Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 91, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BLOW-UP PHENOMENA FOR SECOND-ORDER DIFFERENTIAL INEQUALITIES WITH SHIFTED ARGUMENTS

MOHAMED JLELI, MOKHTAR KIRANE, BESSEM SAMET

Abstract. We provide sufficient conditions for the blow-up of solutions to certain second-order differential inequalities and systems with advanced and delayed arguments. Our proofs are based on the test function method.

1. Introduction

The study of differential equations and inequalities with shifted (advanced or delayed) arguments is an important and significant branch of mathematical analysis. Applications of these equations can be found in physics, mechanics, engineering, biology and so on (see [1, 3, 7, 12] and the references therein). Many results concerning existence and uniqueness of solutions, stability of solutions, oscillation and numerical methods have been published. Results concerning blowing-up solutions in a finite time are however scarce. It seems that the first result is due to Kobayashi [5] who considered a parabolic equations with delay. Later, Lightbourne and Rankin [6] considered also blowing-up solutions to partial differential equations with delay (let us remark in passing that their proof is doubtfull). More recently Ezzinbi and Jazar [4] considered blowing-up solutions to ordinary differential equations with delay, however their method of proof is different from ours. Casal, Diaz and Vegas [2] also considered blow-up for some ordinary and parabolic differential equations with time-delay. Let us mention that many nice results concerning blowing-up solutions for integral equations have been published mainly by Olmstead and his co-authors, we just mention the review paper of Roberts [10] and its references.

Recently, Salieva [11] obtained sufficient conditions for the blow-up of solutions to certain classes of first-order differential equations and inequalities with advanced and delayed arguments using the test function method [8]. Motivated by that work, we consider in this paper various classes of second-order differential inequalities and systems with shifted arguments, for which we study the blow-up phenomenon. As in [11], using the test function method, we provide sufficient conditions for the existence of blow-up. we obtain also an upper bound of the blow-up time.

The paper is organized as follows. In Section 2, we consider a second-order differential inequality with advanced arguments. Sufficient conditions, for which

2010 *Mathematics Subject Classification.* 34K38, 35B44.
Key words and phrases. Blow-up; differential inequalities; advanced argument; delayed argument.

Submitted February 22, 2016. Published March 31, 2016.

any solution of the considered inequality blows-up at a finite time, are provided. Moreover, an estimate of the blow-up time is obtained. In Section 3, we consider second-order differential inequalities with delayed arguments in the right-hand side. Finally, in Section 4, we study the blow-up of solutions for two systems of differential inequalities with advanced arguments.

2. Differential inequality with advanced argument

We consider the differential inequality

$$\begin{gathered} \frac{d^2y(t)}{dt^2} \geq |y(t+\tau)|^q, \quad t>0, \\ y(0)=y_0>0, \quad y'(0)=y_1>0, \end{gathered} \tag{2.1}$$

where $\tau>0$ and $q>1$.

If there exists a $T>0$ such that a function y satisfies (2.1) for every $t\in(0,T)$, then y is called a local solution to (2.1). If a function y satisfies (2.1) for every $t>0$, then it is called a global solution to (2.1). If y is not a global solution, the largest possible $T>0$ is called the blow-up time for the function y. We have the following blow-up result.

Theorem 2.1. *If $q>1$, then any solution of (2.1) blows-up at a finite time.*

Proof. Suppose that y is a global solution of (2.1). Let ω be a real number such that

$$\omega > \frac{2q}{q-1}. \tag{2.2}$$

Then

$$\int_0^\infty \frac{d^2y}{dt^2}\varphi^\omega(t)\,dt \geq \int_0^\infty |y(t+\tau)|^q\varphi^\omega(t)\,dt,$$

for every test function φ, $\varphi(t\geq T)=0$, $\varphi\geq 0$. Integrating by parts,

$$\begin{aligned} &-y_1[\varphi(0)]^\omega+\omega y_0[\varphi(0)]^{\omega-1}\varphi'(0)+\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt \\ &\geq \int_0^\infty |y(t+\tau)|^q\varphi^\omega(t)\,dt, \end{aligned} \tag{2.3}$$

where

$$\phi(t)=\omega\Big((\omega-1)\varphi'^2(t)+\varphi(t)\varphi''(t)\Big), \quad t>0.$$

Observe that from (2.1), we have $y''(t)\geq 0$, which implies

$$y'(t)\geq y'(0)=y_1>0, \quad t>0, \tag{2.4}$$

$$y(t)\geq y(0)=y_0>0, \quad t>0. \tag{2.5}$$

Using (2.5) and Taylor-Lagrange formula, we obtain

$$\begin{aligned} |y(t+\tau)|^q &= y(t+\tau)^q \\ &= y^q(t)+\tau q y^{q-1}(t)y'(t)+\frac{q\tau^2}{2}\Big((q-1)y^{q-2}(t+\theta\tau)y'^2(t+\theta\tau) \\ &\quad + y^{q-1}(t+\theta\tau)y''(t+\theta\tau)\Big), \end{aligned}$$

where $\theta=\theta_t\in(0,1)$. Therefore, by (2.1), (2.4) and (2.5), we obtain

$$
\begin{aligned}
|y(t+\tau)|^q &\geq y^q(t) + \tau q y_0^{q-1} y_1 \\
&\quad + \frac{q\tau^2}{2}\Big((q-1)y_0^{q-2}y_1^2 + y_0^{q-1}y^q(t+(\theta+1)\tau)\Big) \\
&\geq y^q(t) + \tau q y_0^{q-1} y_1 + \frac{q\tau^2}{2}\Big((q-1)y_0^{q-2}y_1^2 + y_0^{2q-1}\Big),
\end{aligned}
$$

i.e.,

$$
|y(t+\tau)|^q \geq y^q(t) + \tau q y_0^{q-1} y_1 + \frac{q\tau^2}{2}\Big((q-1)y_0^{q-2}y_1^2 + y_0^{2q-1}\Big), \quad t>0. \tag{2.6}
$$

Combining (2.3) with (2.6), we obtain

$$
\begin{aligned}
\int_0^\infty y^q(t)\varphi^\omega(t)\,dt &\leq -y_1[\varphi(0)]^\omega + \omega y_0[\varphi(0)]^{\omega-1}\varphi'(0) \\
&\quad + \int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt - \gamma(q,y_0,y_1)\int_0^\infty \varphi^\omega(t)\,dt,
\end{aligned} \tag{2.7}
$$

where

$$
\gamma(q,y_0,y_1) = \tau q y_0^{q-1}y_1 + \frac{q\tau^2}{2}\Big((q-1)y_0^{q-2}y_1^2 + y_0^{2q-1}\Big) > 0.
$$

Using the ε-Young inequality with parameters q and $q' = \frac{q}{q-1}$, for $\varepsilon > 0$, we obtain

$$
\begin{aligned}
&\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt \\
&= \int_0^\infty y(t)\varphi^{\frac{\omega}{q}}(t)[\varphi(t)]^{\omega-2-\frac{\omega}{q}}\phi(t)\,dt \\
&\leq \varepsilon \int_0^\infty y^q(t)\varphi^\omega(t)\,dt + C_\varepsilon \int_0^\infty [\varphi(t)]^{(\omega-2-\frac{\omega}{q})\frac{q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt \\
&= \varepsilon \int_0^\infty y^q(t)\varphi^\omega(t)\,dt + C_\varepsilon \int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt,
\end{aligned} \tag{2.8}
$$

where $C_\varepsilon = (\varepsilon q)^{-\frac{q'}{q}}/q'$. Note that thanks to the choice (2.2) of the parameter ω, we have

$$
\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt < \infty.
$$

Inequalities (2.7) and (2.8) yield

$$
\begin{aligned}
&(1-\varepsilon)\int_0^\infty y^q(t)\varphi^\omega(t)\,dt \\
&\leq -y_1\varphi^\omega(0) + \omega y_0\varphi^{\omega-1}(0)\varphi'(0) \\
&\quad + C_\varepsilon \int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt - \gamma(q,y_0,y_1)\int_0^\infty \varphi^\omega(t)\,dt.
\end{aligned} \tag{2.9}
$$

Now, as a test function, we use

$$
\varphi(t) = \varphi_0(\frac{t}{T}), \quad T>0, \tag{2.10}
$$

where φ_0 is the classical cutoff function, that is, φ_0 is a smooth decreasing function such that

$$
0 \leq \varphi_0 \leq 1, \quad |\varphi_0'(s)| \leq Cs^{-1}, \quad s>0,
$$

$$\varphi_0(s) = \begin{cases} 1 & \text{if } 0 \le s \le \frac{1}{2}, \\ 0 & \text{if } s \ge 1. \end{cases}$$

In this case,

$$\varphi^\omega(0) = 1 \quad \text{and} \quad \varphi'(0) = 0. \tag{2.11}$$

Moreover,

$$\int_0^\infty \varphi^\omega(t)\,dt = T\int_0^1 \varphi_0^\omega(s)\,ds \ge T\int_0^{\frac{1}{2}} \varphi_0^\omega(s)\,ds = \frac{T}{2}. \tag{2.12}$$

Combining (2.9), (2.11) and (2.12), we obtain

$$(1-\varepsilon)\int_0^\infty y^q(t)\varphi^\omega(t)\,dt \le -\Big(y_1 + \frac{T}{2}\gamma(q,y_0,y_1)\Big) + C_\varepsilon \int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt.$$

Taking $\varepsilon = 1/q$, we obtain

$$\int_0^\infty y^q(t)\varphi^\omega(t)\,dt \le -q'\Big(y_1 + \frac{T}{2}\gamma(q,y_0,y_1)\Big) + q'C_\varepsilon \int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt.$$

On the other hand, we have $q'C_\varepsilon = 1$. Therefore,

$$\int_0^\infty y^q(t)\varphi^\omega(t)\,dt \le -q'\Big(y_1 + \frac{T}{2}\gamma(q,y_0,y_1)\Big) + \int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt. \tag{2.13}$$

Using the change of variable $t = Ts$, we obtain

$$\begin{aligned}
&\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}(t)\,dt \\
&\le \frac{1}{T^{\frac{q+1}{q-1}}}\omega^{\frac{2q}{q-1}}\int_0^\infty [\varphi_0(s)]^{\omega-\frac{2q}{q-1}}\Big((\omega-1)\varphi_0'^2(s) + \varphi_0(s)|\varphi_0''(s)|\Big)^{\frac{q}{q-1}}\,ds \qquad (2.14)\\
&= \frac{C'}{T^{\frac{q+1}{q-1}}},
\end{aligned}$$

where

$$C' = \omega^{\frac{2q}{q-1}}\int_0^\infty [\varphi_0(s)]^{\omega-\frac{2q}{q-1}}\Big((\omega-1)\varphi_0'^2(s) + \varphi_0(s)|\varphi_0''(s)|\Big)^{\frac{q}{q-1}}\,ds > 0. \tag{2.15}$$

Then, by (2.13), we have

$$\int_0^\infty y^q(t)\varphi_0^\omega\big(\frac{t}{T}\big)\,dt \le -q'\Big(y_1 + \frac{T}{2}\gamma(q,y_0,y_1)\Big) + \frac{C'}{T^{\frac{q+1}{q-1}}}. \tag{2.16}$$

Passing to the limit as $T \to \infty$ and using the monotone convergence theorem, we obtain

$$\int_0^\infty y^q(t)\,dt \le -\infty,$$

which is a contradiction. Therefore, for any $q > 1$, we have nonexistence of a global solution for (2.1). □

Remark 2.2. Observe that if T^* is the zero of the right-hand side of (2.16), then the blow-up time for (2.1) does not exceed T^*. Moreover, we have the estimate

$$T^* < \mathbb{T} = \Big(\frac{C'}{q'y_1}\Big)^{\frac{q-1}{q+1}},$$

since the right-hand side of (2.16) tends to $+\infty$ as $T \to 0^+$, and it is negative for $t = \mathbb{T}$.

3. Differential inequalities with delayed arguments

We start with the second-order differential inequality

$$\begin{aligned} \frac{d^2y(t)}{dt^2} &\geq |y(t-\tau)|^q, \quad t>0, \\ y(t) &= \psi(t), \quad t\in[-\tau,0], \end{aligned} \tag{3.1}$$

where $\tau>0$, $q>1$ and ψ is a given continuously differentiable real-valued function on the initial interval $[-\tau,0]$. We have the following result.

Theorem 3.1. *Let $q>1$. If*

$$\psi'(0)+\int_{-\tau}^0 |\psi(t)|^q\,dt>0, \tag{3.2}$$

then any solution of (3.1) *has a blow-up.*

Proof. Suppose that y is a global solution of (3.1). Let ω be a real number such that

$$\omega>\frac{2q}{q-1}.$$

As a test function, we take the function φ^ω, where φ is defined by (2.10). By integration by parts, we obtain

$$-\psi'(0)+\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt\geq \int_0^\infty |y(t-\tau)|^q\varphi^\omega(t)\,dt, \tag{3.3}$$

where

$$\phi(t)=\omega\Big((\omega-1)\varphi'^2(t)+\varphi(t)\varphi''(t)\Big),\quad t>0.$$

On the other hand, for $T>2\tau$, we have

$$\begin{aligned} \int_0^\infty |y(t-\tau)|^q\varphi^\omega(t)\,dt &= \int_{-\tau}^\infty |y(t)|^q\varphi^\omega(t+\tau)\,dt \\ &\geq \int_{-\tau}^0 |\psi(t)|^q\,dt+\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we obtain

$$\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt\leq -\Big(\psi'(0)+\int_{-\tau}^0|\psi(t)|^q\,dt\Big)+\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt. \tag{3.5}$$

As in the proof of Theorem 2.1, using the ε-Young inequality with parameters q and $q'=\frac{q}{q-1}$, we obtain

$$\int_0^\infty y(t)\varphi^{\omega-2}(t)\phi(t)\,dt\leq \varepsilon\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt+C_\varepsilon\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt, \tag{3.6}$$

where $C_\varepsilon>0$. Combining (3.5) and (3.6), we obtain

$$\begin{aligned} &(1-\varepsilon)\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt \\ &\leq -\Big(\psi'(0)+\int_{-\tau}^0|\psi(t)|^q\,dt\Big)+C_\varepsilon\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt. \end{aligned}$$

Taking $0 < \varepsilon < 1$ and using (3.2), we obtain

$$\psi'(0) + \int_{-\tau}^{0} |\psi(t)|^q \, dt \leq C_\varepsilon \int_0^\infty [\varphi(t)]^{\omega - \frac{2q}{q-1}} |\phi(t)|^{\frac{q}{q-1}} \, dt. \tag{3.7}$$

From (2.14), we have

$$\int_0^\infty [\varphi(t)]^{\omega - \frac{2q}{q-1}} |\phi(t)|^{\frac{q}{q-1}} \, dt \leq \frac{C'}{T^{\frac{q+1}{q-1}}},$$

where C' is given by (2.15). Therefore, by (3.7), we obtain

$$\psi'(0) + \int_{-\tau}^{0} |\psi(t)|^q \, dt \leq \frac{C_\varepsilon C'}{T^{\frac{q+1}{q-1}}}.$$

Passing to the limit as $T \to \infty$, we obtain

$$\psi'(0) + \int_{-\tau}^{0} |\psi(t)|^q \, dt \leq 0$$

which is a contradiction with (3.2). $\square$

Now, we consider the differential inequality

$$\begin{gathered} \frac{d}{dt}\Big(\frac{dy(t)}{dt} + y(t)\Big) \geq |y(t-\tau)|^q, \quad t > 0, \\ y(t) = \psi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{3.8}$$

where $\tau > 0$, $q > 1$, and ψ is a given continuously differentiable real-valued function on the initial interval $[-\tau, 0]$. We have the following result.

Theorem 3.2. *Let $q > 1$. If*

$$\psi'(0) + \psi(0) + \int_{-\tau}^{0} |\psi(t)|^q \, dt > 0, \tag{3.9}$$

then any solution of (3.8) *has a blow-up.*

Proof. Suppose that y is a global solution of (3.8). Let ω be a real number such that

$$\omega > \frac{2q}{q-1}.$$

As a test function, we take the function φ^ω, where φ is defined by (2.10). Integrating by parts, we obtain

$$\begin{aligned} &-\psi'(0) + \int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t) \, dt - \psi(0) - \omega \int_0^\infty y(t)[\varphi(t)]^{\omega-1}\varphi'(t) \, dt \\ &\geq \int_0^\infty |y(t-\tau)|^q \varphi^\omega(t) \, dt, \end{aligned} \tag{3.10}$$

where

$$\phi(t) = \omega\Big((\omega-1)\varphi'^2(t) + \varphi(t)\varphi''(t)\Big), \quad t > 0.$$

From the proof of Theorem 3.1, for $T > 2\tau$, we have

$$\int_0^\infty |y(t-\tau)|^q \varphi^\omega(t) \, dt \geq \int_{-\tau}^{0} |\psi(t)|^q \, dt + \int_0^\infty |y(t)|^q \varphi^\omega(t) \, dt.$$

Therefore, from (3.10), we obtain

$$\begin{aligned} &-\Big(\psi'(0)+\psi(0)+\int_{-\tau}^{0}|\psi(t)|^q\,dt\Big)+\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt \\ &-\omega\int_0^\infty y(t)[\varphi(t)]^{\omega-1}\varphi'(t)\,dt \\ &\geq \int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt. \end{aligned} \tag{3.11}$$

As in the proof of Theorem 3.1, using the ε-Young inequality with parameters q and $q'=\frac{q}{q-1}$, we obtain

$$\begin{aligned} &\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt \\ &\leq \varepsilon\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt+C_\varepsilon\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt, \end{aligned} \tag{3.12}$$

where $C_\varepsilon>0$. Similarly, we obtain

$$\begin{aligned} &-\omega\int_0^\infty y(t)[\varphi(t)]^{\omega-1}\varphi'(t)\,dt \\ &\leq \omega\varepsilon\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt+\omega C_\varepsilon\int_0^\infty [\varphi(t)]^{\omega-\frac{q}{q-1}}|\varphi'(t)|^{\frac{q}{q-1}}\,dt. \end{aligned} \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we obtain the estimate

$$\begin{aligned} &\big(1-(\omega+1)\varepsilon\big)\int_0^\infty |y(t)|^q\varphi^\omega(t)\,dt \\ &\leq -\Big(\psi'(0)+\psi(0)+\int_{-\tau}^{0}|\psi(t)|^q\,dt\Big)+C\Big(L_1(\varphi)+L_2(\varphi)\Big), \end{aligned} \tag{3.14}$$

where

$$\begin{gathered} C=\omega C_\varepsilon, \\ L_1(\varphi)=\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt, \\ L_2(\varphi)=\int_0^\infty [\varphi(t)]^{\omega-\frac{q}{q-1}}|\varphi'(t)|^{\frac{q}{q-1}}\,dt. \end{gathered}$$

Taking $0<\varepsilon<\frac{1}{\omega+1}$, we deduce from (3.14) that

$$\psi'(0)+\psi(0)+\int_{-\tau}^{0}|\psi(t)|^q\,dt\leq C\Big(L_1(\varphi)+L_2(\varphi)\Big). \tag{3.15}$$

• Estimate of $L_1(\varphi)$. From (2.14), we have

$$L_1(\varphi)\leq\frac{C'}{T^{\frac{q+1}{q-1}}}, \tag{3.16}$$

where C' is given by (2.15).

• Estimate of $L_2(\varphi)$. Using the change of variable $t=Ts$, we obtain

$$L_2(\varphi)=\frac{C''}{T^{q'-1}}, \tag{3.17}$$

where

$$C'' = \int_0^\infty [\varphi_0(s)]^{\omega-\frac{q}{q-1}} |\varphi_0'(s)|^{\frac{q}{q-1}}\, ds.$$

From (3.15), the estimates (3.16) and (3.17), we deduce that

$$\psi'(0) + \psi(0) + \int_{-\tau}^0 |\psi(t)|^q\, dt \le C\Big(\frac{C'}{T^{\frac{q+1}{q-1}}} + \frac{C''}{T^{q'-1}}\Big).$$

Passing to the limit as $T \to \infty$, we obtain

$$\psi'(0) + \psi(0) + \int_{-\tau}^0 |\psi(t)|^q\, dt \le 0,$$

which is a contradiction with (3.9). □

The argument used for the proof of Theorem 3.2 yields the following blow-up result for the more general differential inequality

$$\begin{gathered} \frac{d}{dt}\Big(\frac{dy(t)}{dt} + y(t)\Big) \ge |t-\tau|^p |y(t-\tau)|^q, \quad t > 0, \\ y(t) = \psi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{3.18}$$

where $\tau > 0$, $q > 1$, $p \ge 0$, and ψ is a given continuously differentiable real-valued function on the initial interval $[-\tau, 0]$.

Theorem 3.3. *Let $q > 1$ and $p \ge 0$. If*

$$\psi'(0) + \psi(0) + \int_{-\tau}^0 |t|^p |\psi(t)|^q\, dt > 0,$$

then any solution of (3.18) *has a blow-up.*

4. Systems of differential inequalities with advanced arguments

We consider the system of differential inequalities

$$\begin{gathered} \frac{d^2y(t)}{dt^2} \ge |z(t+\tau_1)|^p, \quad t > 0, \\ \frac{d^2z(t)}{dt^2} \ge |y(t+\tau_2)|^q, \quad t > 0 \end{gathered} \tag{4.1}$$

with the initial conditions

$$y(0) = y_0 > 0, \quad y'(0) = y_1 > 0, \quad z(0) = z_0 > 0, \quad z'(0) = z_1 > 0, \tag{4.2}$$

where $\tau_i > 0$, $i = 1, 2$, $p > 1$, and $q > 1$. We have the following result.

Theorem 4.1. *Let $p > 1$ and $q > 1$. Then a blow-up situation takes place for problem* (4.1)-(4.2).

Proof. Suppose that (y, z) is a global solution of (4.1)-(4.2). Let ω be a real number such that

$$\omega > \max\Big\{\frac{2p}{p-1}, \frac{2q}{q-1}\Big\}.$$

Then

$$\begin{gathered} -y_1 + \int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\, dt \ge \int_0^\infty |z(t+\tau_1)|^p \varphi^\omega(t)\, dt, \\ -z_1 + \int_0^\infty z(t)[\varphi(t)]^{\omega-2}\phi(t)\, dt \ge \int_0^\infty |y(t+\tau_2)|^q \varphi^\omega(t)\, dt, \end{gathered}$$

where φ is the test function defined by (2.10) and

$$\phi(t)=\omega\Big((\omega-1)\varphi'^2(t)+\varphi(t)\varphi''(t)\Big),\quad t>0.$$

Applying the Taylor-Lagrange formula to the functions $|z(t+\tau_1)|^p$ and $|y(t+\tau_2)|^q$, we obtain

$$\begin{aligned}|z(t+\tau_1)|^p &= z(t+\tau_1)^p\\ &= z^p(t)+\tau_1 p z^{p-1}(t)z'(t)+\frac{p\tau_1^2}{2}\Big((p-1)z^{p-2}(t+\theta_1\tau_1)z'^2(t+\theta_1\tau_1)\\ &\quad + z^{p-1}(t+\theta_1\tau_1)z''(t+\theta_1\tau_1)\Big)\end{aligned}$$

and

$$\begin{aligned}|y(t+\tau_2)|^q &= y(t+\tau_2)^q\\ &= y^q(t)+\tau_2 q y^{q-1}(t)y'(t)+\frac{q\tau_2^2}{2}\Big((q-1)y^{q-2}(t+\theta_2\tau_2)y'^2(t+\theta_2\tau_2)\\ &\quad + y^{q-1}(t+\theta_2\tau_2)y''(t+\theta_2\tau_2)\Big),\end{aligned}$$

where $\theta_i=\theta_i(t)\in(0,1)$, $i=1,2$. Therefore, by (4.1), we obtain

$$\begin{aligned}|z(t+\tau_1)|^p &\geq z^p(t)+\tau_1 p z_0^{p-1}z_1\\ &\quad+\frac{p\tau_1^2}{2}\Big((p-1)z_0^{p-2}z_1^2+z_0^{p-1}y^q(t+\theta_1\tau_1+\tau_2)\Big)\\ &\geq z^p(t)+\tau_1 p z_0^{p-1}z_1+\frac{p\tau_1^2}{2}\Big((p-1)z_0^{p-2}z_1^2+z_0^{p-1}y_0^q\Big),\end{aligned}$$

i.e.,

$$|z(t+\tau_1)|^p\geq z^p(t)+\mathcal{A},$$

where

$$\mathcal{A}=\tau_1 p z_0^{p-1}z_1+\frac{p\tau_1^2}{2}\Big((p-1)z_0^{p-2}z_1^2+z_0^{p-1}y_0^q\Big)>0.$$

Similarly,

$$|y(t+\tau_2)|^q\geq y^q(t)+\mathcal{B},$$

where

$$\mathcal{B}=\tau_2 q y_0^{q-1}y_1+\frac{q\tau_2^2}{2}\Big((q-1)y_0^{q-2}y_1^2+y_0^{q-1}z_0^p\Big)>0.$$

Therefore, we have the system of integral inequalities

$$\begin{gathered}\int_0^\infty z^p(t)\varphi^\omega(t)\,dt\leq -y_1+\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt-\mathcal{A}\int_0^\infty\varphi^\omega(t)\,dt,\\ \int_0^\infty y^q(t)\varphi^\omega(t)\,dt\leq -z_1+\int_0^\infty z(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt-\mathcal{B}\int_0^\infty\varphi^\omega(t)\,dt.\end{gathered}$$

Using the ε-Young inequality with parameters $\varepsilon_1>0$, q, $\frac{q}{q-1}$, and the ε-Young inequality with parameters $\varepsilon_2>0$, p, $\frac{p}{p-1}$, we obtain

$$\begin{gathered}\int_0^\infty z^p(t)\varphi^\omega(t)\,dt\leq -y_1+\varepsilon_1\int_0^\infty y^q(t)\varphi^\omega(t)\,dt+C_{\varepsilon_1}L(q,\varphi)-\mathcal{A}\int_0^\infty\varphi^\omega(t)\,dt,\\ \int_0^\infty y^q(t)\varphi^\omega(t)\,dt\leq -z_1+\varepsilon_2\int_0^\infty z^p(t)\varphi^\omega(t)\,dt+C_{\varepsilon_2}L(p,\varphi)-\mathcal{B}\int_0^\infty\varphi^\omega(t)\,dt,\end{gathered}$$

where

$$L(q,\varphi)=\int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt.$$

Combining the two integral inequalities above, we obtain

$$\begin{aligned}
&y_1+\varepsilon_1 z_1+(1-\varepsilon_1\varepsilon_2)\int_0^\infty z^p(t)\varphi^\omega(t)\,dt\\
&\le C_{\varepsilon_1}L(q,\varphi)+\varepsilon_1 C_{\varepsilon_2}L(p,\varphi)-(\mathcal{B}\varepsilon_1+\mathcal{A})\int_0^\infty \varphi^\omega(t)\,dt.
\end{aligned}$$

Following the proof of Theorem 2.1, we have

$$\begin{gathered}
L(q,\varphi)\le \frac{C_1}{T^{\frac{q+1}{q-1}}}\,,\quad L(p,\varphi)\le \frac{C_2}{T^{\frac{p+1}{p-1}}}\,,\\
\int_0^\infty \varphi^\omega(t)\,dt\ge \frac{T}{2}\,,
\end{gathered}$$

where $C_i>0$, $i=1,2$, are constants. Therefore, taking $0<\varepsilon_1\varepsilon_2<1$, we obtain the estimate

$$y_1+\varepsilon_1 z_1\le \frac{C_{\varepsilon_1}C_1}{T^{\frac{q+1}{q-1}}}+\frac{\varepsilon_1 C_{\varepsilon_2}C_2}{T^{\frac{p+1}{p-1}}}-(\mathcal{B}\varepsilon_1+\mathcal{A})\frac{T}{2}\,.$$

Passing to the limit as $T\to\infty$, we obtain $y_1+\varepsilon_1 z_1\le-\infty$ which is a contradiction. ☐

Now, we consider the system of differential inequalities

$$\begin{gathered}
\frac{d^2y(t)}{dt^2}\ge |z(t+\tau_1)|^p,\quad t>0,\\
\frac{dz(t)}{dt}\ge |y(t+\tau_2)|^q,\quad t>0
\end{gathered}\tag{4.3}$$

with the initial conditions

$$y(0)=y_0>0,\quad y'(0)=y_1>0,\quad z(0)=z_0>0,\tag{4.4}$$

where $\tau_i>0$, $i=1,2$, $p>1$, and $q>1$. We have the following result.

Theorem 4.2. *Let $p>1$ and $q>1$. Then a blow-up situation takes place for problem* (4.3)-(4.4).

Proof. Suppose that (y,z) is a global solution of (4.3)-(4.4). Let ω be a real number such that

$$\omega>\max\big\{\frac{p}{p-1},\frac{2q}{q-1}\big\}.$$

Then

$$\begin{gathered}
-y_1+\int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt\ge\int_0^\infty |z(t+\tau_1)|^p\varphi^\omega(t)\,dt,\\
-z_0+\omega\int_0^\infty z(t)[\varphi(t)]^{\omega-1}|\varphi'(t)|\,dt\ge\int_0^\infty |y(t+\tau_2)|^q\varphi^\omega(t)\,dt,
\end{gathered}$$

where φ is the test function defined by (2.10) and

$$\phi(t)=\omega\Big((\omega-1)\varphi'^2(t)+\varphi(t)\varphi''(t)\Big),\quad t>0.$$

Applying the Taylor-Lagrange formula to the functions $|z(t+\tau_1)|^p$ and $|y(t+\tau_2)|^q$, we obtain

$$|z(t+\tau_1)|^p = z(t+\tau_1)^p = z^p(t) + \tau_1 p z^{p-1}(t+\theta_1\tau_1)z'(t+\theta_1\tau_1)$$

and

$$\begin{aligned}|y(t+\tau_2)|^q &= y(t+\tau_2)^q \\ &= y^q(t) + \tau_2 q y^{q-1}(t)y'(t) + \frac{q\tau_2^2}{2}\Big((q-1)y^{q-2}(t+\theta_2\tau_2)y'^2(t+\theta_2\tau_2) \\ &\quad + y^{q-1}(t+\theta_2\tau_2)y''(t+\theta_2\tau_2)\Big),\end{aligned}$$

where $\theta_i = \theta_i(t) \in (0,1)$, $i = 1,2$. Therefore, by (4.3), we obtain

$$|z(t+\tau_1)|^p \geq z^p(t) + \mathcal{A},$$

where

$$\mathcal{A} = \tau_1 p z_0^{p-1} y_0^q > 0.$$

Similarly,

$$|y(t+\tau_2)|^q \geq y^q(t) + \mathcal{B},$$

where

$$\mathcal{B} = \tau_2 q y_0^{q-1} y_1 + \frac{q\tau_2^2}{2}\Big((q-1)y_0^{q-2}y_1^2 + y_0^{q-1}z_0^p\Big) > 0.$$

Therefore, we obtain the system of integral inequalities

$$\begin{aligned}\int_0^\infty z^p(t)\varphi^\omega(t)\,dt &\leq -y_1 + \int_0^\infty y(t)[\varphi(t)]^{\omega-2}\phi(t)\,dt - \mathcal{A}\int_0^\infty \varphi^\omega(t)\,dt, \\ \int_0^\infty y^q(t)\varphi^\omega(t)\,dt &\leq -z_0 + \omega\int_0^\infty z(t)[\varphi(t)]^{\omega-1}|\varphi'(t)|\,dt - \mathcal{B}\int_0^\infty \varphi^\omega(t)\,dt.\end{aligned}$$

Using the ε-Young inequality with parameters $\varepsilon_1 > 0$, q, $\frac{q}{q-1}$, and the ε-Young inequality with parameters $\varepsilon_2 > 0$, p, $\frac{p}{p-1}$, we obtain

$$\begin{aligned}\int_0^\infty z^p(t)\varphi^\omega(t)\,dt &\leq -y_1 + \varepsilon_1\int_0^\infty y^q(t)\varphi^\omega(t)\,dt + C_{\varepsilon_1}L(q,\varphi) - \mathcal{A}\int_0^\infty \varphi^\omega(t)\,dt, \\ \int_0^\infty y^q(t)\varphi^\omega(t)\,dt &\leq -z_0 + \omega\varepsilon_2\int_0^\infty z^p(t)\varphi^\omega(t)\,dt + \omega C_{\varepsilon_2}L_p(\varphi) - \mathcal{B}\int_0^\infty \varphi^\omega(t)\,dt,\end{aligned}$$

where

$$\begin{aligned}L(q,\varphi) &= \int_0^\infty [\varphi(t)]^{\omega-\frac{2q}{q-1}}|\phi(t)|^{\frac{q}{q-1}}\,dt, \\ L_p(\varphi) &= \int_0^\infty [\varphi(t)]^{\omega-\frac{p}{p-1}}|\varphi'(t)|^{\frac{p}{p-1}}\,dt.\end{aligned}$$

Combining the above inequalities and taking $0 < \varepsilon_1\varepsilon_2 < \omega^{-1}$, we obtain

$$y_1 + \varepsilon_1 z_0 \leq \omega\varepsilon_1 C_{\varepsilon_2}L_p(\varphi) - (\varepsilon_1\mathcal{B} + \mathcal{A})\int_0^\infty \varphi^\omega(t)\,dt + C_{\varepsilon_1}L(q,\varphi),$$

which yields

$$y_1 + \varepsilon_1 z_0 \leq \frac{\omega\varepsilon_1 C_{\varepsilon_2}C_1}{T^{\frac{1}{p-1}}} - \frac{(\varepsilon_1\mathcal{B} + \mathcal{A})T}{2} + \frac{C_{\varepsilon_1}C_2}{T^{\frac{q+1}{q-1}}},$$

for some constants $C_i > 0$, $i = 1,2$. Passing to the limit as $T \to \infty$, we obtain $y_1 + \varepsilon_1 z_0 \leq -\infty$ which is a contradiction. $\square$

Acknowledgements. : The third authors extends his appreciation to Distinguished Scientist Fellowship Program (DSFP) at King Saud University (Saudi Arabia).

References

[1] O. Arino, M. L. Hbid, E. Ait Das (editors); *Delay differential equations and applications*, NATO sciences series. Berlin: Springer, 2006.
[2] A. C. Casal, J. I. Diaz, J. M. Vegas; *Blow-up in some ordianry and parabolic differential equations with time-delay*, Dynamic Systems and Applications, 18 (2009), 29-46.
[3] T. Erneux; *Applied delay differential equations*, New York: Springer, 2009.
[4] K. Ezzinbi, M. Jazar; *Blow-up results for some nonlinear delay differential equations*, Positivity 10 (2006), 329-341.
[5] K. Kobayashi; *On the semilinear heat equations with time lag*, Hiroshima Math. J., 7 (1977), 459-472.
[6] J. H. Lightbourne III, S. M. Rankin III; *Global existence for a delay differential equation*, Journal of Differetnial Equations, 40 (1981), 186-192.
[7] MacDonald; *Biological delay systems: Linear stability thoery*, Cambridge University Press, New York, 1989.
[8] E. Mitidieri, S. Pohozaev; *A priori estimates and nonexistence of solutions of nonlinear partial differential equations and inequalities*, Proceedings of Steklov Math. Inst., 234 (2003), 1–383.
[9] R. D. Parshad, S. Bhommick, E. Quansah, R. Agrawal, R. K. Upadhyay; *Finite time blow up in population model with commpetitive interference and time delay*, ArXiv: 1511.07498v1 [math.AP] 23 Nov 2015.
[10] C. Roberts; *Recent results on blow-up and quanching for nonlinear Volterra equations*, Journal of Computational and Applied Mathematics, 205 (2007), 736-743.
[11] O. Salieva; *Blow-up of solutions to some differential equations and inequalities with shifted arguments*, Electron. J. Differential Equations., 57 (2016), 1–8.
[12] V. Volterra; *Sur la théorie mathématique des phénom enes héreéditaires*, J. Math. Pures et Appl. 7 (1028), 249-298.
[13] Z. Yang, H. Brunner; *Blow-up behavior of Hammerstein-type delay Volterra integral equations*, Front. Math. China, 2013 8(2), 261-280.

Mohamed Jleli
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: jleli@ksu.edu.sa

Mokhtar Kirane
LaSIE, Faculté des Sciences et Technologies, Université de La Rochelle, Avenue M. Crépeau, 17042 La Rochelle, France
E-mail address: mokhtar.kirane@univ-lr.fr

Bessem Samet
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: bsamet@ksu.edu.sa