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STABILIZATION OF FRACTIONAL-ORDER OF IVGTT GLUCOSE-INSULIN INTERACTION

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Abstract. In this work we considered a discretization for a fractional-order model based on the Intra-Venous Glucose Tolerance Test (IVGTT). We prove the existence and uniqueness of the solution of the model, and the non-negativity and boundedness of solutions. Moreover, we establish sufficient conditions of stability or instability for the proposed fractional-order model and its discretization. Numerical computations are carried out for illustrating the analytic results.

1. Introduction and statement of main results

The relation between glucose and insulin, its regulatory hormone, has been studied by many researchers for several years. The glucose and insulin system problem arises when the level of glucose concentration is far from the normal range. Insulin reduces the liver's production of spontaneous glucose and also allows tissues to increase glucose uptake. Many scientists are concerned with mathematical models concerned with glucose-insulin dynamics and clinical studies to account for a range of significant markers of Diabetes Mellitus growth. The complex glucose-insulin relationship has been studied; see [7, 8, 11, 27, 29, 31, 32, 33, 36]. These models consist of simply linear ordinary differential equations and were considered unacceptable for different reasons, such as parameters have poor fits to experimental data or are not identifiable [26]. To diagnose a diabetic individual, various glucose tolerance tests have been applied in the clinics and experimental researches. Bolie [9], Ackerman et al [1, 2], Gatewood et al [20], Bergman et al [8], Steil et al [34], Caumo et al [13], Gaetano and Arino [18], Gresl et al [21] offered the glucose-insulin linear models homeostasis based on Intra-Venous Glucose Tolerance Test (IVGTT) method. The "Minimal Model" was proposed in 1980 by Bergman et al [7, 8], and was updated in 1986. This model, which describes IVGTT experimental data well with the smallest collection of [7, 8, 29, 36] identifiable and meaningful parameters, can be considered to be the most famous model used in glucose metabolism physiological research. In [18, 23], Gaetano and Arino and Li et al had reinvestigated the dynamical behavior of the "Minimal Model" in both modeling and physiological

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aspect to understanding blood glucose regulatory system:

$$\begin{aligned} \frac{dG(t)}{dt} &= -p_1 G(t) - \frac{p_4 I(t) G(t)}{\beta G(t) + 1} + p_7, \quad G(0) = G_b + p_0, \\ \frac{dI(t)}{dt} &= p_6 G(t) - b_2 I(t), \quad I(0) = I_b + p_0 p_3. \end{aligned} \tag{1.1}$$

with $G_i = G_b$, for $t \in [-p_5, 0)$, where $G(t)$ [mg/dL], $I(t)$ [mU/L] are the concentration of blood glucose and insulin, G_b [mg/dL] is the concentration of basal blood glucose, I_b [mU/L] is the concentration of basal blood insulin, p_1 [1/min] is the Insulin independent glucose clearance rate, p_2 [1/min] is the active insulin clearance rate (upt. decrease), p_3 [L/(min2mU)] is the increase caused by insulin in uptake ability, p_4 [1/min] is the destroy rate of blood insulin, p_5 [mg/dL] is the aim glucose level, p_6 [mUdL/Lmgmin] is the Pancreatic free rate after glucose bolus, and p_7 (mg/dl)[1/min] is the concentration at time 0 of the Plasma insulin, above basal insulinemia, immediately after the glucose bolus intake.

Recently, some real processes in physics, biology, epidemiology and other scientific fields have been modelled by many scientists using fractional calculus; see [3, 4, 5, 6, 12, 14, 15, 16, 22, 30]. There is a significant potential for the principle of fractional calculus to transform the way we see the model and regulate the environment around us. It is naturally that the fractional order differential equations are used because it is related to systems with memory which exists in most biological systems [35]. Also they are, at least, as stable as their integer-order counterpart [19, 24]. Hence, we suggest to establish a system of fractional glucose-insulin for modeling (1.1), based on the model presented in [23]:

$$\begin{aligned} D^q G(t) &= -p_1 G(t) - \frac{p_4 I(t) G(t)}{\beta G(t) + 1} + p_7, \\ D^q I(t) &= p_6 G(t) - p_2 I(t), \end{aligned} \tag{1.2}$$

with $G(0) = G_b + p_0$, $I(0) = I_b + p_0 p_3$, $G_i = G_b$, for $t \in [-p_5, 0)$.

We are also interested in applying, motivated by the above works, the discretization method of piecewise constant arguments to the model (1.2):

$$\begin{aligned} G_{n+1} &= G_n + \frac{m^q}{\Gamma(q+1)} \big[-p_1 G_n - \frac{p_4 I_n G_n}{\beta G_n + 1} + p_7 \big], \\ I_{n+1} &= I_n + \frac{m^q}{\Gamma(q+1)} [p_6 G_n - p_2 I_n], \end{aligned} \tag{1.3}$$

where $m > 0$ represents the time interval of production.

In this article, the fractional glucose-insulin model (1.2) and its discretized (1.3) are investigated. The existence, uniqueness, non-negativity and boundedness of the solutions for the model (1.2) is proved. Moreover, sufficient conditions of stability or instability of the models (1.2) and (1.3) are obtained. Some conditions on the existence of bifurcation of systems (1.2) and (1.3) are presented by using bifurcation theory. Furthermore, the numerical diagrams are carried out for illustrating the analytic results.

2. Preliminaries

The Caputo fractional-order derivative is used in this paper.

Definition 2.1 ([30])**.** If $q \in \mathbb{R}^+$ is a non integer order, the fractional integral $I^q f(t)$ of the function $f(t)$, $t > 0$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ is the Euler gamma function.

Definition 2.2 ([30])**.** The Caputo fractional derivative $D^q f(t)$ of order $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$ is defined as

$$D^q f(t) = \begin{cases} \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds, & , n-1 < q < n, \\ \frac{d^n}{dt^n} f(t), & q = n. \end{cases}$$

By using a method from by Li et al [25], we study the existence and uniqueness of the solutions of the fractional system (1.2) in $\Omega \times (0, T]$ with

$$\Omega = \{(G, I) \in \mathbb{R}^2 : \max(|G|, |I|) \leq \Phi\}.$$

Theorem 2.3. *For any $X_0 = (G_0, I_0) \in \Omega$, there is a unique solution $X(t) \in \Omega$ with initial condition X_0 of the model* (1.2)*, for all $t \geq 0$.*

Proof. For $X, \overline{X} \in \Omega$, suppose a mapping $H(X) = (H_1(X), H_2(X))$ is defined as follows:

$$\begin{gathered} H_1(X) = -p_1 G(t) - \frac{p_4 I(t) G(t)}{\beta G(t) + 1} + p_7, \\ H_2(X) = p_6 G(t) - p_2 I(t). \end{gathered} \tag{2.1}$$

Therefore,

$$\begin{aligned} &\|H(X) - H(\overline{X})\| \\ &= |H_1(X) - H_1(\overline{X})| + |H_2(X) - H_2(\overline{X})| \\ &= \Big| - p_1 G - \frac{p_4 I G}{\beta G + 1} + p_7 + p_1 \overline{G} + \frac{p_4 \overline{I}\,\overline{G}}{\beta \overline{G} + 1} - p_7 \Big| + |p_6 G - p_2 I - p_6 \overline{G} + p_2 \overline{I}| \\ &= \Big| - p_1 (G - \overline{G}) - p_4 \frac{\beta I G \overline{G} + I G - \beta G \overline{I G} - \overline{I G}}{(\beta G + 1)(\beta \overline{G} + 1)} \Big| + |p_6 (G - \overline{G}) + p_2 (I - \overline{I})| \\ &\leq (p_1 + p_6 + p_4 \Phi) \eta |G - \overline{G}| + (p_2 + p_4 \beta \Phi^2) |I - \overline{I}| \\ &\leq \Gamma \|X - \overline{X}\|, \end{aligned} \tag{2.2}$$

where

$$\Gamma = \max\{p_1 + p_6 + p_4 \Phi, p_2 + p_4 \beta \Phi^2\}.$$

The Lipschitz condition on $H(X)$ is thus fulfilled. This proves that the model's solutions (1.2) exist and are unique. □

To study the non-negativity and boundedness of the solutions of the model (1.2), the method used by [25] is utilized.

Theorem 2.4. *The solutions of* (1.2)*, which start from $\mathbb{R}^2_+$ are nonnegative and uniformly bounded.*

Proof. From (1.2), one has

$$\begin{aligned} D^qG(t)|_{G=0} &= p_7 \ge 0,\\ D^qI(t)|_{I=0} &= p_6G_b \ge 0. \end{aligned} \tag{2.3}$$

Following [10, Lemmas 5 and 6], one can derive that the non-negativity of the solutions of (1.2). By taking $H(t) = G(t) + I(t)$, one obtains

$$D^qH(t) = (p_6 - p_1)G(t) - p_2I - \frac{p_4I(t)G(t)}{\beta G(t) + 1} + p_7\,. \tag{2.4}$$

Hence, for all $\lambda > 0$,

$$D^qH(t) + \lambda H(t) \le (\lambda + p_6 - p_1)G(t) + (\lambda - p_2)I + p_7. \tag{2.5}$$

One can choose $\lambda < \min\{p_1 - p_6, p_2\}$. Thus

$$D^qH(t) + \lambda H(t) \le p_7\,. \tag{2.6}$$

Following [17, Lemma 9], one obtains

$$0 \le H(t) \le H(0)E_q(-\lambda(t)^q) + r(t)^qE_{q,q+1}(-\lambda(t)^q)), \tag{2.7}$$

where E_q is the Mittag-Leffler function. Also following [17, Lemma 5 and Corollary 6], one obtains

$$0 \le H(t) \le \frac{p_7}{\lambda}, \quad t \to \infty \tag{2.8}$$

Hence, the solutions of (1.2) starting from $\mathbb{R}^2_+$ are uniformly bounded in the open region V, where

$$V = \big\{(G, I) \in \mathbb{R}^2_+ : H(t) \le \frac{p_7}{\lambda} + \varepsilon, \text{ for all } \varepsilon > 0\big\}.$$

□

3. Model description and its discretization

We apply the discretization method to model (1.2). Following the piecewise constant arguments, the discretization of model (1.2) is given as

$$\begin{aligned} D^qG(t) &= -p_1G\big(m\big[\frac{t}{m}\big]\big) - \frac{p_4I(m[\frac{t}{m}])G(m[\frac{t}{m}])}{\beta G(m[\frac{t}{m}]) + 1} + p_7\,,\\ D^qI(t) &= p_6G\big(m\big[\frac{t}{m}\big]\big) - p_2I\big(m\big[\frac{t}{m}\big]\big), \end{aligned} \tag{3.1}$$

with $G(0) = G_0$, $I(0) = I_0$. The steps of the suggested discretization method are the following:

Step 1. Let $t \in [0, m)$, then $\frac{t}{m} \in [0, 1)$. Thus, one gets

$$\begin{aligned} D^qG_1 &= -p_1G_0 - \frac{p_4I_0G_0}{\beta G_0 + 1} + p_7,\\ D^qI_1 &= p_6G_0 - p_2I_0. \end{aligned}$$

The solution of (3.1) is given as

$$\begin{aligned} G_1(t) &= G_0 + \frac{m^q}{\Gamma(q+1)}\Big(- p_1G_0 - \frac{p_4I_0G_0}{\beta G_0 + 1} + p_7\Big),\\ I_1(t) &= I_0 + \frac{m^q}{\Gamma(q+1)}\Big(p_6G_0 - p_2I_0\Big). \end{aligned}$$

Step 2. Let $t \in [m, 2m)$. Then $\frac{t}{m} \in [1, 2)$. Thus, one obtains

$$D^q G_2(t) = -p_1 G_1(m) - \frac{p_4 I_1(m) G_1(m)}{\beta G_1(m) + 1} + p_7,$$
$$D^q I_2(t) = p_6 G_1(m) - p_2 I_1(m).$$

The solution of (3.1) is given by

$$G_2(t) = G_1(m) + \frac{(t-m)^q}{\Gamma(q+1)}\Big(- p_1 G_1(m) - \frac{p_4 I_1(m) G_1(m)}{\beta G_1(m) + 1} + p_7\Big),$$
$$I_2(t) = I_1(m) + \frac{(t-m)^q}{\Gamma(q+1)}\big(p_6 G_1(m) - p_2 I_1(m)\big).$$

Thus, by repeating the process, we can deduce that the solution of (3.1) is

$$G_{n+1}(t) = G_n(nm) + \frac{(t-nm)^q}{\Gamma(q+1)}\Big(- p_1 G_n(nm) - \frac{p_4 I_n(nm) G_1(nm)}{\beta G_n(nm) + 1} + p_7\Big),$$
$$I_{n+1}(t) = I_n(nm) + \frac{(t-nm)^q}{\Gamma(q+1)}\big[p_6 G_n(nm) - p_2 I_n(nm)\big].$$

Thus (1.3) follows.

4. Fixed point and stability of equilibria

To find the fixed points, let

$$D^q G(t) = 0,$$
$$D^q I(t) = 0.$$

Thus, the equilibrium point $E_1 = (G^*, I^*)$ of (1.2) is given by

$$G^* = \frac{(qp_2p_7 - p_1p_2) \pm \sqrt{(qp_2p_7 - p_1p_2)^2 + 4p_2p_7(qp_1p_2 + p_4p_6)}}{2(qp_2p_7 - p_1p_2)},$$
$$I^* = \frac{p_6}{p_2} G^*.$$

To linearize the fractional model (1.2), about $E_1 = (G^*, I^*)$, we remove the nonlinear terms. Then, one obtains the linear variational system

$$D^q G(t) = -A_1 G(t) - A_2 I(t),$$
$$D^q I(t) = A_3 G(t) - A_4 I(t).$$

where

$$A_1 = \Big(p_1 + \frac{p_4 I^*}{qG^* + 1} - \frac{qp_4 I^* G^*}{(qG^* + 1)^2}\Big), \quad A_2 = \frac{p_4 G^*}{qG^* + 1}, \quad A_3 = p_6, \quad A_4 = p_2.$$

Then, the Jacobian matrix $J(E_1)$ at E_1 for the fractional model (1.2) is

$$J(E_1) = \begin{bmatrix} -A_1 & -A_2 \\ A_3 & -A_4 \end{bmatrix}$$

Its characteristic equation is

$$P_1(\lambda) = \lambda^2 + (A_1 + A_4)\lambda + A_1 A_4 + A_3 A_2 = 0. \tag{4.1}$$

Its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\Big(\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}^2(J) - 4\Delta}\Big),$$

with $\operatorname{tr}(J) = -(A_1 + A_4)$, and $\Delta = A_1A_4 + A_2A_3$. Therefore,

$$|\arg\lambda_1| > \frac{q\pi}{2}, \quad |\arg\lambda_2| > \frac{q\pi}{2},$$

i.e.,

$$\Big|\frac{\sqrt{4\Delta - \operatorname{tr}^2(J)}}{\operatorname{tr}(J)}\Big| > \tan\frac{q\pi}{2},$$

i.e.,

$$\Big|\frac{\sqrt{4[A_1A_4 + A_2A_3] - (A_1 + A_4)^2}}{A_1 + A_4}\Big| > \tan\frac{q\pi}{2}$$

is a sufficient condition for the local asymptotic stability of E_1. Also, the Hopf bifurcation of system (1.2) occurs when

$$|\arg\lambda_1| = \frac{q\pi}{2}, \quad |\arg\lambda_2| = \frac{q\pi}{2},$$

i.e.,

$$\Big|\frac{\sqrt{4\Delta - \operatorname{tr}^2(J)}}{\operatorname{tr}(J)}\Big| = \tan\frac{q\pi}{2},$$

then

$$\Big|\frac{\sqrt{4[A_1A_4 + A_2A_3] - (A_1 + A_4)^2}}{A_1 + A_4}\Big| = \tan\frac{q\pi}{2}.$$

Equation (4.1) can be converted to

$$\lambda^2 + D_1\lambda + D_2 = 0, \tag{4.2}$$

where $D_1 = A_1 + A_4$, $D_2 = A_1A_4 + A_2A_3$. Let $\Psi_i > 0$ $(i = 1, 2)$,

$$\Psi_1 = D_1, \quad \Psi_2 = D_1D_2.$$

Lemma 4.1. *If $\Psi_1 > 0$ and $\Psi_2 > 0$, then the equilibrium point of the fractional model* (1.2) *is asymptotically stable.*

Remark 4.2. The conditions $\Psi_1 > 0$ and $\Psi_2 > 0$ are sufficient condition for Lemma 4.1. If all the roots of equation (4.1) satisfy $|\arg\lambda| > q\pi/2$. then Lemma 4.1 may still hold.

5. Stability of the discretized fractional-order model

Throughout this section, we analyze the stability of the fixed point of the fractional model (1.3), which has the fixed equilibrium point $E_1 = (G^*, I^*)$, where

$$G^* = \frac{(qp_2p_7 - p_1p_2) \pm \sqrt{(qp_2p_7 - p_1p_2)^2 + 4p_2p_7(qp_1p_2 + p_4p_6)}}{2(qp_2p_7 - p_1p_2)},$$
$$I^* = \frac{p_6}{p_2}G^*.$$

Then

$$J(E_1) = \begin{bmatrix} 1 - p_1h - \frac{p_4I^*h}{(\beta G^*+1)^2} & -\frac{p_4G^*h}{(\beta G^*+1)} \\ hp_6 & 1 - p_2h \end{bmatrix},$$

is the Jacobian matrix of system (1.3), at E_1 with $h = m^q/\Gamma(q + 1)$, and its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\Big(\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}^2(J) - 4\Delta}\Big),$$

where

$$\operatorname{tr}(J) = 2 - \Big(p_1 + p_2 + \frac{p_4 I^*}{(\beta G^* + 1)^2}\Big)h,$$

$$\Delta = 1 - \Big(p_1 + p_2 + \frac{p_4 I^* h}{(\beta G^* + 1)^2}\Big)h + \Big(p_1 p_2 + \frac{p_2 p_4 I^*}{(\beta G^* + 1)^2} + \frac{p_4 p_6 G^*}{\beta G^* + 1}\Big)h^2.$$

In section 4, a sufficient condition for the local asymptotic stability of E_1 is given by

$$|\arg\lambda_1| > \frac{q\pi}{2}, \quad |\arg\lambda_2| > \frac{q\pi}{2},$$

i.e.,

$$\Big|\frac{\sqrt{4\Delta - \operatorname{tr}^2(J)}}{\operatorname{tr}(J)}\Big| > \tan\frac{q\pi}{2}.$$

The Hopf bifurcation of system (1.3) (see [28]) occurs when

$$|\arg\lambda_1| = \frac{q\pi}{2}, \quad |\arg\lambda_2| = \frac{q\pi}{2},$$

i.e.,

$$\Big|\frac{\sqrt{4\Delta - \operatorname{tr}^2(J)}}{\operatorname{tr}(J)}\Big| = \tan\frac{q\pi}{2}$$

6. Numerical simulations

Numerical simulations of system (1.2) are given to verify our analytical results by using the matlab programm.

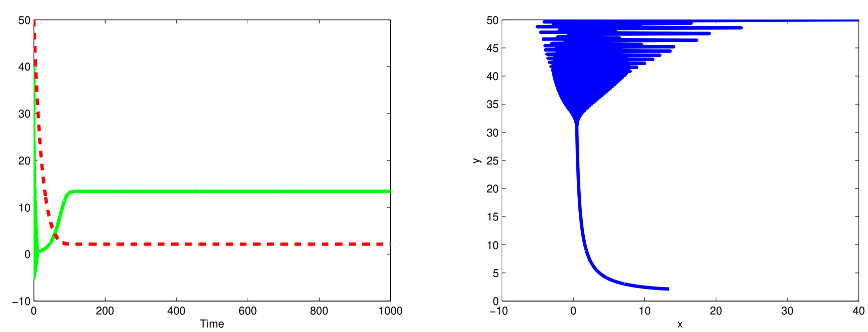

Figure 1. Dynamics and phase plain of the Glucose-Insulin dynamics for $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 4.33$, $q = 0.99$, $f = .09$, $s = .2$

Example 6.1. Let $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times 10^{-5}$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 4.33$, $q = 0.99$, $f = .09$, $s = .2$. The corresponding eigenvalues are $\lambda_1 = -0.0591 + 0.0131i$, $\lambda_2 = -0.0591 - 0.0131i$ for $q = 0.90$, which satisfy the condition $|\arg\lambda| = 2.9239 > q\pi/2 = 1.5551$. Therefore, system (1.2) is stable on $E_1 = (1.0296, 0.1647)$, See Figure 1.

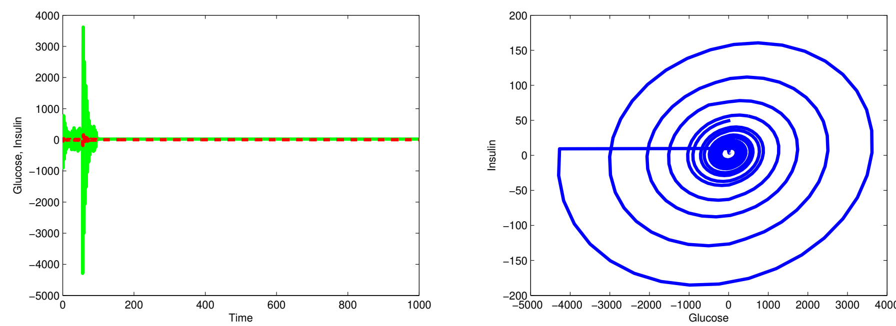

FIGURE 2. Dynamics and phase plain of the Glucose-Insulin dynamics for $p_1 = 0.0002$, $p_2 = 0.02093$, $p_3 = 1.062\times(10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 4.33$, $q = 0.75$, $f = .200111119$, $s = 3.23$

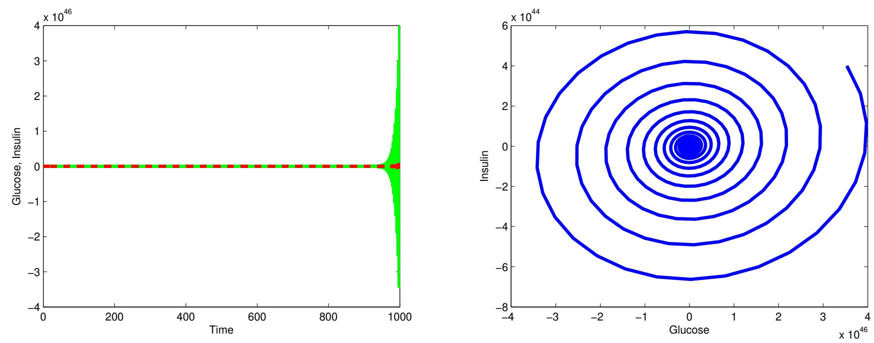

FIGURE 3. Dynamics and phase plain of the Glucose-Insulin dynamics for $p_1 = 0.0002$, $p_2 = 0.02093$, $p_3 = 1.062\times(10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 4.33$, $q = 0.05$, $f = 0.029$, $s = 2.23$

Example 6.2. Let $p_1 = 0.0002$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 4.33$, $q = 0.75$, $f = .200111119$, $s = 3.23$. The corresponding eigenvalues are $\lambda_1 = -0.0295 + 0.0302i$, $\lambda_2 = -0.0295 - 0.0302i$ at $q = 0.99$. Thus, the condition $|\arg\lambda| = 2.3443 > q\pi/2 = 1.1781$ is satisfied. Therefore, system (1.2) is stable on $E_1 = (1.2261, 0.1962)$; see Figure 2.

Example 6.3. Let $p_1 = 0.0002$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 4.33$, $q = 0.05$, $f = 0.029$, $s = 2.23$. The corresponding eigenvalues are $\lambda_1 = -0.0576$, $\lambda_2 = -0.1229$ for $q = 0.99$, which satisfy the condition $|\arg\lambda| = 3.1416 > q\pi/2 = 0.0785$. Therefore, system (1.2) is stable on $E_1 = (4.1709, 0.6674)$, see Figure 3.

Example 6.4. Let $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = .33$, $f = 0.009$, $s = 0.2$. The corresponding eigenvalues are $\lambda_1 = 0.1984$, $\lambda_2 = -0.0402$ for $q = 0.75$, which satisfy

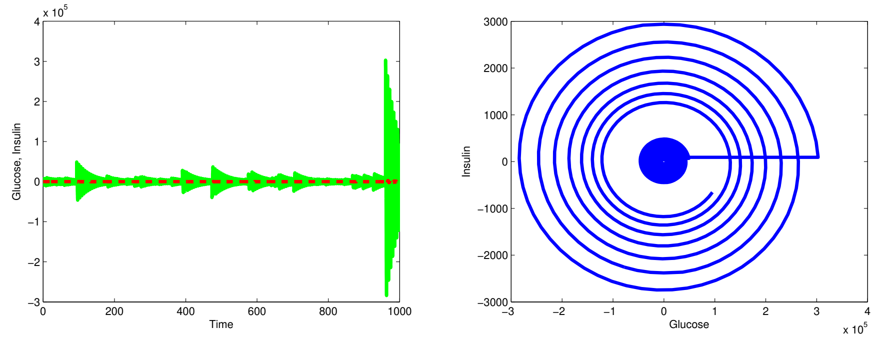

FIGURE 4. Dynamics and phase plain of the Glucose-Insulin dynamics for $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 0.33$, $q = 0.75$, $f = 0.009$, $s = 0.2$

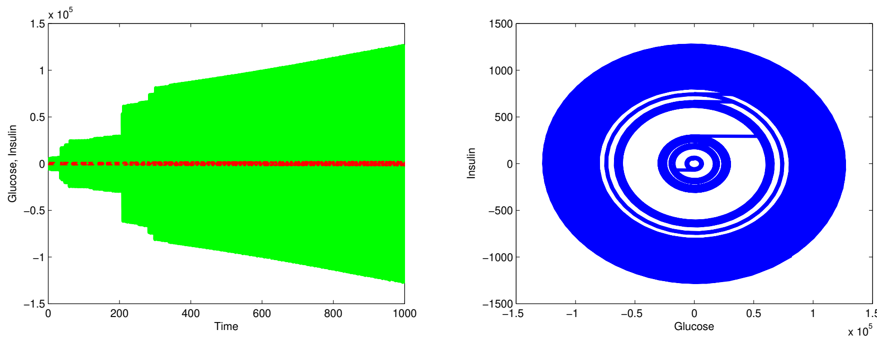

FIGURE 5. Dynamics and phase plain of the Glucose-Insulin dynamics for $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = .33$, $q = 0.55$, $f = 0.009$, $s = 0.2$

the condition $|arg\lambda| = 0 < q\pi/2 = 1.1781$. Therefore, system (1.2) is unstable on $E_1 = (-4.0598, -0.6496)$, see Figure 4.

Example 6.5. Assume that $p_1 = 0.03082$, $p_2 = 0.02093$, $p_3 = 1.062 \times (10^{-5})$, $I_b = 7.3$, $G_b = 92$, $p_4 = 0.3$, $p_5 = 94$, $p_6 = 0.003349$, $p_7 = 0.33$, $f = 0.009$, $s = 0.2$. The corresponding eigenvalues are $\lambda_1 = 0.1984$, $\lambda_2 = -0.0402$ for $q = 0.55$, which satisfy the condition $|\arg\lambda| = 0 < q\pi/2 = 0.8639$. Therefore, system (1.2) is unstable on $E_1 = (-4.0598, -0.6496)$, see Figure 5.

Conclusion. In this work, the fractional-order model (1.2) based on the IVGTT was analyzed to learn the dynamics of the interaction of the glucose and insulin in the human body. A simple discretization scheme was applied to discretize fractional-order system model (1.2). Our results suggested the conditions on the parameters, such the existence of the periodic solution surrounding the equilibrium point. A Hopf bifurcation arises in the analysis. Numerical simulations are carried out to demonstrate the results obtained. Three concrete examples of stability and two examples of instability of certain equilibria are given. From the above discussions, one can deduce that the model is physiologically consistent and the suggested model may be a useful tool for further research on Diabetes Mellitus.

The experimental data in this work was taken from the reference [29].

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