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THIRD-ORDER PRODUCT-TYPE SYSTEMS OF DIFFERENCE EQUATIONS SOLVABLE IN CLOSED FORM

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Abstract. It is shown that a class of third order product-type systems of difference equations is solvable in closed form if initial values and multipliers are complex numbers, whereas the exponents are integers, by finding the formulas for the general solution in all possible cases. The main results complement some quite recent ones in the literature. The presented class of systems is the last one for whose investigation is not needed use of some associated polynomials of degree three or more, completing the investigation of such product-type systems.

1. Introduction

Many recent publications are devoted to the study of nonlinear difference equations and systems of difference equations; see for example [1]-[3], [6]-[8], [11]-[35]. Papaschinopoulos and Schinas essentially initiated a serious study of some classes of concrete systems of difference equations in [13, 14, 15], which was later continued by several authors in numerous other papers; see for example [3, 12, 16, 17, 19, 22, 23, 25, 26, 28, 29, 30, 31, 32, 34, 35] and the related references therein. The study of solvability of difference equations and systems, which is a classical topic [4, 5, 9, 10], has re-attracted some recent interest; see for example [1]-[3], [18], [25]-[28], [30]-[35], where several methods have been used. One of them is transforming the original equation, which have been used and developed in several directions; see for example [1, 3, 18, 27, 30, 31, 32, 33] and the related references therein.

Having studied real-valued difference equations and systems whose right-hand sides are essentially obtained by acting with translations or some operators with maximum on product-type expressions [24, 29], we started studying some systems of difference equations in the complex domain (with complex initial values and/or parameters). One of the basic classes of difference equations and systems are product-type ones. The main obstacle in studying the equations and systems on the complex domain is the fact that many complex-valued functions are not single valued. Hence, we need to pose some conditions to prevent such a situation to obtain uniquely defined solutions. Also, the transformation method or its modifications [3, 30, 31, 32, 33] cannot be directly applied to product-type systems on the complex domain.

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In [28] we studied the following two-dimensional class of product-type systems of difference equations

$$z_{n+1}=\frac{w_n^a}{z_{n-1}^b},\quad w_{n+1}=\frac{z_n^c}{w_{n-1}^d},\quad n\in\mathbb{N}_0,$$

where $a,b,c,d\in\mathbb{Z}$, $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}\setminus\{0\}$, and showed that it is solvable in closed form (a three dimensional extension of the system was investigated in [26]). A related product-type system was studied later in [34], whereas in paper [27] appeared some product-type equations during the study of a general difference equation. Soon after the publication of [26, 28, 34] we realized that some multipliers could be added in product-type systems so that the solvability of the systems is preserved. The first system of this type was studied in our paper [25]. Quite recently we have presented another such a system in [35]. Another thing that we have realized is that there is only a few product-type systems of difference equations which are solvable in closed form, which is connected to the inability of solving the polynomial equations of the degree five or more by radicals. This means that finding all the product-type systems of difference equations which are solvable in closed form is of some interest and importance.

The purpose of this paper is to continue this research, by presenting another solvable product-type system of difference equations. More precisely, we will investigate the solvability of the system

$$z_n=\alpha z_{n-2}^a w_{n-1}^b,\quad w_n=\beta w_{n-2}^c z_{n-3}^d,\quad n\in\mathbb{N}_0,\tag{1.1}$$

where $a,b,c,d\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{C}\setminus\{0\}$, $z_{-3},z_{-2},z_{-1},w_{-2},w_{-1}\in\mathbb{C}\setminus\{0\}$. Cases when $\alpha=0$ or $\beta=0$ or if some of the initial values $z_{-3},z_{-2},z_{-1},w_{-2},w_{-1}$ is equal to zero are quite simple or produce not well-defined solutions, which is why they are excluded from our consideration. The presented class of systems is the last one for whose investigation is not needed use of some associated polynomials of degree three or more, completing the investigation of such product-type systems.

2. Main results

In this section we prove our main results, which concern the solvability of system (1.1). Essentially there are six different results, but we incorporate them all in a theorem. Some of the formulas presented in the theorem hold on set $\mathbb{N}_0$, some hold on set $\mathbb{N}$, or even on the set $\mathbb{N}\setminus\{1\}$ (such a situation appears, for example, if we have an expression of the form x^{n-1} and x can be equal to zero and $n=1$). We will not specify which formula holds on which set and leave the minor observatory problem to the reader. What is interesting is that closed form formulas for solutions to the system of difference equations, although relatively complicated, are obtained in a more or less compact form, which is a rare case (one can note that it was not the case for the equation treated in [34]).

Theorem 2.1. *Consider system of difference equation* (1.1) *where* $a,b,c,d\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ *and* $z_{-3},z_{-2},z_{-1},w_{-2},w_{-1}\in\mathbb{C}\setminus\{0\}$. *Then the following statements hold.*

(a) If $ac=bd$, $a+c\neq 1$, *then the general solution to system* (1.1) *is given by*

$$z_{2n}=\alpha^{\frac{1-c-a(a+c)^n}{1-a-c}}\beta^{b\frac{1-(a+c)^n}{1-a-c}}w_{-1}^{b(a+c)^n}z_{-2}^{a(a+c)^n},\tag{2.1}$$

$$z_{2n+1}=\alpha^{\frac{1-c-a(a+c)^n}{1-a-c}}\beta^{b\frac{1-(a+c)^{n+1}}{1-a-c}}w_{-2}^{bc(a+c)^n}z_{-3}^{bd(a+c)^n}z_{-1}^{a(a+c)^n},\tag{2.2}$$

$$w_{2n} = \alpha^{d\frac{1-(a+c)^{n-1}}{1-a-c}} \beta^{\frac{1-a-c(a+c)^{n}}{1-a-c}} w_{-2}^{c^2(a+c)^{n-1}} z_{-3}^{cd(a+c)^{n-1}} z_{-1}^{d(a+c)^{n-1}}, \tag{2.3}$$

$$w_{2n-1} = \alpha^{d\frac{1-(a+c)^{n-1}}{1-a-c}} \beta^{\frac{1-a-c(a+c)^{n-1}}{1-a-c}} w_{-1}^{c(a+c)^{n-1}} z_{-2}^{d(a+c)^{n-1}}. \tag{2.4}$$

(b) If $ac = bd$, $a + c = 1$, then the general solution to system (1.1) is given by

$$z_{2n} = \alpha^{an+1}\beta^{bn} w_{-1}^{b} z_{-2}^{a}, \tag{2.5}$$

$$z_{2n+1} = \alpha^{an+1}\beta^{b(n+1)} w_{-2}^{bc} z_{-3}^{bd} z_{-1}^{a}, \tag{2.6}$$

$$w_{2n} = \alpha^{d(n-1)}\beta^{(1-a)n+1} w_{-2}^{c^2} z_{-3}^{cd} z_{-1}^{d}, \tag{2.7}$$

$$w_{2n-1} = \alpha^{d(n-1)}\beta^{(1-a)n+a} w_{-1}^{c} z_{-2}^{d}. \tag{2.8}$$

(c) If $ac \neq bd$, $(a+c)^2 \neq 4(ac-bd)$ and $bd \neq (a-1)(c-1)$, then the general solution to system (1.1) is given by

$$\begin{aligned} z_{2n} = \alpha&^{\frac{(t_2-1)(t_1-c)t_1^{n+1}-(t_1-1)(t_2-c)t_2^{n+1}+(t_1-t_2)(1-c)}{(t_1-1)(t_2-1)(t_1-t_2)}} \\ &\beta^{b\frac{(t_2-1)t_1^{n+1}-(t_1-1)t_2^{n+1}+t_1-t_2}{(t_1-1)(t_2-1)(t_1-t_2)}} \\ &\times w_{-1}^{b\frac{t_1^{n+1}-t_2^{n+1}}{t_1-t_2}} z_{-2}^{\frac{(t_1-c)t_1^{n+1}-(t_2-c)t_2^{n+1}}{t_1-t_2}}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} z_{2n+1} = {}&\alpha^{\frac{(t_2-1)(t_1-c)t_1^{n+1}-(t_1-1)(t_2-c)t_2^{n+1}+(t_1-t_2)(1-c)}{(t_1-1)(t_2-1)(t_1-t_2)}} \beta^{b\frac{(t_2-1)t_1^{n+2}-(t_1-1)t_2^{n+2}+t_1-t_2}{(t_1-1)(t_2-1)(t_1-t_2)}} \\ &\times w_{-2}^{bc\frac{t_1^{n+1}-t_2^{n+1}}{t_1-t_2}} z_{-3}^{bd\frac{t_1^{n+1}-t_2^{n+1}}{t_1-t_2}} z_{-1}^{\frac{(t_1-c)t_1^{n+1}-(t_2-c)t_2^{n+1}}{t_1-t_2}}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} w_{2n} = {}&\alpha^{d\frac{(t_2-1)t_1^{n}-(t_1-1)t_2^{n}+t_1-t_2}{(t_1-1)(t_2-1)(t_1-t_2)}} \\ &\times \beta^{\frac{(t_2-1)(t_1-a)t_1^{n+1}-(t_1-1)(t_2-a)t_2^{n+1}+(t_1-t_2)(1-a)}{(t_1-1)(t_2-1)(t_1-t_2)}} \\ &\times w_{-2}^{c\frac{(t_1-a)t_1^{n}-(t_2-a)t_2^{n}}{t_1-t_2}} z_{-3}^{d\frac{(t_1-a)t_1^{n}-(t_2-a)t_2^{n}}{t_1-t_2}} z_{-1}^{d\frac{t_1^{n}-t_2^{n}}{t_1-t_2}}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} w_{2n-1} = {}&\alpha^{d\frac{(t_2-1)t_1^{n}-(t_1-1)t_2^{n}+t_1-t_2}{(t_1-1)(t_2-1)(t_1-t_2)}} \beta^{\frac{(t_2-1)(t_1-a)t_1^{n}-(t_1-1)(t_2-a)t_2^{n}+(t_1-t_2)(1-a)}{(t_1-1)(t_2-1)(t_1-t_2)}} \\ &\times w_{-1}^{\frac{(t_1-a)t_1^{n}-(t_2-a)t_2^{n}}{t_1-t_2}} z_{-2}^{d\frac{t_1^{n}-t_2^{n}}{t_1-t_2}}, \end{aligned} \tag{2.12}$$

where

$$t_{1,2} = \frac{a+c \pm \sqrt{(a+c)^2 - 4(ac-bd)}}{2}. \tag{2.13}$$

(d) If $ac \neq bd$, $(a+c)^2 = 4(ac-bd)$, $bd \neq (a-1)(c-1)$, then the general solution to system (1.1) is given by

$$\begin{aligned} z_{2n} = {}&\alpha^{\frac{1-c+t_1^{n}((n+1)t_1^2-(n(c+1)+2)t_1+c(n+1))}{(1-t_1)^2}} \beta^{b\frac{1-(n+1)t_1^{n}+nt_1^{n+1}}{(1-t_1)^2}} \\ &\times w_{-1}^{b(n+1)t_1^{n}} z_{-2}^{(n(t_1-c)+2t_1-c)t_1^{n}}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} z_{2n+1} = {}&\alpha^{\frac{1-c+t_1^{n}((n+1)t_1^2-(n(c+1)+2)t_1+c(n+1))}{(1-t_1)^2}} \beta^{b\frac{1-(n+2)t_1^{n+1}+(n+1)t_1^{n+2}}{(1-t_1)^2}} \\ &\times w_{-2}^{bc(n+1)t_1^{n}} z_{-3}^{bd(n+1)t_1^{n}} z_{-1}^{(n(t_1-c)+2t_1-c)t_1^{n}}, \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
w_{2n} = {} & \alpha^{d\frac{1-nt_1^{n-1}+(n-1)t_1^n}{(1-t_1)^2}} \beta^{\frac{1-a+t_1^n((n+1)t_1^2-((1+a)n+2)t_1+a(n+1))}{(1-t_1)^2}} \\
& \times w_{-2}^{c(n(t_1-a)+t_1)t_1^{n-1}} z_{-3}^{d(n(t_1-a)+t_1)t_1^{n-1}} z_{-1}^{dnt_1^{n-1}},
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
w_{2n-1} = {} & \alpha^{d\frac{1-nt_1^{n-1}+(n-1)t_1^n}{(1-t_1)^2}} \beta^{\frac{1-a+t_1^{n-1}(nt_1^2-((1+a)n+1-a)t_1+an)}{(1-t_1)^2}} \\
& \times w_{-1}^{(n(t_1-a)+t_1)t_1^{n-1}} z_{-2}^{dnt_1^{n-1}},
\end{aligned} \tag{2.17}
$$

where

$$
t_1 = \frac{a+c}{2}.
$$

(e) If $ac \neq bd$, $(a+c)^2 \neq 4(ac-bd)$, $bd = (a-1)(c-1)$, $a+c \neq 2$, then the general solution to system (1.1) is given by

$$
\begin{aligned}
z_{2n} = {} & \alpha^{\frac{(t_1-c)t_1^{n+1}+((c-1)n+c-2)t_1+(1-c)n+1}{(1-t_1)^2}} \beta^{b\frac{t_1^{n+1}-(n+1)t_1+n}{(1-t_1)^2}} \\
& \times w_{-1}^{b\frac{t_1^{n+1}-1}{t_1-1}} z_{-2}^{\frac{(t_1-c)t_1^{n+1}+c-1}{t_1-1}},
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
z_{2n+1} = {} & \alpha^{\frac{(t_1-c)t_1^{n+1}+((c-1)n+c-2)t_1+(1-c)n+1}{(1-t_1)^2}} \beta^{b\frac{t_1^{n+2}-(n+2)t_1+n+1}{(1-t_1)^2}} \\
& \times w_{-2}^{bc\frac{t_1^{n+1}-1}{t_1-1}} z_{-3}^{bd\frac{t_1^{n+1}-1}{t_1-1}} z_{-1}^{\frac{(t_1-c)t_1^{n+1}+c-1}{t_1-1}},
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
w_{2n} = {} & \alpha^{d\frac{t_1^n-nt_1+n-1}{(1-t_1)^2}} \beta^{\frac{(t_1-a)t_1^{n+1}+((a-1)n+a-2)t_1+(1-a)n+1}{(1-t_1)^2}} \\
& \times w_{-2}^{c\frac{(t_1-a)t_1^n+a-1}{t_1-1}} z_{-3}^{d\frac{(t_1-a)t_1^n+a-1}{t_1-1}} z_{-1}^{d\frac{t_1^n-1}{t_1-1}},
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
w_{2n-1} = {} & \alpha^{d\frac{t_1^n-nt_1+n-1}{(1-t_1)^2}} \beta^{\frac{(t_1-a)t_1^n+((a-1)n-1)t_1+(1-a)n+a}{(1-t_1)^2}} \\
& \times w_{-1}^{\frac{(t_1-a)t_1^n+a-1}{t_1-1}} z_{-2}^{d\frac{t_1^n-1}{t_1-1}},
\end{aligned} \tag{2.21}
$$

where $t_1 = a+c-1$.

(f) If $ac \neq bd$, $(a+c)^2 = 4(ac-bd)$, $bd = (a-1)(c-1)$, and $a+c = 2$, then the general solution to system (1.1) is given by

$$
z_{2n} = \alpha^{\frac{(n+1)((1-c)n+2)}{2}} \beta^{b\frac{n(n+1)}{2}} w_{-1}^{b(n+1)} z_{-2}^{(1-c)n+2-c}, \tag{2.22}
$$

$$
z_{2n+1} = \alpha^{\frac{(n+1)((1-c)n+2)}{2}} \beta^{b\frac{(n+1)(n+2)}{2}} w_{-2}^{bc(n+1)} z_{-3}^{bd(n+1)} z_{-1}^{(1-c)n+2-c}, \tag{2.23}
$$

$$
w_{2n} = \alpha^{d\frac{(n-1)n}{2}} \beta^{\frac{(n+1)((1-a)n+2)}{2}} w_{-2}^{c((1-a)n+1)} z_{-3}^{d((1-a)n+1)} z_{-1}^{dn}, \tag{2.24}
$$

$$
w_{2n-1} = \alpha^{d\frac{(n-1)n}{2}} \beta^{\frac{n((1-a)n+1+a)}{2}} w_{-1}^{(1-a)n+1} z_{-2}^{dn}. \tag{2.25}
$$

Proof. Since $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-3}, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$, using (1.1) and induction we easily get

$$
z_n \neq 0 \quad \text{for } n \geq -3, \quad \text{and} \quad w_n \neq 0 \quad \text{for } n \geq -2.
$$

Hence, from (1.1) we have

$$
w_{n-1}^b = \frac{z_n}{\alpha z_{n-2}^a}, \quad n \in \mathbb{N}_0, \tag{2.26}
$$

$$
w_n^b = \beta^b w_{n-2}^{bc} z_{n-3}^{bd}, \quad n \in \mathbb{N}_0. \tag{2.27}
$$

From (2.26) and (2.27) it follows that

$$z_{n+1} = \alpha^{1-c}\beta^b z_{n-1}^{a+c} z_{n-3}^{bd-ac}, \quad n \in \mathbb{N}. \tag{2.28}$$

Let $\eta := \alpha^{1-c}\beta^b$,

$$u_1 = 1, \quad a_1 = a + c, \quad b_1 = bd - ac. \tag{2.29}$$

From (2.28) we have

$$z_{2(n+1)+i} = \eta^{u_1} z_{2n+i}^{a_1} z_{2(n-1)+i}^{b_1}, \tag{2.30}$$

for $n \in \mathbb{N}_0$ and $i = 0, 1$. From (2.30) it follows that

$$\begin{aligned} z_{2(n+1)+i} &= \eta^{u_1} (\eta z_{2(n-1)+i}^{a_1} z_{2(n-2)+i}^{b_1})^{a_1} z_{2(n-1)+i}^{b_1} \\ &= \eta^{u_1+a_1} z_{2(n-1)+i}^{a_1 a_1 + b_1} z_{2(n-2)+i}^{b_1 a_1} \\ &= \eta^{u_2} z_{2(n-1)+i}^{a_2} z_{2(n-2)+i}^{b_2}, \end{aligned} \tag{2.31}$$

for $n \in \mathbb{N}$ and $i = 0, 1$, where

$$u_2 := u_1 + a_1, \quad a_2 := a_1 a_1 + b_1, \quad b_2 := b_1 a_1. \tag{2.32}$$

Assume that for a $k \geq 2$ it holds

$$z_{2(n+1)+i} = \eta^{u_k} z_{2(n-k+1)+i}^{a_k} z_{2(n-k)+i}^{b_k}, \tag{2.33}$$

for $n \geq k - 1$ and $i = 0, 1$, where

$$u_k := u_{k-1} + a_{k-1}, \quad a_k := a_1 a_{k-1} + b_{k-1}, \quad b_k := b_1 a_{k-1}. \tag{2.34}$$

Using (2.30) in (2.33), it follows that

$$\begin{aligned} z_{2(n+1)+i} &= \eta^{u_k} z_{2(n-k+1)+i}^{a_k} z_{2(n-k)+i}^{b_k} \\ &= \eta^{u_k} (\eta z_{2(n-k)+i}^{a_1} z_{2(n-k-1)+i}^{b_1})^{a_k} z_{2(n-k)+i}^{b_k} \\ &= \eta^{u_k+a_k} z_{2(n-k)+i}^{a_1 a_k + b_k} z_{2(n-k-1)+i}^{b_1 a_k} \\ &= \eta^{u_{k+1}} z_{2(n-k)+i}^{a_{k+1}} z_{2(n-k-1)+i}^{b_{k+1}}, \end{aligned} \tag{2.35}$$

for $n \geq k$ and $i = 0, 1$, where

$$u_{k+1} := u_k + a_k, \quad a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k. \tag{2.36}$$

Equalities (2.31), (2.32), (2.35), (2.36) along with induction show that (2.33) and (2.34) hold for all $k, n \in \mathbb{N}$ such that $2 \leq k \leq n + 1$.

From (2.33) we have

$$z_{2n+i} = \eta^{u_n} z_i^{a_n} z_{i-2}^{b_n},$$

for $n \in \mathbb{N}$ and $i = 0, 1$, from which along with

$$z_0 = \alpha z_{-2}^a w_{-1}^b, \quad z_1 = \alpha z_{-1}^a w_0^b = \alpha z_{-1}^a (\beta w_{-2}^c z_{-3}^d)^b = \alpha\beta^b w_{-2}^{bc} z_{-3}^{bd} z_{-1}^a,$$

it follows that

$$\begin{aligned} z_{2n} &= \eta^{u_n} z_0^{a_n} z_{-2}^{b_n} = (\alpha^{1-c}\beta^b)^{u_n} (\alpha z_{-2}^a w_{-1}^b)^{a_n} z_{-2}^{b_n} \\ &= \alpha^{(1-c)u_n + a_n} \beta^{bu_n} w_{-1}^{ba_n} z_{-2}^{aa_n + b_n} \\ &= \alpha^{u_{n+1} - cu_n} \beta^{bu_n} w_{-1}^{ba_n} z_{-2}^{a_{n+1} - ca_n}, \end{aligned} \tag{2.37}$$

$$\begin{aligned} z_{2n+1} &= \eta^{u_n} z_1^{a_n} z_{-1}^{b_n} = (\alpha^{1-c}\beta^b)^{u_n} (\alpha\beta^b w_{-2}^{bc} z_{-3}^{bd} z_{-1}^a)^{a_n} z_{-1}^{b_n} \\ &= \alpha^{(1-c)u_n + a_n} \beta^{bu_n + ba_n} w_{-2}^{bca_n} z_{-3}^{bda_n} z_{-1}^{aa_n + b_n} \\ &= \alpha^{u_{n+1} - cu_n} \beta^{bu_{n+1}} w_{-2}^{bca_n} z_{-3}^{bda_n} z_{-1}^{a_{n+1} - ca_n}, \end{aligned} \tag{2.38}$$

for $n \in \mathbb{N}$.

From (2.34) and since $u_1 = 1$, we have

$$a_k = a_1 a_{k-1} + b_1 a_{k-2}, \quad k \geq 3, \tag{2.39}$$

$$u_k = 1 + \sum_{j=1}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{2.40}$$

Case $ac = bd$**.** Since $b_1 = bd - ac = 0$ equation (2.39) is reduced to

$$a_k = a_1 a_{k-1} = (a + c)a_{k-1}, \quad k \geq 3,$$

which implies

$$a_k = a_2(a + c)^{k-2} = (a + c)^k, \tag{2.41}$$

for $k \in \mathbb{N}$ (for $k = 1, 2$ this is directly verified).

Equalities (2.40) and (2.41) yield

$$u_k = 1 + \sum_{j=1}^{k-1} (a + c)^j, \quad k \in \mathbb{N},$$

so that

$$u_k = \frac{1 - (a + c)^k}{1 - a - c}, \quad k \in \mathbb{N}, \tag{2.42}$$

if $a + c \neq 1$, whereas

$$u_k = k, \quad k \in \mathbb{N}, \tag{2.43}$$

if $a + c = 1$.

If $a + c \neq 1$, then from (2.37), (2.38), (2.41), (2.42) and since

$$u_{n+1} - cu_n = \frac{1 - c - a(a + c)^n}{1 - a - c}$$
$$a_{n+1} - ca_n = a(a + c)^n$$

we obtain formulas (2.1) and (2.2), for $n \geq 2$.

If $a + c = 1$, then from (2.37), (2.38), (2.41), (2.43) and since

$$u_{n+1} - cu_n = (1 - c)n + 1 = an + 1$$

we obtain formulas (2.5) and (2.6), for $n \in \mathbb{N}$.

Case $ac \neq bd$**.** Let $t_{1,2}$ be the roots of the characteristic polynomial

$$P(t) = t^2 - (a + c)t + ac - bd, \tag{2.44}$$

associated with difference equation (2.39). Note that they are given by the formulas in (2.13). We have

$$a_n = c_1 t_1^n + c_2 t_2^n, \quad n \in \mathbb{N},$$

where $c_1, c_2 \in \mathbb{R}$, if $(a + c)^2 \neq 4(ac - bd)$, whereas

$$u_n = (d_1 n + d_2)t_1^n, \quad n \in \mathbb{N},$$

where $d_1, d_2 \in \mathbb{R}$, if $(a + c)^2 = 4(ac - bd)$.

Since $a_1 = t_1 + t_2$ and $a_2 = (t_1 + t_2)^2 - t_1 t_2 = t_1^2 + t_1 t_2 + t_2^2$, it is easily obtained that

$$a_k = \frac{t_1^{k+1} - t_2^{k+1}}{t_1 - t_2}, \quad k \in \mathbb{N}, \tag{2.45}$$

if $(a+c)^2 \neq 4(ac-bd)$, whereas

$$a_k = (k+1)t_1^k, \quad k \in \mathbb{N}, \tag{2.46}$$

if $(a+c)^2 = 4(ac-bd)$.

From (2.40) and (2.45) we have

$$u_k = 1 + \sum_{j=1}^{k-1} \frac{t_1^{j+1} - t_2^{j+1}}{t_1 - t_2} = \frac{(t_2-1)t_1^{k+1} - (t_1-1)t_2^{k+1} + t_1 - t_2}{(t_1-1)(t_2-1)(t_1-t_2)}, \tag{2.47}$$

for $k \in \mathbb{N}$, if $(a+c)^2 \neq 4(ac-bd)$ and $bd \neq (a-1)(c-1)$.

From (2.40) and (2.46) it follows that

$$u_k = 1 + \sum_{j=1}^{k-1} (j+1)t_1^j = \frac{1 - (k+1)t_1^k + kt_1^{k+1}}{(1-t_1)^2}, \tag{2.48}$$

for $k \in \mathbb{N}$, if $(a+c)^2 = 4(ac-bd)$ and $bd \neq (a-1)(c-1)$.

If $(a+c)^2 \neq 4(ac-bd)$, $bd = (a-1)(c-1)$ and $a+c \neq 2$, then polynomial (2.44) has exactly one zero equal to one, say t_2. From (2.40) and (2.45) we have

$$u_k = 1 + \sum_{j=1}^{k-1} \frac{t_1^{j+1} - 1}{t_1 - 1} = \frac{t_1^{k+1} - (k+1)t_1 + k}{(t_1-1)^2}, \tag{2.49}$$

for $k \in \mathbb{N}$.

Finally, if $(a+c)^2 = 4(ac-bd)$, $bd = (a-1)(c-1)$ and $a+c=2$, then both zeros of polynomial (2.44) are equal to one. From (2.40) and (2.46) it follows that

$$u_k = 1 + \sum_{j=1}^{k-1} (j+1) = \frac{k(k+1)}{2}, \tag{2.50}$$

for $k \in \mathbb{N}$.

If $(a+c)^2 \neq 4(ac-bd)$ and $bd \neq (a-1)(c-1)$, then from (2.37), (2.38), (2.45), (2.47) and since

$$u_{n+1} - cu_n = \frac{(t_2-1)(t_1-c)t_1^{n+1} - (t_1-1)(t_2-c)t_2^{n+1} + (t_1-t_2)(1-c)}{(t_1-1)(t_2-1)(t_1-t_2)}$$
$$a_{n+1} - ca_n = \frac{(t_1-c)t_1^{n+1} - (t_2-c)t_2^{n+1}}{t_1-t_2},$$

we obtain formulas (2.9) and (2.10).

If $(a+c)^2 = 4(ac-bd)$ and $bd \neq (a-1)(c-1)$, then from (2.37), (2.38), (2.46), (2.48) and since

$$u_{n+1} - cu_n = \frac{1 - c + t_1^n((n+1)t_1^2 - ((c+1)n+2)t_1 + c(n+1))}{(1-t_1)^2}$$
$$a_{n+1} - ca_n = ((t_1-c)n + 2t_1 - c)t_1^n,$$

we obtain formulas (2.14) and (2.15).

If $(a+c)^2 \neq 4(ac-bd)$, $bd = (a-1)(c-1)$ and $a+c \neq 2$, then from (2.37), (2.38), (2.45) with $t_2 = 1$, (2.49) and since

$$u_{n+1} - cu_n = \frac{(t_1-c)t_1^{n+1} + ((c-1)n + c - 2)t_1 + (1-c)n + 1}{(1-t_1)^2}$$

$$a_{n+1} - ca_n = \frac{(t_1 - c)t_1^{n+1} + c - 1}{t_1 - 1},$$

we obtain formulas (2.18) and (2.19), where $t_1 = ac - bd = a + c - 1$.

If $(a+c)^2 = 4(ac - bd)$, $bd = (a-1)(c-1)$ and $a + c = 2$, then from (2.37), (2.38), (2.46) with $t_1 = 1$, (2.50) and since

$$u_{n+1} - cu_n = \frac{(n+1)((1-c)n+2)}{2}$$
$$a_{n+1} - ca_n = (1-c)n + 2 - c,$$

we obtain formulas (2.22) and (2.23).

From (1.1) we also have

$$z_{n-3}^d = \frac{w_n}{\beta w_{n-2}^c}, \quad n \in \mathbb{N}_0, \tag{2.51}$$

$$z_n^d = \alpha^d z_{n-2}^{ad} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{2.52}$$

Thus, from (2.51) and (2.52) it follows that

$$w_{n+3} = \alpha^d \beta^{1-a} w_{n+1}^{a+c} w_{n-1}^{bd-ac}, \quad n \in \mathbb{N}_0. \tag{2.53}$$

Note that difference equations (2.28) and (2.53) have only different constant multipliers.

Let $\nu := \alpha^d \beta^{1-a}$,

$$\hat{u}_1 = 1, \quad \hat{a}_1 = a + c, \quad \hat{b}_1 = bd - ac. \tag{2.54}$$

As above it is proved that for any $k \in \mathbb{N}$ it holds

$$w_{2(n+1)+i} = \nu^{\hat{u}_k} w_{2(n-k+1)+i}^{\hat{a}_k} w_{2(n-k)+i}^{\hat{b}_k}, \tag{2.55}$$

for $n \geq k$ and $i = -1, 0$, where

$$\hat{u}_k := \hat{u}_{k-1} + \hat{a}_{k-1}, \quad \hat{a}_k := \hat{a}_1 \hat{a}_{k-1} + \hat{b}_{k-1}, \quad \hat{b}_k = \hat{b}_1 \hat{a}_{k-1}. \tag{2.56}$$

Since initial conditions (2.54) and system (2.56) are the same as those in (2.29) and (2.34), it follows that

$$\hat{a}_k = a_k, \quad \hat{b}_k = b_k, \quad \hat{u}_k = u_k, \tag{2.57}$$

for every $k \in \mathbb{N}$. From (2.55) we have

$$w_{2n+i} = \nu^{u_{n-1}} w_{2+i}^{a_{n-1}} w_i^{b_{n-1}},$$

for $n \geq 2$ and $i = -1, 0$, from which along with

$$w_0 = \beta w_{-2}^c z_{-3}^d, \qquad w_1 = \beta w_{-1}^c z_{-2}^d,$$
$$w_2 = \beta w_0^c z_{-1}^d = \beta(\beta w_{-2}^c z_{-3}^d)^c z_{-1}^d = \beta^{1+c} w_{-2}^{c^2} z_{-3}^{cd} z_{-1}^d,$$

it follows that

$$\begin{aligned} w_{2n} &= \nu^{u_{n-1}} w_2^{a_{n-1}} w_0^{b_{n-1}} \\ &= (\alpha^d \beta^{1-a})^{u_{n-1}} (\beta^{1+c} w_{-2}^{c^2} z_{-3}^{cd} z_{-1}^d)^{a_{n-1}} (\beta w_{-2}^c z_{-3}^d)^{b_{n-1}} \\ &= \alpha^{du_{n-1}} \beta^{(1-a)u_{n-1} + (1+c)a_{n-1} + b_{n-1}} w_{-2}^{c^2 a_{n-1} + cb_{n-1}} z_{-3}^{cda_{n-1} + db_{n-1}} z_{-1}^{da_{n-1}} \\ &= \alpha^{du_{n-1}} \beta^{u_{n+1} - au_n} w_{-2}^{c(a_n - aa_{n-1})} z_{-3}^{d(a_n - aa_{n-1})} z_{-1}^{da_{n-1}}, \end{aligned} \tag{2.58}$$

$$\begin{aligned} w_{2n-1} &= \nu^{u_{n-1}} w_1^{a_{n-1}} w_{-1}^{b_{n-1}} \\ &= (\alpha^d \beta^{1-a})^{u_{n-1}} (\beta w_{-1}^c z_{-2}^d)^{a_{n-1}} w_{-1}^{b_{n-1}} \\ &= \alpha^{du_{n-1}} \beta^{(1-a)u_{n-1}+a_{n-1}} w_{-1}^{ca_{n-1}+b_{n-1}} z_{-2}^{da_{n-1}} \\ &= \alpha^{du_{n-1}} \beta^{u_n - au_{n-1}} w_{-1}^{a_n - aa_{n-1}} z_{-2}^{da_{n-1}}, \end{aligned} \tag{2.59}$$

for $n \geq 2$.

Case $ac = bd$**.** If $a + c \neq 1$, then from (2.41), (2.42), (2.58), (2.59) and since

$$u_n - au_{n-1} = \frac{1 - a - c(a+c)^{n-1}}{1 - a - c}$$
$$a_n - aa_{n-1} = c(a+c)^{n-1}$$

we obtain formulas (2.3) and (2.4).

If $a + c = 1$, then from (2.41) with $a + c = 1$, (2.43), (2.58), (2.59) and since

$$u_n - au_{n-1} = (1-a)n + a$$

we obtain formulas (2.7) and (2.8).

Case $ac \neq bd$**.** If $ac \neq bd$, $(a+c)^2 \neq 4(ac - bd)$ and $bd \neq (a-1)(c-1)$, then from (2.45), (2.47), (2.58), (2.59) and since

$$u_n - au_{n-1} = \frac{(t_2 - 1)(t_1 - a)t_1^n - (t_1 - 1)(t_2 - a)t_2^n + (t_1 - t_2)(1 - a)}{(t_1 - 1)(t_2 - 1)(t_1 - t_2)}$$
$$a_n - aa_{n-1} = \frac{(t_1 - a)t_1^n - (t_2 - a)t_2^n}{t_1 - t_2}$$

we obtain formulas (2.11) and (2.12).

If $ac \neq bd$, $(a+c)^2 = 4(ac - bd)$, $bd \neq (a-1)(c-1)$, then from (2.46), (2.48), (2.58), (2.59) and since

$$u_n - au_{n-1} = \frac{1 - a + t_1^{n-1}(nt_1^2 - ((1+a)n + 1 - a)t_1 + an)}{(1 - t_1)^2}$$
$$a_n - aa_{n-1} = (n(t_1 - a) + t_1)t_1^{n-1}$$

we obtain formulas (2.16) and (2.17).

If $ac \neq bd$, $(a+c)^2 \neq 4(ac - bd)$, $bd = (a-1)(c-1)$ and $a + c \neq 2$, then from (2.45) with $t_2 = 1$, (2.49), (2.58), (2.59) and since

$$u_n - au_{n-1} = \frac{(t_1 - a)t_1^n + ((a-1)n - 1)t_1 + (1-a)n + a}{(1 - t_1)^2}$$
$$a_n - aa_{n-1} = \frac{(t_1 - a)t_1^n + a - 1}{t_1 - 1}$$

we obtain formulas (2.20) and (2.21), where $t_1 = ac - bd = a + c - 1$.

If $ac \neq bd$, $(a+c)^2 = 4(ac - bd)$, $bd = (a-1)(c-1)$ and $a + c = 2$, then from (2.46) with $t_1 = 1$, (2.50), (2.58), (2.59) and since

$$u_n - au_{n-1} = \frac{n((1-a)n + 1 + a)}{2}$$
$$a_n - aa_{n-1} = (1-a)n + 1$$

we obtain formulas (2.24) and (2.25).

By some standard but tedious and time-consuming calculations it is checked that all the formulas in the theorem really present general solution to system (1.1) (in each of these six cases), completing the proof of the theorem. □

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