, z_2 \in \mathcal{S}_\alpha$ and $t \in [0,T_0]$, we have

$$\begin{aligned}
&\|(Qz_1)(t)-(Qz_2)(t)\|_\alpha \\
&\le \|G(t,z_1(a(t)))-G(t,z_2(a(t)))\|_\alpha \\
&\quad + K(\alpha)\int_0^t (t-s)^{-\alpha}[\|F(s,z_1(s),z_1(h_1(s,z_1(s)))) \\
&\quad - F(s,z_2(s),z_2(h_1(s,z_2(s))))\|]ds \\
&\quad + \sum_{0<t_i<t} \|U(t,t_i)[I_i(z_1(t_i^-)) - I_i(z_2(t_i^-))]\|_\alpha.
\end{aligned} \tag{3.31}$$

conteq1

Now, we estimate

$$\begin{aligned}
&\|F(t,z_1(t),z_1(h_1(t,z_1(t)))) - F(t,z_2(t),z_2(h_1(t,z_2(t))))\| \\
&\le L_F[\|z_1(t)-z_2(t)\|_\alpha + \|z_1(h_1(t,z_1(t))) - z_2(h_1(t,z_2(t)))\|_{\alpha-1}] \\
&\le L_F[\|z_1(t)-z_2(t)\|_\alpha + \|A^{-1}\|\,\|z_1(h_1(t,z_2(t))) - z_2(h_1(t,z_2(t)))\|_\alpha \\
&\quad + \|z_1(h_1(t,z_1(t))) - z_1(h_1(t,z_2(t)))\|_{\alpha-1}].
\end{aligned} \tag{3.32}$$

Let

$$h_j(t,u(t)) = b_j(t,u(b_{j+1}(t,\dots u(t,b_\delta(t,u(t)))\dots))), \quad j=1,2,\dots,\delta,\ u \in \mathcal{S}_\alpha,$$

with $h_{\delta+1}(t,u(t)) = t$ [29, p. 2183].

Using the bounded inclusion $E_\alpha \hookrightarrow E_{\alpha-1}$, we obtain

$$\begin{aligned}
&\|z_1(h_j(t,z_2(t))) - z_2(h_j(t,z_2(t)))\|_{\alpha-1} \\
&= \|A^{\alpha-1}z_1(h_j(t,z_2(t))) - z_2(h_j(t,z_2(t)))\|, \\
&\le \|A^{-1}\| \times \|z_1(h_j(t,z_2(t))) - z_2(h_j(t,z_2(t)))\|_\alpha.
\end{aligned} \tag{3.33}$$

Since $h_j \in \mathbb{R}^+$, we have

$$\|z_1(t)-z_2(t)\|_\alpha = \sup_{h_j(t,z_2(t))\in[0,t]} \|z_1(h_j(t,z_2(t))) - z_2(h_j(t,z_2(t)))\|_\alpha.$$

Therefore,

$$\begin{aligned}\|z_1(h_j(t,z_2(t))) - z_2(h_j(t,z_2(t)))\|_{\alpha-1} &\le \|(A)^{-1}\| \sup_{t\in[0,T_0]} \|z_1(t)-z_2(t)\|_\alpha,\\ &\le \|A^{-1}\| \times \|z_1-z_2\|_{\mathcal{PC},\alpha}.\end{aligned}$$

Thus, we can estimate

$$\begin{aligned}&|h_1(t,z_1(t)) - h_1(t,z_2(t))|\\ &= |b_1(t,z_1(h_2(t,z_1(t)))) - b_1(t,z_2(h_2(t,z_2(t))))|,\\ &\le L_{b_1}\|z_1(h_2(t,z_1(t))) - z_2(h_2(t,z_2(t)))\|_{\alpha-1},\\ &\le L_{b_1}[\|z_1(h_2(t,z_1(t))) - z_1(h_2(t,z_2(t)))\|_{\alpha-1},\\ &\quad + \|z_1(h_2(t,z_2(t))) - z_2(h_2(t,z_2(t)))\|_{\alpha-1}],\\ &\le L_{b_1}[\mathcal{L}|b_2(t,z_1(h_3(t,z_1(t)))) - b_2(t,z_2(h_3(t,z_2(t))))| + \|A^{-1}\| \times \|z_1-z_2\|_{\mathcal{PC},\alpha}],\\ &\dots,\\ &\le [\mathcal{L}^{\delta-1}L_{b_1}\dots L_{b_\delta} + \mathcal{L}^{\delta-2}L_{b_1}\dots L_{b_{\delta-1}} + \dots + \mathcal{L}L_{b_1}L_{b_2}\\ &\quad + L_{b_1}]\|A^{-1}\| \times \|z_1-z_2\|_{\mathcal{PC},\alpha}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|F(t,z_1(t),z_1(h_1(t,z_1(t)))) - F(t,z_2(t),z_2(h_1(t,z_2(t))))\|\\ &\le L_F(2+\mathcal{L}L_b\|A^{-1}\|)\|z_1-z_2\|_{\mathcal{PC},\alpha}\\ &\le L_F(2+\mathcal{L}L_b)\|z_1-z_2\|_{\mathcal{PC},\alpha},\end{aligned} \tag{3.34}$$

feq3

where $L_b = [\mathcal{L}^{\delta-1}L_{b_1}\dots L_{b_\delta} + \mathcal{L}^{\delta-2}L_{b_1}\dots L_{b_{\delta-1}} + \dots + \mathcal{L}L_{b_1}L_{b_2} + L_{b_1}] > 0$. Similarly,

$$\|G(t,z_1(a(t))) - G(t,z_2(a(t)))\|_\alpha \le \|A(0)^{\alpha-\beta}\|L_G\big[\|z_1(t)-z_2(t)\|_\alpha\big]. \tag{3.35}$$

geq3

Using inequalities (3.34), (3.35) in (3.31), we deduce that

$$\begin{aligned}\|(Qz_1)(t) - (Qz_2)(t)\|_\alpha &\le \Big[\|A(0)^{\alpha-\beta}\|L_G + K(\alpha)L_F(2+\mathcal{L}L_b)\frac{T_0^{1-\alpha}}{(1-\alpha)}\\ &\quad + K(\alpha)\sum_{i=1}^{\delta} L_i\Big] \sup_{t\in J}\|z_1(t)-z_2(t)\|_\alpha\end{aligned} \tag{3.36}$$

Thus, for $t \in [0,T_0]$,

$$\begin{aligned}&\|(Qz_1) - (Qz_2)\|_{\mathcal{PC},\alpha}\\ &\le \Big[\|A(0)^{\alpha-\beta}\|L_G + K(\alpha)L_F(2+\mathcal{L}L_b)\frac{T_0^{1-\alpha}}{(1-\alpha)} + K(\alpha)\sum_{i=1}^{\delta} L_i\Big]\|z_1-z_2\|_{\mathcal{PC},\alpha}.\end{aligned}$$

From inequality (3.9), we get that Q is a contraction map. Since $\mathcal{S}_\alpha$ is a closed subset of Banach space $Y = PC([0,T_0];E_\alpha)$, therefore $\mathcal{S}_\alpha$ is a complete metric space. Thus, by Banach fixed point theorem, there exists a unique fixed point $u \in \mathcal{S}_\alpha$ of map Q which is unique fixed point, i.e., $Qu(t) = u(t)$. From the Theorem (2.1), we conclude that u is a solution for system (1.1)-(1.3) on $[0,T_0]$. □

4. Example

In this section, we consider an example to illustrate the discussed theory. We study the following differential equation with deviated argument

$$\begin{aligned}
&\partial_t[v(t,x)+\partial_x\mathcal{F}_1(t,v(b(t),x))]-\partial_x(p(t,x)\partial_x)[v(t,x)+\partial_x\mathcal{F}_1(t,v(b(t),x))],\\
&=\widetilde{H}(x,v(t,x))+\widetilde{G}(t,x,v(t,x));\quad 0<x<1,\ t\in(0,\frac{1}{2})\cup(\frac{1}{2},1)
\end{aligned}\tag{4.1}$$

example1

$$v(t,0)=v(t,1)=0,\quad t>0,\tag{4.2}$$

$$v(0,x)=u_0(x),\quad x\in(0,1),\tag{4.3}$$

$$\Delta v|_{t=1/2}=\frac{v(\frac{1}{2})^-}{5+v(\frac{1}{2})^-},\tag{4.4}$$

example2

where

$$\widetilde{H}(x,v(x,t))=\int_0^x\mathcal{K}(x,y)v(y,N(t))dy,$$
$$N(t)=g_1(t)|v(x,g_2(t)|v(x,\dots g_\delta(t)|v(x,t)|)|)|,$$

and the map $\widetilde{G}\in C(\mathbb{R}_+\times[0,1]\times\mathbb{R};\mathbb{R})$ is locally Lipschitz continuous in v, locally Hölder continuous in t, measurable and uniformly continuous in x. Here, we assume that functions $g_i:\mathbb{R}_+\to\mathbb{R}_+$, $(i=1,2,\dots,\delta)$ are locally Hölder continuous in t such that $g_i(0)=0$ and $\mathcal{K}:[0,1]\times[0,1]\to\mathbb{R}$ is continuously differentiable function i.e., $\mathcal{K}\in C^1([0,1]\times[0,1],\mathbb{R})$.

We assume that p is a function which is positive and has continuous partial derivative p_x such that for each $\tau\in[0,\infty)$ and $0<x<1$, we have

- (i) $0<p_0\le p(\tau,x)<p_0'$,
- (ii) $|p_x(\tau,x)|\le p_1$,
- (iii) $|p(\tau,x)-p(s,x)|\le C|\tau-s|^\epsilon$,
- (iv) $|p_x(\tau,x)-p_x(s,x)|\le C|\tau-s|^\epsilon$,

for some $\epsilon\in(0,1]$ and some constants $p_0,p_0',p_1,C>0$. Let us consider $E=L^2((0,1);\mathbb{R})$ and

$$-\frac{\partial}{\partial x}(p(t,x)\frac{\partial}{\partial x}u(t,x))=A(t)u(t,x),$$

with $E_1=D(A(0))=H^2(0,1)\cap H_0^1(0,1)$, $E_{1/2}=D((A(0))^{1/2})=H_0^1(0,1)$. Clearly, the family $\{A(t):t>0\}$ satisfies the hypotheses (P1)–(P3) on each bounded interval $[0,T]$.

Now, we define the function $f:\mathbb{R}_+\times H^2(0,1)\times E_{-1/2}\to E$ as

$$f(t,\xi,\zeta)(x)=\widetilde{H}(x,\zeta)+\widetilde{G}(t,x,\xi),\quad \text{for } x\in(0,1),\tag{4.5}$$

where $\widetilde{H}:[0,1]\times E_{-1/2}\to E$ is defined as

$$\widetilde{H}(x,\zeta)=\int_0^x\mathcal{K}(x,y)\zeta(y)dy,\tag{4.6}$$

and $\widetilde{G}:\mathbb{R}_+\times[0,1]\times E_{1/2}\to E$ satisfies following condition

$$\|\widetilde{G}(t,x,\xi)\|\le W(x,t)(1+\|\xi\|_{1/2}),\tag{4.7}$$

where Q is continuous in t and $Q(\cdot,t)\in X$. Also, we assume that the map $G:\mathbb{R}_+\times H_0^1(0,1)\to L^2(0,1)$ is such that

$$G(t,v(b(t)))(x)=\partial_x\mathcal{F}_1(t,v(b(t),x))$$

and satisfies the assumption (P7). There are some possibilities of the map b as follows:

(i) $b(t) = lt$ for $t \in [0, T]$ and $0 < l \le 1$;
(ii) $b(t) = lt^n$ for $t \in [0, 1]$, $n \in \mathbb{N}$ and $0 < l \le 1$;
(iii) $b(t) = l\sin(t)$ for $t \in [0, \pi/2]$ and $0 < l \le 1$.

For $v \in D(A)$ and $\lambda \in \mathbb{R}$, with $Av = \lambda v$, we have that

$$-\frac{\partial}{\partial x}(p(t,x)\frac{\partial}{\partial x}v(x)) = \lambda v(x),$$

which is the standard Strum-Liouville problem having real eigenvalues. For $v \in D(A)$ and $\lambda \in \mathbb{R}$, with $Av = \lambda v = -\frac{\partial}{\partial x}(p(t,x)\frac{\partial}{\partial x}v(x))$, we have that $\langle Av, v\rangle = \langle \lambda v, v\rangle$; that is,

$$\langle -\frac{d}{dx}(p(t,x)v'), v\rangle = \langle p(t,x)v', v'\rangle \ge p_0\|v'\|^2_{L^2} \tag{4.8}$$

Since we assume that p is a positive function with $p_0' > p(t,x) > p_0 > 0$, where p_0 is constant. Thus, we get $\lambda|y|^2_{L^2} \ge p_0\|v'\|^2_{L^2} > 0$. So $\lambda > 0$. In particular case for $p(t,x) = 1$, we have

$$v'' + \lambda v = 0. \tag{4.9}$$

FQ1

Case 1 $\lambda = 0$. Then solution of above equation is $v = C_1x + C_2$. Using boundary condition $v(0) = v(1) = 0$, we get $C_1 = C_2 = 0$. Thus, $v(x) = 0$ be the solution of $v'' = 0$, which is not an eigenfunction.

Case 2 Let $\lambda = -\mu^2$ and $\mu \ne 0$. Then equation (4.9) reduce to

$$[D^2 - \mu^2]v = 0 \tag{4.10}$$

FQ2

whose auxiliary equation is $D^2 - \mu^2 = 0$ i.e. $D = \pm\mu$. Thus solution of (4.10) is

$$v(x) = C_1e^{\mu x} + C_2e^{-\mu x}, \tag{4.11}$$

FQ3

Using the boundary conditions, we get $C_1 = C_2 = 0$. Thus, (4.11) gives $v = 0$ which is not an eigenfunction.

Cases 3 Let $\lambda = \mu^2$ with $\mu \ne 0$. Thus, equation (4.9) reduce to

$$[D^2 + \mu^2]v = 0. \tag{4.12}$$

FQ4

Therefore, the solution of (4.12) is

$$y = C_1\sin(\mu x) + C_2\cos(\mu x). \tag{4.13}$$

FQ5

Using the condition $v(0) = v(1) = 0$, we get $C_2 = 0$ and $C_1\sin(\mu) = 0$. For the non-trivial solution, we have $C_1 \ne 0$ and $\sin(\mu) = 0$. Thus, $\mu = n\pi$. Therefore $\lambda_n = \mu^2 = n^2\pi^2$, $n \in \mathbb{N}$. Hence, (4.13) reduces to $v(x) = C_1\sin(n\pi x)$ for $n = 1, \dots,$ and then $\lambda = \mu^2 = n^2\pi^2$, $n = 1, 2, \dots,$. Hence the required eigenfunction $v_n(x)$ with the corresponding eigenfunction λ_n are given by

$$v_n = C_1\sin(\sqrt{\lambda_n}x), \quad \lambda_n = n^2\pi^2, \quad n = 1, 2, \dots.$$

Next, we show that $\widetilde{H} : [0,1] \times E_{-1/2} \to E$ is defined as

$$\widetilde{H}(x, \zeta(x,t)) = \int_0^x \mathcal{K}(x,y)\zeta(y,t)dy, \tag{4.14}$$

where $\zeta(x,t) = v(x, h_1(t, v(x,t)))$. It is easy to verify that $f = \widetilde{H} + \widetilde{G}$ satisfies the assumption $(P4)$. Similarly, we show that the maps $b_i : [0,T] \times E_{-1/2} \to [0,T]$

defined as $b_i(t) = g_i(t)|\xi(x,\cdot)|$ for $i = 1, 2, \dots, \delta$ and satisfies the assumption (P5). For each $t \in [0,T]$, we get

$$|b_i(t,\xi)| = |g_i(t)||\xi(x,\cdot)| \le |g_i|_\infty \|\xi\|_{L^\infty(0,1)} \le N\|\xi\|_{-1/2},$$

where N is a positive constant, depending on the bounds on g_i's and we use the embedding $H_0^1(0,1) \subset C[0,1]$. Since we have that g_i satisfies the condition

$$|g_i(t) - g_i(s)| \le L_{g_i}|t-s|^\mu, \quad t, s \in [0,T], \tag{4.15}$$

where L_{g_i} is a positive constant and $\mu \in (0,1]$. For $z_1, z_2 \in X_{-1/2}$ and $t \in [0,T]$

$$\begin{aligned} |b_i(t,z_1) - b_i(t,z_2)| &\le \|g_i\|_\infty \|z_1 - z_2\|_{L^\infty(0,1)} + L_{g_i}|t-s|^\mu \|z_2\|_{L^\infty(0,1)}, \\ &\le N\|g_i\|_\infty \|z_1 - z_2\|_{-1/2} + L_{g_i}|t-s|^\mu \|z_2\|_{-1/2}, \\ &\le \max\{N\|g_i\|_\infty, L_{g_i\|z_2\|_\infty}\}(\|z_1 - z_2\|_{-1/2} + |t-s|^\mu). \end{aligned}$$

For $z_1, z_2 \in D((-A)^{-1/2})$, then

$$\|I_i(z_1) - I_i(z_2)\|_{1/2} \le \frac{\|z_1 - z_2\|_{1/2}}{\|(5+z_1)(5+z_2)\|_{1/2}} \le \frac{1}{25}\|z_1 - z_2\|_{1/2}. \tag{4.16}$$

Thus, we can apply the results of previous sections to obtain the existence result of the solution for (4.1)-(4.4).

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