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PERIODIC SOLUTIONS FOR CONFORMABLE TYPE NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this article we study a type of conformable non-instantaneous impulsive equation with periodic effects. We find a Cauchy matrix that can provide solutions of linear and nonlinear problems and prove some of their properties. Also we study the existence of periodic solution of different types of conformable non-instantaneous impulsive differential equation. Some examples also are given to illustrate our theoretical results.

1. Introduction

Hernández and O'Regan [11] introduced the concept of non-instantaneous impulsive equations. After that many results about this equations have been obtained, including periodic differential equations with periodic non-instantaneous impulses; see for example [1, 4, 6, 7, 10, 12, 15, 17, 18, 20, 23, 24, 27]. In [9] an effective framework is given for obtaining periodic solutions of first-order periodic non-instantaneous impulsive problems. Fečkan et al. [19] studied the periodic solutions of second order non-instantaneous impulsive problems. Wang et al. [13, 14, 22, 25, 28, 29, 30] established many results about periodic solutions with non-instantaneous impulses. Abdeljawad [2] introduced the concept of conformable derivatives and studied its basic theory. Recently, many articles about conformable derivatives have appeared, see [3, 5, 8, 16, 21, 26].

In this article, we study the conformable homogeneous linear non-instantaneous impulsive differential equation

$$\begin{gathered}
\mathfrak{D}_{\beta}^{\sigma_k} z(\iota) = Pz(\iota), \quad \iota \in (\sigma_k, \iota_{k+1}],\ k = 0, 1, 2, \dots, \\
z(\iota_k^+) = Qz(\iota_k^-), \quad k = 1, 2, \dots, \\
z(\iota) = Qz(\iota_k^-), \quad \iota \in (\iota_k, \sigma_k],\ k = 1, 2, \dots, \\
z(\sigma_k^+) = z(\sigma_k^-), \quad k = 1, 2, \dots, \\
z(a) = z_a \in \mathbb{R}^n,
\end{gathered} \tag{1.1}$$

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the conformable non-homogeneous linear non-instantaneous impulsive differential equation

$$\begin{gathered}
\mathfrak{D}_{\beta}^{\sigma_k} z(\iota)=P z(\iota)+h(\iota), \quad \iota \in (\sigma_k, \iota_{k+1}],\ k=0,1,2,\dots \\
z(\iota_k^{+})=Q z(\iota_k^{-})+d_k, \quad k=1,2,\dots, \\
z(\iota)=Q z(\iota_k^{-})+d_k, \quad \iota \in (\iota_k, \sigma_k],\ k=1,2,\dots, \\
z(\sigma_k^{+})=z(\sigma_k^{-}), \quad k=1,2,\dots, \\
z(a)=z_a \in \mathbb{R}^n,
\end{gathered} \tag{1.2}$$

and the conformable nonlinear non-instantaneous impulsive differential equation

$$\begin{gathered}
\mathfrak{D}_{\beta}^{\sigma_k} z(\iota)=P z(\iota)+h(t, z(\iota)), \quad \iota \in (\sigma_k, \iota_{k+1}],\ k=0,1,2,\dots \\
z(\iota_k^{+})=Q z(\iota_k^{-})+d_k, \quad k=1,2,\dots, \\
z(\iota)=Q z(\iota_k^{-})+d_k, \quad \iota \in (\iota_k, \sigma_k],\ k=1,2,\dots, \\
z(\sigma_k^{+})=z(\sigma_k^{-}), \quad k=1,2,\dots, \\
z(a)=z_a \in \mathbb{R}^n,
\end{gathered} \tag{1.3}$$

where P and Q are $n \times n$ constant matrices with $PQ = QP$, $0 < \beta < 1$. ι_k and σ_k satisfy $a = \sigma_0 < \iota_1 < \sigma_1 < \dots < \iota_k < \sigma_k < \iota_{k+1} \dots, k = 1, 2, \dots,$ $d_k \in \mathbb{R}^n$. Let E be the unit matrix, $\mathbb{I} = \bigcup_{k=0}^{\infty}(\sigma_k, \iota_{k+1}]$ and $\mathbb{J} = \bigcup_{k=1}^{\infty}(\iota_k, \sigma_k]$ and $h(\cdot) \in C(\mathbb{I}, \mathbb{R}^n)$, $h(\cdot,\cdot) \in C(\mathbb{I} \times \mathbb{R}^n, \mathbb{R}^n)$.

We use the assumption

(A1) ι_k and σ_k satisfy $\iota_{k+q} = \iota_k + T$, $\sigma_{k+q} = \sigma_k + T$ for $n(a, T) = q$ in which $n(a, T)$ denotes the number of impulsive points existing in (a, T). d_k are constant vectors with $d_{k+q} = d_k + T$.

We set $I = [a, +\infty)$ and $PC(I, \mathbb{R}^n) := \{y : I \to \mathbb{R}^n : y \in C\big((\iota_k, \iota_{k+1}], \mathbb{R}^n\big)$, $k = 0, 1, \dots$. There exists $z(\iota_k^-)$ and $z(\iota_k^+)$, $k = 1, 2, \dots$ with $z(\iota_k^-) = z(\iota_k)\}$, where $C\big((\iota_k, \iota_{k+1}], \mathbb{R}^n\big)$ denotes the space of all continuous functions from $(\iota_k, \iota_{k+1}]$ into $\mathbb{R}^n$. We denote a vector $\theta = (\theta_1, \dots, \theta_n)^\top \in \mathbb{R}^n$ with its norm $\|\theta\| = \sum_{i=1}^n |\theta_i|$ and a matrix $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ with its matrix norm $\|\kappa\| = \max_{\|y\|=1} \|\kappa y\|$.

This article is organized as follows: In Section 2, we introduce some basic theory and give the solution of (1.1) and (1.2) for non-instantaneous impulsive Cauchy matrix $W(\cdot,\cdot)$. Also we give some properties of $W(\cdot,\cdot)$. Section 3 concerns the existence of T-periodic solutions of (1.2) with two types of conditions. In Section 4, two lemmas prove the existence of T-periodic solutions of (1.3).

2. Preliminaries

Definition 2.1 (see [2, Definition 2.1])**.** The conformable derivative with lower index a of a function $x : [a, \infty) \to \mathbb{R}$ is defined as

$$\begin{gathered}
\mathfrak{D}_{\beta}^{a} x(\iota)=\lim_{\varepsilon \to 0} \frac{x(\iota+\varepsilon(\iota-a)^{1-\beta})-x(\iota)}{\varepsilon},\ \iota>a,\ 0<\beta<1, \\
\mathfrak{D}_{\beta}^{a} x(a)=\lim_{\iota \to a^{+}} \mathfrak{D}_{\beta}^{a} x(\iota).
\end{gathered}$$

Remark 2.2 ([2])**.** If $\mathfrak{D}_{\beta}^{a} x(\iota_a)$ exists and is finite, we say that x is β-differentiable at ι_a. If $x \in C^1((a, \infty], \mathbb{R})$, then $\mathfrak{D}_{\beta}^{a} x(\iota) = (\iota - a)^{1-\beta} x'(\iota)$. The conformable derivative $\mathfrak{D}_{\beta}^{a} x(\iota)$ exists if and only if x is differentiable at ι and $\mathfrak{D}_{\beta}^{a} x(\iota) = (\iota - a)^{1-\beta} x'(\iota)$ for $\iota > a$.

Definition 2.3 (see [2, Notation]). The conformable integral with lower index a of a function $x:[a,\infty)\to\mathbb{R}$ is written as

$$\mathfrak{I}_\beta^a x(\iota)=\int_a^\iota x(\sigma)d_\beta(\sigma,a)=\int_a^\iota(\sigma-a)^{\beta-1}x(\sigma)d\sigma,\ \iota\geq a, 0<\beta<1,$$

if $a=0$, then we write $d_\beta(\sigma,a)$ as $d_\beta(\sigma)$.

Lemma 2.4. *The solution $z(\cdot,\cdot,\cdot)\in PC([a,\infty)\times[a,\infty),\mathbb{R}^n)$ of (1.1) with the initial condition $z(\sigma)=z_\sigma$ has the form*

$$z(\iota):=z(\iota,\sigma,z_\sigma)=W(\iota,\sigma)z_\sigma,\quad \iota\geq a, \tag{2.1}$$

in which

$$W(\iota,\sigma)=Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+-\left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^+ +\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}, \tag{2.2}$$

where $j^+:=\max\{0,j\}$, $j\in\mathbb{R}$. When $n(a,\iota)=n(a,\sigma)$, we have $\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}=0$. In particular with $\sigma=a$,

$$z(\iota,a,z_a)=Q^{n(a,\iota)}e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+ +\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}((\iota_{k+1}-\sigma_k)^\beta\right]}z_a=W(\iota,a)z_a. \tag{2.3}$$

Proof. There are several cases to be considered.

Case 1:: $n(a,\iota)$ and $n(a,\sigma)$ satisfy $n(a,\iota)=n(a,\sigma)$.

(i) For any $t,\sigma\in(\sigma_k,\iota_{k+1}]$, $k=0,1,2,\dots$, when $\iota\in(a,\iota_1]$, we have

$$z(\iota)=e^{\frac{P}{\beta}(\iota-\sigma_0)^\beta}z_a.$$

When $\iota\in(\iota_1,\sigma_1]$, we have

$$z(\iota)=Qz(\iota_1^-)=Qe^{\frac{P}{\beta}(\iota_1-\sigma_0)^\beta}z_a.$$

Then, for $\iota,\sigma\in(\sigma_1,\iota_2]$, we have

$$z(\iota)=e^{\frac{P}{\beta}(\iota-\sigma_1)^\beta}z(\sigma_1)=e^{\frac{P}{\beta}(\iota-\sigma_1)^\beta}Qe^{\frac{P}{\beta}(\iota_1-\sigma_0)^\beta}z_a,$$
$$z(\sigma)=e^{\frac{P}{\beta}(\sigma-\sigma_1)^\beta}z(\sigma_1)=e^{\frac{P}{\beta}(\sigma-\sigma_1)^\beta}Qe^{\frac{P}{\beta}(\iota_1-\sigma_0)^\beta}z_a,$$

so

$$W(\iota,\sigma)=e^{\frac{P}{\beta}\left((\iota-\sigma_1)^\beta-(\sigma-\sigma_1)^\beta\right)}.$$

In summary for any $\iota,\sigma\in(\sigma_k,\iota_{k+1}]$, it holds

$$W(\iota,\sigma)=e^{\frac{P}{\beta}\left((\iota-\sigma_k)^\beta-(\sigma-\sigma_k)^\beta\right)}.$$

(ii) For any $\iota,\sigma\in(\iota_k,\sigma_k]$, $k=1,2,\dots$, we have $z(\iota)=Qz(\iota_k^-)$, $k=1,2,\dots$, so $z(\iota)=z(\sigma)$.

(iii) For any $\iota\in(\sigma_{n(a,\iota)},\iota_{n(a,\iota)+1})$ and any $\sigma\in(\iota_{n(a,\sigma)},\sigma_{n(a,\sigma)}]$, we have

$$z(\iota)=e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma_{n(a,\iota)}^+)=e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma_{n(a,\iota)}^-)=e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma),$$

so

$$W(\iota,\sigma)=e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}.$$

Case 2: $n(a,\iota)$ and $n(a,\sigma)$ satisfy $n(a,\iota)=n(a,\sigma)+1$.

(i) For any $\iota \in (\iota_{n(a,\iota)}, \sigma_{n(a,\iota)}]$ and any $\sigma \in (\sigma_{n(a,\sigma)}, \iota_{n(a,\sigma)+1}]$, we have $z(\iota) = Qz(\iota^-_{n(a,\iota)})$ and

$$z(\iota) = Qz(\iota^-_{n(a,\iota)}) = Qe^{\frac{P}{\beta}\left((\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta\right)}z(\sigma),$$

so

$$W(\iota,\sigma) = Qe^{\frac{P}{\beta}\left((\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta\right)}.$$

(ii) For any $\iota \in (\sigma_{n(a,\iota)}, \iota_{n(a,\iota)+1}]$ and any $\sigma \in (\sigma_{n(a,\sigma)}, \iota_{n(a,\sigma)+1}]$, we have

$$\begin{aligned}
z(\iota) &= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma^+_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma^-_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}Qz(\iota^-_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}Qe^{\frac{P}{\beta}\left((\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta\right)}z(\sigma),
\end{aligned}$$

so

$$W(\iota,\sigma) = Qe^{\frac{P}{\beta}\left((\iota-\sigma_{n(a,\iota)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta+(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta\right)}.$$

(iii) For any $\iota \in (\iota_{n(a,\iota)}, \sigma_{n(a,\iota)}]$ and any $\sigma \in (\iota_{n(a,\sigma)}, \sigma_{n(a,\sigma)}]$,there is

$$\begin{aligned}
z(\iota) &= Qz(\iota^-_{n(a,\iota)})\\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta}z(\sigma^+_{n(a,\sigma)})\\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta}z(\sigma^-_{n(a,\sigma)})\\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta}z(\sigma),
\end{aligned}$$

so

$$W(\iota,\sigma) = Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta}.$$

(iv) For any $\iota \in (\sigma_{n(a,\iota)}, \iota_{n(a,\iota)+1}]$ and any $\sigma \in (\iota_{n(a,\sigma)}, \sigma_{n(a,\sigma)}]$, we have

$$\begin{aligned}
z(\iota) &= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma^+_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}z(\sigma^-_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}Qz(\iota^-_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta}Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta}z(\sigma^+_{n(a,\iota)})\\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta)}Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta}z(\sigma),
\end{aligned}$$

so

$$W(\iota,\sigma) = Qe^{\frac{P}{\beta}\left((\iota-\sigma_{n(a,\iota)})^\beta+(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)})^\beta\right)}.$$

Case 3: $n(a,\iota)$ and $n(a,\sigma)$ satisfy $n(a,\iota) = n(a,\sigma)+2$.

(i) For any $\iota \in (\iota_{n(a,\iota)}, \sigma_{n(a,\iota)}]$ and any $\sigma \in (\sigma_{n(a,\sigma)}, \iota_{n(a,\sigma)+1}]$, we have

$$\begin{aligned}
z(\iota) &= Qz(\iota^-_{n(a,\iota)})\\
&= Qe^{\frac{p}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta}z(\sigma^+_{n(a,\sigma)+1})\\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta}z(\sigma^-_{n(a,\sigma)+1})\\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta}Qz(\iota^-_{n(a,\sigma)+1})
\end{aligned}$$

$$= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} Qe^{\frac{P}{\beta}\left((\iota_{n(a,\sigma)+1}-\sigma_{n(a,\sigma)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta\right)} z(\sigma),$$

so that

$$W(\iota,\sigma) = Q^2 e^{\frac{P}{\beta}\left(-(\sigma-\sigma_{n(a,\sigma)})^\beta+(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta+(\iota_{n(a,\sigma)+1}-\sigma_{n(a,\sigma)})^\beta\right)}.$$

(ii) For any $\iota \in (\sigma_{n(a,\iota)}, \iota_{n(a,\iota)+1}]$ and any $\sigma \in (\sigma_{n(a,\sigma)}, \iota_{n(a,\sigma)+1}]$, we have

$$\begin{aligned}
z(\iota) &= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} z(\sigma^+_{n(a,\iota)}) \\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} z(\sigma^-_{n(a,\iota)}) \\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} Qz(\iota^-_{n(a,\iota)}) \\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} z(\sigma^+_{n(a,\iota)-1}) \\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} z(\sigma^-_{n(a,\sigma)+1}) \\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta)} Qz(\iota^-_{n(a,\sigma)+1}) \\
&= e^{\frac{P}{\beta}(\iota-\sigma_{n(a,\iota)})^\beta} Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} Qe^{\frac{P}{\beta}\left((\iota_{n(a,\sigma)+1}-\sigma_{n(a,\sigma)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta\right)} z(\sigma),
\end{aligned}$$

so

$$W(\iota,\sigma) = Q^2 e^{\frac{P}{\beta}\left((\iota-\sigma_{n(a,\iota)})^\beta+(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta+(\iota_{n(a,\sigma)+1}-\sigma_{n(a,\sigma)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta\right)}.$$

(iii) For any $\iota \in (\iota_{n(a,\iota)}, \sigma_{n(a,\iota)}]$ and any $\sigma \in (\iota_{n(a,\sigma)}, \sigma_{n(a,\sigma)}]$, we have

$$\begin{aligned}
z(\iota) &= Qz(\iota^-_{n(a,\iota)}) \\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} z(\sigma^+_{n(a,\sigma)+1}) \\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} z(\sigma^-_{n(a,\sigma)+1}) \\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta)} Qz(\iota^-_{n(a,\sigma)+1}) \\
&= Qe^{\frac{P}{\beta}(\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta} Qe^{\frac{P}{\beta}(\iota_{n(a,\sigma)+1}-\sigma_{n(a,\sigma)})^\beta} z(\sigma),
\end{aligned}$$

so

$$W(\iota,\sigma) = Q^2 e^{\frac{P}{\beta}\left((\iota_{n(a,\iota)}-\sigma_{n(a,\sigma)+1})^\beta+(\iota_{n(a,\sigma)+1}-\sigma_{n(a,\sigma)})^\beta\right)}.$$

Case 4: General $n(a,\iota)$ and $n(a,\sigma)$.

(i) For any $\iota \in (\sigma_{n(a,\iota)}, \iota_{n(a,\iota)+1}]$ and any $\sigma \in (\iota_{n(a,\sigma)}, \sigma_{n(a,\sigma)}]$, we have

$$W(\iota,\sigma) = Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\left[(\iota-\sigma_{n(a,\iota)})^\beta+\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}.$$

(ii) For any $\iota \in (\iota_{n(a,\iota)}, \sigma_{n(a,\iota)}]$ and any $\sigma \in (\sigma_{n(a,\sigma)}, \iota_{n(a,\sigma)+1}]$, we have

$$W(\iota,\sigma) = Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\left[-(\sigma-\sigma_{n(a,\sigma)})^\beta+\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta)\right]}.$$

(iii) For any $\iota \in (\sigma_{n(a,\iota)}, \iota_{n(a,\iota)+1}]$ and any $\sigma \in (\sigma_{n(a,\sigma)}, \iota_{n(a,\sigma)+1}]$, we have

$$W(\iota,\sigma) = Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\left[(\iota-\sigma_{n(a,\iota)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta+\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}.$$

(iv) For any $\iota \in (\iota_{n(a,\iota)}, \sigma_{n(a,\iota)}]$ and any $\sigma \in (\iota_{n(a,\sigma)}, \sigma_{n(a,\sigma)}]$, we have

$$W(\iota,\sigma) = Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta}.$$

Thus $W(\iota,\sigma)$ can be written in the form

$$W(\iota,\sigma)=\begin{cases} e^{\frac{P}{\beta}\left((\iota-\sigma_k)^\beta-(\sigma-\sigma_k)^\beta\right)},\\ \quad \text{if } \iota,\sigma\in(\sigma_k,\iota_{k+1}],\ k=0,1,2,\dots;\\ E,\quad \text{if } \iota,\sigma\in(\iota_k,\sigma_k],\ k=1,2,\dots,n(a,\iota);\\ Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\left[-(\sigma-\sigma_{n(a,\sigma)})^\beta+\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]},\\ \quad \text{if } \sigma\in(\sigma_{n(a,\sigma)},\iota_{n(a,\sigma)+1}],\ \iota\in(\iota_{n(a,\iota)},\sigma_{n(a,\iota)}];\\ Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\left[(\iota-\sigma_{n(a,\iota)})^\beta+\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]},\\ \quad \text{if } \sigma\in(\iota_{n(a,\sigma)},\sigma_{n(a,\sigma)}],\ \iota\in(\sigma_{n(a,\iota)},\iota_{n(a,\iota)+1}];\\ Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\left[((\iota-\sigma_{n(a,\iota)})^\beta-(\sigma-\sigma_{n(a,\sigma)})^\beta+\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}((\iota_{k+1}-\sigma_k)^\beta\right]},\\ \quad \text{if } \sigma\in(\sigma_{n(a,\sigma)},\iota_{n(a,\sigma)+1}],\ \iota\in(\sigma_{n(a,\iota)},\iota_{n(a,\iota)+1}];\\ Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta},\\ \quad \text{if } \sigma\in(\iota_{n(a,\sigma)},\sigma_{n(a,\sigma)}],\ \iota\in(\iota_{n(a,\iota)},\sigma_{n(a,\iota)}], \end{cases}$$

so $W(\iota,\sigma)$ can be written as

$$\begin{aligned} &W(\iota,\sigma)\\ &=Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+-\left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^++\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}. \end{aligned} \tag{2.4}$$

In particular, if $\sigma=a$, we have

$$W(\iota,a)=Q^{n(a,\iota)}e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^++\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}.$$

The proof is complete. □

Lemma 2.5. *For $\iota,\sigma\in I,\eta\in I$, if* (A1) *holds, then*

$$W(\iota,\sigma)=W(\iota,\eta)W(\eta,\sigma),\quad \sigma\le\eta\le\iota.$$

Proof. From $\iota,\sigma\in I$ and $\eta\in I$, we obtain

$$\begin{aligned} W(\iota,\sigma)&=Q^{n(a,\iota)-n(a,\sigma)}e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+-\left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^++\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}\\ &=Q^{n(a,\iota)-n(a,\eta)}e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+-\left((\eta-\sigma_{n(a,\eta)})^\beta\right)^++\sum_{k=n(a,\eta)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}\\ &\quad\times Q^{n(a,\eta)-n(a,\sigma)}e^{\frac{P}{\beta}\left[\left((\eta-\sigma_{n(a,\eta)})^\beta\right)^+-\left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^++\sum_{k=n(a,\sigma)}^{n(a,\eta)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}\\ &=W(\iota,\eta)W(\eta,\sigma). \end{aligned}$$

The proof is complete. □

Lemma 2.6. *If* (A1) *holds, then $W(\cdot+T,\cdot+T)=W(\cdot,\cdot)$.*

Proof. It is clear that $n(a,\iota+T)=n(a,T)+n(T,\iota+T)=q+n(T,\iota+T)=q+n(a,\iota)$, so $\sigma_{n(a,\iota+T)}=\sigma_{n(a,\iota)+q}=\sigma_{n(a,\iota)}+T$. According (2.4), for $a\le\sigma<\iota\le T$, we obtain

$$\begin{aligned} &W(\iota+T,\sigma+T)\\ &=Q^{n(a,\iota+T)-n(a,\sigma+T)}e^{\Big\{\frac{P}{\beta}\left[\left((\iota+T-\sigma_{n(a,\iota+T)})^\beta\right)^+\right.} \end{aligned}$$

$$
\begin{aligned}
&- \Big(\big(\sigma+T-\sigma_{n(a,\sigma+T)}\big)^{\beta}\Big)^{+} + \sum_{k=n(a,\sigma+T)}^{n(a,\iota+T)-1} (\iota_{k+1}-\sigma_k)^{\beta}\Big]\Big\} \\
&= Q^{n(a,T)+n(T,\iota+T)-(n(a,T)+n(T,\sigma+T))} e^{\big\{\frac{P}{\beta}\Big[\Big(\big(\iota+T-\sigma_{n(a,\iota+T)}\big)^{\beta}\Big)^{+}} \\
&- \Big(\big(\sigma+T-\sigma_{n(a,\sigma+T)}\big)^{\beta}\Big)^{+} + \sum_{k=n(a,\sigma)+q}^{n(a,\iota)+q-1} (\iota_{k+1}-\sigma_k)^{\beta}\Big]\Big\} \\
&= Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\Big[\Big(\big(\iota-\sigma_{n(a,\iota)}\big)^{\beta}\Big)^{+} - \Big(\big(\sigma-\sigma_{n(a,\sigma)}\big)^{\beta}\Big)^{+} + \sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\Big]}.
\end{aligned}
$$

The proof is complete. □

Lemma 2.7. *Suppose that* (A1) *holds, and for every* $t,\sigma \in I, \eta \in I$ *with* $\sigma \le \eta \le \iota$, *then* $W(\iota,a)W(T,\sigma) = W(\iota,\sigma)W(T,a)$.

Proof. Calculating each side of the equality yields

$$
\begin{gathered}
W(\iota,a)W(T,\sigma) = W(\iota+T,T)W(T,\sigma) = W(\iota+T,\sigma), \\
W(\iota,\sigma)W(T,a) = W(\iota,\sigma)W(\iota+T,\iota) = W(\iota+T,\iota)W(\iota,\sigma) = W(\iota+T,\sigma).
\end{gathered}
$$

So $W(\iota,a)W(T,\sigma) = W(\iota,\sigma)W(T,a)$. and he proof is complete. □

Lemma 2.8. *A solution* $z \in PC(I,\mathbb{R}^n)$ *of* (1.2) *with* $z(a) = z_a \in \mathbb{R}^n$ *has the form*

$$
\begin{aligned}
z(\iota,a,z_a) = W(\iota,a)z_a &+ \sum_{k=0}^{n(a,\iota)-1} \int_{\sigma_k}^{\iota_{k+1}} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1} d\sigma \\
&+ \int_{\sigma_{n(a,\iota)}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1} d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k.
\end{aligned} \tag{2.5}
$$

Set

$$
\chi(\iota) = \begin{cases} (\iota-\sigma_k)^{\beta-1}, & \iota \in (\sigma_k,\iota_{k+1}],\ k=0,1,2,\dots, \\ 0, & \iota \in (\iota_k,\sigma_k],\ k=1,2,\dots. \end{cases}
$$

So (2.5) *can be rewritten as*

$$
z(\iota,a,z_a) = W(\iota,a)z_a + \int_a^{\iota} \chi(\iota)W(\iota,\sigma)h(\sigma)d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k.
$$

Proof. When $\iota \in [\sigma_0,\iota_1]$, it holds $\mathfrak{D}_{\beta}^{\sigma_0} z(\iota) = Pz(\iota)$, and the solution of the above equation is

$$
z(\iota) = W(\iota,a)z_a.
$$

When $z_a = z_a(\iota)$, we obtain

$$
\begin{aligned}
\mathfrak{D}_{\beta}^{\sigma_0} z(\iota) &= \mathfrak{D}_{\beta}^{\sigma_0} W(\iota,a)z_a(\iota) + W(\iota,a)\mathfrak{D}_{\beta}^{\sigma_0} z_a(\iota) \\
&= Pz(\iota) + W(\iota,a)(\iota-\sigma_0)^{1-\beta} z_a'(\iota) \\
&= Pz(\iota) + c(\iota).
\end{aligned}
$$

Then

$$
\begin{gathered}
y_a'(\iota) = W^{-1}(\iota,a)c(\iota)(\iota-\sigma_0)^{\beta-1}, \\
z_a(\iota) = \int_{\sigma_0}^{\iota} W^{-1}(\sigma,a)h(\sigma)(\sigma-\sigma_0)^{\beta-1} d\sigma + z_a.
\end{gathered}
$$

By comparing both side of the equation, we obtain

$$\begin{aligned} z(\iota) &= W(\iota, a)\Big[\int_{\sigma_0}^{\iota} W^{-1}(\sigma, a)h(\sigma)(\sigma-\sigma_0)^{\beta-1}d\sigma + z_a\Big] \\ &= W(\iota, a)z_a + \int_{\sigma_0}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_0)^{\beta-1}d\sigma. \end{aligned}$$

When $\iota \in (\iota_1, \sigma_1]$, we have

$$z(\iota) = Qz(\iota_1^-) + d_1 = NW(\iota_1, a)z_a + N\int_{\sigma_0}^{\iota_1} W(\iota_1,\sigma)h(\sigma)(\sigma-\sigma_0)^{\beta-1}d\sigma + d_1.$$

When $\iota \in (\sigma_1, \iota_2]$, we have

$$\begin{aligned} z(\iota) &= W(\iota,\sigma_1)z(\sigma_1) + \int_{\sigma_1}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_1)^{\beta-1}d\sigma \\ &= W(\iota,\sigma_1)QW(\iota_1,a)z_a + W(\iota,\sigma_1)Q\int_{\sigma_0}^{\iota_1} W(\iota_1,\sigma)h(\sigma)(\sigma-\sigma_0)^{\beta-1}d\sigma \\ &\quad + W(\iota,\sigma_1)d_1 + \int_{\sigma_1}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_1)^{\beta-1}d\sigma \\ &= W(\iota,a)z_a + \int_{\sigma_0}^{\iota_1} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_0)^{\beta-1}d\sigma \\ &\quad + \int_{\sigma_1}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_1)^{\beta-1}d\sigma + W(\iota,\sigma_1)d_1. \end{aligned} \tag{2.6}$$

For a positive integer n and $\iota \in (\sigma_{n(a,\iota)}, \iota_{n(a,\iota)+1}]$, it holds

$$\begin{aligned} z(\iota) &= W(\iota,a)z_a + \sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\ &\quad + \int_{\sigma_{n(a,\iota)}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k. \end{aligned}$$

When $\iota \in (\iota_{n(a,\iota)+1}, \sigma_{n(a,\iota)+1}]$, we have

$$\begin{aligned} z(\iota) &= Qz(\iota_{n(a,\iota)+1}^-) + d_{n(a,\iota)+1} \\ &= Q\Big[W(\iota_{n(a,\iota)+1},\sigma_0)z_a + \sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota_{n(a,\iota)+1},\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\ &\quad + \int_{\sigma_{n(a,\iota)}}^{\iota_{n(a,\iota)+1}} W(\iota_{n(a,\iota)+1},\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma \\ &\quad + \sum_{k=1}^{n(a,\iota)} W(\iota_{n(a,\iota)+1},\sigma_k)d_k\Big] + d_{n(a,\iota)+1}. \end{aligned}$$

When $\iota \in (\sigma_{n(a,\iota)+1}, \sigma_{n(a,\iota)+2}]$, we have

$$\begin{aligned} z(\iota) &= W(\iota,\sigma_{n(a,\iota)+1})z(\sigma_{n(a,\iota)+1}) + \int_{\sigma_{n(a,\iota)+1}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)+1})^{\beta-1}d\sigma \\ &= W(\iota,\sigma_{n(a,\iota)+1})QW(\iota_{n(a,\iota)+1},a)z_a \end{aligned}$$

$$
\begin{aligned}
&+ W(\iota,\sigma_{n(a,\iota)+1})Q\sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota_{n(a,\iota)+1},\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma\\
&+ W(\iota,\sigma_{n(a,\iota)+1})Q\int_{\sigma_{n(a,\iota)}}^{\iota_{n(a,\iota)+1}} W(\iota_{n(a,\iota)+1},\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma\\
&+ W(\iota,\sigma_{n(a,\iota)+1})Q\sum_{k=1}^{n(a,\iota)} W(\iota_{n(a,\iota)+1},\sigma_k)d_k\\
&+ W(\iota,\sigma_{n(a,\iota)+1})d_{n(a,\iota)+1} + \int_{\sigma_{n(a,\iota)+1}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)+1})^{\beta-1}d\sigma\\
&= W(\iota,a)z_a + \sum_{k=0}^{n(a,\iota)}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma\\
&+ \int_{\sigma_{n(a,\iota)+1}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)+1})^{\beta-1}d\sigma + \sum_{k=1}^{n(a,\iota)+1} W(\iota,\sigma_k)d_k.
\end{aligned}
$$

Using induction, we can show that the solution of (1.2) with $z(a) = z_a \in \mathbb{R}^n$ has the form

$$
\begin{aligned}
z(\iota) &= W(\iota,a)z_a + \sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma\\
&\quad + \int_{\sigma_{n(a,\iota)}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k \qquad (2.7)\\
&= W(\iota,a)z_a + \int_a^{\iota}\chi(\iota)W(\iota,\sigma)h(\sigma)d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k.
\end{aligned}
$$

The proof is complete. □

A function $z(\cdot,a,z_a) \in PC([a,\infty),\mathbb{R}^n)$ is T-periodic if $z(\iota,a,z_a) = z(\iota+T,a,z_a)$, $\iota \geq a$. We define

$$PC_T(I,\mathbb{R}^n) = \{z \in PC(I,\mathbb{R}^n) : z(\iota) = z(\iota+T), \iota \geq a\}.$$

Theorem 2.9. *If* (A1) *holds, then* (1.1) *has a solution* $z \in PC_T(I,\mathbb{R}^n)$ *if an only if* $(E - W(T,a))z_a = 0$.

Proof. Lemma 2.4 gives $z(\iota,a,z_a) = W(\iota,a)z_a$, so that

$$
\begin{aligned}
z(\iota+T,a,z_a) = z(\iota,a,z_a) &\Longleftrightarrow W(\iota+T,a)z_a = W(\iota,a)z_a\\
&\Longleftrightarrow W(\iota+T,T)W(T,a)z_a = W(\iota,a)z_a\\
&\Longleftrightarrow W(\iota,a)W(T,a)z_a = W(\iota,a)z_a\\
&\Longleftrightarrow (E - W(T,a))z_a = 0.
\end{aligned}
$$

The proof is complete. □

Example 2.10. Consider (1.1) and let $\beta = 1/2$, $\sigma_0 = 0$, $s_k = k$, $\iota_k = k - \frac{1}{2}$, $k = 1,2,\dots,T = 1$, $q = 1$. We set

$$P = \begin{pmatrix} 2 & 1\\ 0 & 1\end{pmatrix},\quad Q = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\quad z_0 = \begin{pmatrix} 1\\ 1\end{pmatrix},$$

so that
$$e^{Pt\beta} = \begin{pmatrix} e^{4t} & e^{4t} - e^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$
And we can obtain
$$\begin{aligned}
W(\iota,0) &= Q^{n(0,\iota)} e^{\frac{P}{\beta}\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \\
&= e^{4\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \begin{pmatrix} 1 & n(0,\iota) \\ 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 1 - e^{-2\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \\ 0 & e^{-2\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \end{pmatrix} \\
&= e^{4\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \\
&\quad \times \begin{pmatrix} 1 & 1 + (n(0,\iota)-1) e^{-2\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \\ 0 & e^{-2\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \end{pmatrix}.
\end{aligned}$$
Then
$$\begin{aligned}
z(\iota,0,z_0) &= W(\iota,0)z_0 \\
&= e^{4\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \\
&\quad \times \begin{pmatrix} 2 + (n(0,\iota)-1) e^{-2\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \\ e^{-2\big[\big((\iota-\sigma_{n(0,\iota)})^{\beta}\big)^{+} + \sum_{k=0}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\big]} \end{pmatrix}.
\end{aligned}$$
Then
$$W(1,0) = \begin{pmatrix} e^{2^{3/2}} & e^{2^{3/2}} \\ 0 & e^{2^{1/2}} \end{pmatrix}, \quad (E - W(1,0))z_a \neq 0,$$
so (1.1) only has the trivial 1-periodic solution.

3. Nonhomogeneous linear non-instantaneous impulsive problem

In this section, we study the existence of T-periodic solution of (1.2) in this section and consider the following assumptions:

- (A2) $\det(E - W(T,a)) \neq 0$;
- (A3) $\det(E - W(T,a)) = 0$;
- (A4) there are constants $u \in \mathbb{R}$ and $J \geq 1$ such that $\|\exp\{A\iota\}\| \leq Je^{ut}$, $\iota \geq a$;
- (A5) for any $\iota \in \mathbb{I}$, it holds $h(\iota + T) = h(\iota)$.

Lemma 3.1. *Assume* (A1), (A2), (A5). *Then the solution $z \in PC([a,T],\mathbb{R}^n)$ of* (1.2) *with $z(T) = z(a)$ has the form*
$$z(\iota,a,z_a) = \int_a^T \chi(\iota)\phi(\iota,\sigma)h(\sigma)d\sigma + \sum_{k=1}^{q} \phi(\iota,\sigma_k)d_k,$$
where
$$\phi(\iota,\sigma) = \begin{cases} \big(W(T,a)(E - W(T,a))^{-1} + E\big)W(\iota,\sigma), & a < s < t, \\ W(\iota,a)(E - W(T,a))^{-1}W(T,\sigma), & \iota \leq \sigma \leq T. \end{cases} \tag{3.1}$$

Proof. From Lemma (2.4), we obtain

$$
\begin{aligned}
z(T,a,z_a) &= W(T,a)z_a + \sum_{k=0}^{q-1}\int_{\sigma_k}^{\iota_{k+1}} W(T,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_q}^{T} W(T,\sigma)h(\sigma)(\sigma-\sigma_q)^{\beta-1}d\sigma + \sum_{k=1}^{q} W(T,\sigma_k)d_k = z_a,
\end{aligned}
$$

so that

$$
\begin{aligned}
z_a &= (E-W(T,a))^{-1}\Big(\sum_{k=0}^{q-1}\int_{\sigma_k}^{\iota_{k+1}} W(T,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_q}^{T} W(T,\sigma)h(\sigma)(\sigma-\sigma_q)^{\beta-1}d\sigma + \sum_{k=1}^{q} W(T,\sigma_k)d_k\Big)
\end{aligned}
$$

Then, the solution of (1.2) has the form

$$
\begin{aligned}
& z(\iota,a,z_a) \\
&= W(\iota,a)(E-W(T,a))^{-1}\Big(\sum_{k=0}^{q-1}\int_{\sigma_k}^{\iota_{k+1}} W(T,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_q}^{T} W(T,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma + \sum_{k=1}^{q} W(T,\sigma_k)d_k\Big) \\
&\quad + \sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_{n(a,\iota)}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k \\
&= \Big(\sum_{k=0}^{q-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_q}^{T} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma)h(\sigma)(\sigma-\sigma_q)^{\beta-1}d\sigma \\
&\quad + \sum_{k=1}^{q} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma_k)d_k\Big) \\
&\quad + \sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_{n(a,\iota)}}^{\iota} W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma + \sum_{k=1}^{n(a,\iota)} W(\iota,\sigma_k)d_k \\
&= \sum_{k=0}^{n(a,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}} \big(W(T,a)(E-W(T,a))^{-1}+E\big)W(\iota,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma \\
&\quad + \int_{\sigma_{n(a,\iota)}}^{\iota} \big(W(T,a)(E-W(T,a))^{-1}+E\big)W(\iota,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\iota}^{\iota_{n(a,\iota)}+1} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma)h(\sigma)(\sigma-\sigma_{n(a,\iota)})^{\beta-1}d\sigma\\
&+\sum_{k=n(a,\iota)+1}^{q-1}\int_{\sigma_k}^{\iota_{k+1}} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma)h(\sigma)(\sigma-\sigma_k)^{\beta-1}d\sigma\\
&+\int_{\sigma_q}^{T} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma)h(\sigma)(\sigma-\sigma_q)^{\beta-1}d\sigma\\
&+\sum_{k=1}^{n(a,\iota)}\big(W(T,a)(E-W(T,a))^{-1}+E\big)W(\iota,\sigma_k)d_k\\
&+\sum_{k=n(a,\iota)+1}^{q} W(\iota,a)(E-W(T,a))^{-1}W(T,\sigma_k)d_k\\
=&\int_a^T \chi(\iota)\phi(\iota,\sigma)h(\sigma)d\sigma+\sum_{k=1}^{q}\phi(\iota,\sigma_k)d_k.
\end{aligned}
$$

The proof is complete. □

Then we consider $\det(E-W(T,a))=0$. We assume that Q is invertible and consider the adjoint system of (1.1) with the form

$$
\begin{gathered}
\mathfrak{D}_{\beta}^{\sigma_k}x(\iota)=-P^{\top}x(\iota),\quad \iota\in(\sigma_k,\iota_{k+1}],\ k=0,1,2,\dots,\\
x(\iota_k^+)=[Q^{\top}]^{-1}x(\iota_k^-),\quad k=1,2,\dots,\\
x(\iota)=[Q^{\top}]^{-1}x(\iota_k^-),\quad \iota\in(\iota_k,\sigma_k],\ k=1,2,\dots,\\
x(\sigma_k^+)=x(\sigma_k^-),\quad k=1,2,\dots,\\
x(a)=x_a\in\mathbb{R}^n.
\end{gathered}
\tag{3.2}
$$

Theorem 3.2. *Let y and x be the solution of* (1.1) *and* (3.2)*, respectively. Then $\langle z(\iota),x(\iota)\rangle=c$ for $\iota\ge a$, where c is a constant.*

Proof. Let $\iota\in(\sigma_k,\iota_{k+1}]$, $k=0,1,\dots$. Then

$$
\begin{aligned}
\mathfrak{D}_{\beta}^{\sigma_k}\langle z(\iota),x(\iota)\rangle&=\langle \mathfrak{D}_{\beta}^{\sigma_k}z(\iota),x(\iota)\rangle+\langle z(\iota),\mathfrak{D}_{\beta}^{\sigma_k}x(\iota)\rangle\\
&=\langle Pz(\iota),x(\iota)\rangle+\langle z(\iota),-P^{\top}x(\iota)\rangle\\
&=\langle z(\iota),P^{\top}x(\iota)\rangle+\langle z(\iota),-P^{\top}x(\iota)\rangle=0.
\end{aligned}
$$

Let $\iota\in(\iota_k,\sigma_k]$, $k=1,2,\dots$. Then

$$
\begin{aligned}
\langle z(\iota),x(\iota)\rangle&=\langle Qz(\iota_k^-),[Q^{\top}]^{-1}x(\iota_k^-)\rangle\\
&=\langle z(\iota_k^-),Q^{\top}[Q^{\top}]^{-1}x(\iota_k^-)\rangle\\
&=\langle z(\iota_k^-),x(\iota_k^-)\rangle.
\end{aligned}
$$

Let $t=\iota_k$, $k=1,2,\dots$. Then

$$
\begin{aligned}
\langle z(\iota_k^+),x(\iota_k^+)\rangle&=\langle Qz(\iota_k^-),[Q^{\top}]^{-1}x(\iota_k^-)\rangle\\
&=\langle z(\iota_k^-),Q^{\top}[Q^{\top}]^{-1}x(\iota_k^-)\rangle\\
&=\langle z(\iota_k^-),x(\iota_k^-)\rangle.
\end{aligned}
$$

Therefore $\langle z(\iota),x(\iota)\rangle=c$. The proof is complete. □

Lemma 3.3. *Suppose that* (A1), (A3) *hold and* $\operatorname{rank}(E - W(T,a)) = n - l$ *with* $1 \le l \le n$. *Then the adjoint system* (3.2) *has l linearly independent T-periodic solutions.*

Proof. By (A3) and $\operatorname{rank}(E - W(T,a)) = n - l$, Equation (1.1) has linearly independent solutions. And the solution of (3.2) is $x(\iota) = W^\top(\iota,a)x_a$, where

$$W^\top(\iota,a) = [(Q^\top)^{-1}]^{n(a,\iota)} e^{-\frac{P^\top}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}.$$

Now

$$x(\iota) = \left[e^{\frac{P^\top}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}(Q^\top)^{n(a,\iota)}\right]^{-1} x_a.$$

Then

$$\begin{aligned}
x(T) = x_a &\iff \left[e^{\frac{P^\top}{\beta}\left[\left((T-\sigma_{n(a,\iota)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,T)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}(Q^\top)^{n(a,T)}\right]^{-1} x_a = x_a \\
&\iff \left[E - e^{\frac{P^\top}{\beta}\left[\left((T-\sigma_{n(a,\iota)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,T)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}(Q^\top)^{n(a,T)}\right] x_a = 0 \\
&\iff x_a \in \ker\left[E - Q^{n(a,T)} e^{\frac{P}{\beta}\left[\left((T-\sigma_{n(a,\iota)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,T)-1}(\iota_{k+1}-\sigma_k)^\beta\right]}\right]^\top \\
&\quad = \ker(E - W(T,a))^\top.
\end{aligned}$$

So that

$$\dim\ker(E - W(T,a))^\top = n - \operatorname{rank}(E - W(T,a))^\top = n - \operatorname{rank}(E - W(T,a)) = n - l.$$

The proof is complete. □

Theorem 3.4. *Suppose that* (A1), (A3) *hold. Then*(1.2) *has a T-periodic solution if and only if $\langle x_a, \phi_q\rangle = 0$, for every initial value x_a of a T-periodic solution of* (3.2)*, in which*

$$\phi_q = \int_a^T \chi(\iota)W(T,\sigma)h(\sigma)d\sigma + \sum_{k=1}^{n(a,T)} W(T,\sigma_k)d_k.$$

Proof. By Lemma 2.4, we obtain

$$\begin{aligned}
&z(T) = W(T,a)z_a + \phi_q = z_a \\
&\iff \phi_q = (E - W(T,a))z_a \\
&\iff \phi_q \in \operatorname{Im}(E - W(T,a)) \\
&\iff \phi_q \in [\ker(E - W(T,a))^\top]^\perp \\
&\iff \phi_q \in [\ker(E - W(T,a)^\top)]^\perp.
\end{aligned}$$

The proof is complete. □

Example 3.5. Consider (1.2) with $\beta = 1/2$, $\sigma_0 = 0$, $\sigma_k = k$, $\iota_k = k - 1/2$, $k \in N$, $T = 1$, $q = 1$, and

$$h(\iota) = \begin{cases} ((\iota - k)^{1/2}, 0)^\top, & \iota \in (k, k+\frac{1}{2}],\ k = 0,1,\dots \\ ((-t - k + 1)^{1/2}, 0)^\top, & \iota \in (k+\frac{1}{2}, k+1],\ k = 0,1,\dots. \end{cases}$$

Put

$$P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad d_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that

$$e^{Pt/\beta} = \begin{pmatrix} e^{2t} & 4te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

It is easy to obtain

$$\begin{aligned}
&W(\iota,\sigma)\\
&= Q^{n(0,\iota)-n(0,\sigma)} e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(0,\iota)})^{\beta}\right)^{+}-\left((\sigma-\sigma_{n(0,\sigma)})^{\beta}\right)^{+}+\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\right]}\\
&= e^{2\left[\left((\iota-\sigma_{n(0,\iota)})^{\beta}\right)^{+}-\left((\sigma-\sigma_{n(0,\sigma)})^{\beta}\right)^{+}+\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\right]} \begin{pmatrix} 1 & n(0,\iota)-n(0,\sigma) \\ 0 & 1 \end{pmatrix}\\
&\quad\times \begin{pmatrix} 1 & 4\left[\left((\iota-\sigma_{n(0,\iota)})^{\beta}\right)^{+}-\left((\sigma-\sigma_{n(0,\sigma)})^{\beta}\right)^{+}+\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\right] \\ 0 & 1 \end{pmatrix}\\
&= e^{2\left[\left((\iota-\sigma_{n(0,\iota)})^{\beta}\right)^{+}-\left((\sigma-\sigma_{n(0,\sigma)})^{\beta}\right)^{+}+\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\right]} \begin{pmatrix} 1 & \widetilde{B} \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{B} = 4\Big[&\left((\iota-\sigma_{n(0,\iota)})^{\beta}\right)^{+}-\left((\sigma-\sigma_{n(0,\sigma)})^{\beta}\right)^{+}+\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{\beta}\Big]\\
&+ n(0,\iota)-n(0,\sigma).
\end{aligned}$$

Then

$$\begin{aligned}
\phi_1 &= \int_0^1 \chi(\sigma)W(1,\sigma)h(\sigma)d\sigma + \sum_{k=1}^{n(0,1)} W(1,\sigma_k)d_k\\
&= \int_0^{1/2} W(1,\sigma)h(\sigma)s^{-\frac{1}{2}}d\sigma + (0,0)^{\top}\\
&= (\frac{e^{2^{1/2}}}{2} - \frac{1}{2} - \frac{2^{1/2}}{2}, 0)^{\top}.
\end{aligned}$$

From

$$W(1,0) = \begin{pmatrix} e^{2^{1/2}} & (1+2^{5/2})e^{2^{1/2}} \\ 0 & e^{2^{1/2}} \end{pmatrix}$$

and

$$(E-W(1,0))^{-1} = \begin{pmatrix} \frac{1}{1-e^{2^{1/2}}} & -\frac{(1+2^{5/2})e^{2^{1/2}}}{1-e^{2^{1/2}}} \\ 0 & \frac{1}{1-e^{2^{1/2}}} \end{pmatrix},$$

we have

$$z_0 = (E-W(1,0))^{-1}\phi_1 = \Big(\frac{e^{2^{1/2}}-1-2^{1/2}}{2(1-e^{2^{1/2}})}, 0\Big)^{\top}.$$

Therefore,

$$\begin{aligned}
&z(\iota,0,z_0)\\
&= W(\iota,0)z_0 + \int_0^{\iota} \chi(\sigma)W(\iota,\sigma)h(\sigma)d\sigma + \sum_{k=1}^{n(0,\iota)} W(1,\sigma_k)d_k\\
&= e^{2(\iota-\sigma_{n(0,\iota)})^{1/2}+2^{1/2}n(0,\iota)}\Big(\frac{e^{2^{1/2}}-1-2^{1/2}}{2(1-e^{2^{1/2}})}, 0\Big)^{\top}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n(0,\iota)-1}\int_{\sigma_k}^{\iota_{k+1}}\Big(e^{2\big[\big((\iota-\sigma_{n(0,\iota)})^{1/2}\big)^+-\big((\sigma-\sigma_{n(0,\sigma)})^{1/2}\big)^++\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{1/2}\big]},\\
&0\Big)^\top d\sigma\\
&+\int_{\sigma_{n(0,\iota)}}^{\iota}\Big(e^{2\big[\big((\iota-\sigma_{n(0,\iota)})^{1/2}\big)^+-\big((\sigma-\sigma_{n(0,\sigma)})^{1/2}\big)^++\sum_{k=n(0,\sigma)}^{n(0,\iota)-1}(\iota_{k+1}-\sigma_k)^{1/2}\big]},0\Big)^\top d\sigma\\
&=e^{2(\iota-\sigma_{n(0,\iota)})^{1/2}}\Big(e^{2^{1/2}n(0,\iota)}\frac{e^{2^{1/2}}-1-2^{1/2}}{2(1-e^{2^{1/2}})}\\
&+\sum_{k=0}^{n(0,\iota)-1}\int_{k}^{k+\frac12}e^{-2\big((\sigma-\sigma_{n(0,\sigma)})^{1/2}\big)^++(n(0,\iota)-k)2^{1/2}}\\
&+\int_{\sigma_{n(0,\iota)}}^{\iota}e^{-2\big((\sigma-\sigma_{n(0,\sigma)})^{1/2}\big)^+}d\sigma,0\Big)^\top\\
&=e^{2(\iota-\sigma_{n(0,\iota)})^{1/2}}\Big(e^{2^{1/2}n(0,\iota)}\frac{e^{2^{1/2}}-1-2^{1/2}}{2(1-e^{2^{1/2}})}\\
&+\big(-\frac{2^{1/2}}{2}e^{-2^{1/2}}-\frac{e^{-2^{1/2}}}{2}+\frac12\big)\frac{e^{2^{1/2}}(1-e^{2^{\frac12 n(0,\iota)}})}{1-e^{2^{1/2}}}\\
&-(\iota-\sigma_{n(0,\iota)})^{1/2}e^{-2(\iota-\sigma_{n(0,\iota)})^{1/2}}-\frac12 e^{-2(\iota-\sigma_{n(0,\iota)})^{1/2}}+\frac12,0\Big)^\top\\
&=\Big(e^{2(\iota-\sigma_{n(0,\iota)})^{1/2}}\frac{e^{2^{1/2}}}{2(1-e^{2^{1/2}})}-(\iota-\sigma_{n(0,\iota)})^{1/2}-\frac12+\frac12 e^{2(\iota-\sigma_{n(0,\iota)})^{1/2}},0\Big)^\top.
\end{aligned}
$$

Then

$$
\begin{aligned}
z(\iota+1,0,z_0)&=\Big(e^{2(\iota+1-\sigma_{n(0,t+1)})^{1/2}}\frac{e^{2^{1/2}}}{2(1-e^{2^{1/2}})}-(\iota+1-\sigma_{n(0,t+1)})^{1/2}\\
&\quad-\frac12+\frac12 e^{2(\iota+1-\sigma_{n(0,\iota)+1})^{1/2}},0\Big)^\top\\
&=z(\iota,0,z_0),
\end{aligned}
$$

so there is a 1-periodic solution.

4. Nonlinear non-instantaneous impulsive problems

In this section, we study the nonlinear non-instantaneous impulsive problem (1.3). We use the following assumptions:

- (A6) For all $\iota\in\mathbb{I}$ and $x\in\mathbb{R}^n$, we have $h(\iota+T,x)=h(\iota,x)$;
- (A7) there is a constant $L_h>0$ such that $\|h(\iota,x_1)-h(\iota,x_2)\|\le L_h\|x_1-x_2\|$ for all $\iota\in\mathbb{I}$ and $x_1,x_2\in\mathbb{R}^n$;
- (A8) there are constant $A,B\ge 0$ such that $\|h(\iota,x)\|\le A\|x\|+B$ for any $\iota\in\mathbb{I}$ and $x\in\mathbb{R}^n$.

Two important lemmas are given first.

Lemma 4.1. *When $\iota\in[a,T]$ and Lemma 3.1 holds, we obtain*

$$
\sum_{i=1}^{q}\|\phi(\iota,\sigma_i)d_i\|
$$

$$\leq F_u := \begin{cases} e^{uqT^\beta}\max\{\|Q\|^q,1\}\max\{\|W(T,a)\|,1\} \\ \times J\big[\|(E-W(T,a))^{-1}\|+1\big]\sum_{i=1}^q\|d_i\|, \quad \textit{if} \;\; u>0, \\ \max\{\|Q\|^q,1\}\max\{\|W(T,a)\|,1\} \\ \times J\big[\|(E-W(T,a))^{-1}\|+1\big]\sum_{i=1}^q\|d_i\|, \quad \textit{if}\; u\leq 0. \end{cases}$$

Proof. Using (3.1) we have

$$\begin{aligned}
&\sum_{i=1}^q\|\phi(\iota,\sigma_i)d_i\| \\
&\leq \sum_{i=1}^q\|\phi(\iota,\sigma_i)\|\|d_i\| \\
&= \sum_{a<\sigma_i<t}\|\phi(\iota,\sigma_i)\|\|d_i\| + \sum_{\iota\leq\sigma_i<T}\|\phi(\iota,\sigma_i)\|\|d_i\| \\
&\leq \sum_{a<\sigma_i<t}\|W(T,a)\|\|(E-W(T,a))^{-1}\|\|W(\iota,\sigma_i)\|\|d_i\| + \|W(\iota,\sigma_i)\|\|d_i\| \\
&\quad + \sum_{\iota\leq\sigma_i<T}\|W(\iota,a)\|\|(E-W(T,a))^{-1}\|\|W(T,\sigma_i)\|\|d_i\| \\
&= \sum_{0<i<n(a,\iota)}\|(E-W(T,a))^{-1}\|\|W(T,a)\|\Big\|Q^{n(a,\iota)-n(a,\sigma_i)}\exp\Big\{\frac{P}{\beta}\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ \\
&\quad - \big((\sigma_i-\sigma_{n(a,\sigma_i)})^\beta\big)^+ + \sum_{k=n(a,\sigma_i)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]\Big\}\Big\|\|d_i\| \\
&\quad + \sum_{0<i<n(a,\iota)}\Big\|Q^{n(a,\iota)-n(a,\sigma_i)}e^{\{}\frac{P}{\beta}\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ - \big((\sigma_i-\sigma_{n(a,\sigma_i)})^\beta\big)^+ \\
&\quad + \sum_{k=n(a,\sigma_i)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]\}\Big\|\|d_i\| \\
&\quad + \sum_{n(a,\iota)\leq i<q}\|(E-W(T,a))^{-1}\|\Big\|Q^{n(a,\iota)}e^{\frac{P}{\beta}\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ + \sum_{k=0}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\| \\
&\quad \times \Big\|Q^{q-n(a,\sigma_i)}e^{\frac{P}{\beta}\big[\big((T-\sigma_q)^\beta\big)^+ - \big((\sigma_i-\sigma_{n(a,\sigma_i)})^\beta\big)^+ + \sum_{k=n(a,\sigma_i)}^{q-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|\|d_i\| \\
&\leq \max\{\|Q\|^q,1\}\max\{\|W(T,a)\|,1\}J\Big[\sum_{0<i<n(a,\iota)}\|(E-W(T,a))^{-1}\| \\
&\quad \times \Big\|e^{\frac{P}{\beta}\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ - \big((\sigma_i-\sigma_{n(a,\sigma_i)})^\beta\big)^+ + \sum_{k=n(a,\sigma_i)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|\|d_i\| \\
&\quad + \sum_{0<i<n(a,\iota)}\Big\|e^{\frac{P}{\beta}\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ - \big((\sigma_i-\sigma_{n(a,\sigma_i)})^\beta\big)^+ + \sum_{k=n(a,\sigma_i)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|\|d_i\| \\
&\quad + \sum_{n(a,\iota)\leq i<q}\|(E-W(T,a))^{-1}\|\Big\|e^{\frac{P}{\beta}\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ + \sum_{k=0}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\| \\
&\times \Big\|e^{\frac{P}{\beta}\big[\big((T-\sigma_q)^\beta\big)^+ - \big((\sigma_i-\sigma_{n(a,\sigma_i)})^\beta\big)^+ + \sum_{k=n(a,\sigma_i)}^{q-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|\|d_i\|\Big]
\end{aligned}$$

If $u > 0$, then

$$
\begin{aligned}
&\sum_{i=1}^{q} \|\phi(\iota, \sigma_i)\|\|d_i\| \\
&\leq \max\{\|Q\|^q, 1\} \max\{\|W(T,a)\|, 1\} J\Big[\sum_{a<i<n(a,\iota)} \|(E - W(T,a))^{-1}\| \times e^{uqT^\beta} \|d_i\| \\
&\quad + \sum_{a<i<n(a,\iota)} e^{uqT^\beta} \|d_i\| + \sum_{n(a,\iota)\leq i<q} \|(E - W(T,a))^{-1}\| e^{uqT^\beta} \|d_i\| \Big] \\
&\leq e^{uqT^\beta} \max\{\|Q\|^q, 1\} \max\{\|W(T,a)\|, 1\} J\big[\|(E - W(T,a))^{-1}\| + 1\big] \sum_{i=1}^{q} \|d_i\|.
\end{aligned}
$$

If $u \leq 0$, then

$$
\begin{aligned}
&\sum_{i=1}^{q} \|\phi(\iota, \sigma_i)\|\|d_i\| \\
&\leq \max\{\|Q\|^q, 1\} \max\{\|W(T,a)\|, 1\} J\big[\|(E - W(T,a))^{-1}\| + 1\big] \sum_{i=1}^{q} \|d_i\|.
\end{aligned}
$$

The proof is complete. □

Lemma 4.2. *When $\iota \in [a,T]$ and Lemma 3.1 holds, we obtain*

$$
\begin{aligned}
&\int_a^T \chi(\sigma)\phi(\iota,\sigma) d\sigma \\
&\leq K_u := \begin{cases} T^\beta \max\{\|Q\|^q, 1\} J e^{uqT^\beta} (\|(E - W(T,a))^{-1}\| + 1), & u > 0, \\ T^\beta \max\{\|Q\|^q, 1\} J (\|(E - W(T,a))^{-1}\| + 1), & u \leq 0. \end{cases}
\end{aligned}
$$

Proof. Using (3.1), we have

$$
\begin{aligned}
&\int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| d\sigma \\
&\leq \int_a^T \|\phi(\iota,\sigma) T^{\beta-1}\| d\sigma \\
&\leq T^{\beta-1}\Big[\int_a^\iota \|(E - W(T,a))^{-1}\|\|W(\iota,\sigma)\| + \|W(\iota,\sigma)\| d\sigma \\
&\quad + \int_\iota^T \|W(\iota,a)\|\|(E - W(T,a))^{-1}\|\|W(T,\sigma)\| d\sigma\Big] \\
&= T^{\beta-1}\Big[\int_a^\iota \|(E - W(T,a))^{-1}\| \\
&\quad \times \Big\|Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+ - \left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,\iota)-1} (\iota_{k+1}-\sigma_k)^\beta\right]}\Big\| \\
&\quad + \Big\|Q^{n(a,\iota)-n(a,\sigma)} e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+ - \left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,\iota)-1} (\iota_{k+1}-\sigma_k)^\beta\right]}\Big\| d\sigma \\
&\quad + \int_\iota^T \Big\|Q^{n(a,\iota)} e^{\frac{P}{\beta}\left[\left((\iota-\sigma_{n(a,\iota)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,\iota)-1} (\iota_{k+1}-\sigma_k)^\beta\right]}\Big\| \|(E - W(T,a))^{-1}\| \\
&\quad \times \Big\|Q^{n(a,T)-n(a,\sigma)} e^{\frac{P}{\beta}\left[\left((T-\sigma_{n(a,T)})^\beta\right)^+ - \left((\sigma-\sigma_{n(a,\sigma)})^\beta\right)^+ + \sum_{k=n(a,\sigma)}^{n(a,T)-1} (\iota_{k+1}-\sigma_k)^\beta\right]}\Big\| d\sigma\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\le T^{\beta-1}\max\{\|Q\|^q,1\}J\int_a^\iota \|(E-W(T,a))^{-1}\|\\
&\quad\times \Big\|e^{u\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+-\big((\sigma-\sigma_{n(a,\sigma)})^\beta\big)^+ +\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|\\
&\quad+\Big\|e^{u\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+-\big((\sigma-\sigma_{n(a,\sigma)})^\beta\big)^+ +\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|d\sigma\\
&\quad+\int_\iota^T \Big\|e^{u\big[\big((\iota-\sigma_{n(a,\iota)})^\beta\big)^+ +\sum_{k=n(a,\sigma)}^{n(a,\iota)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|\|(E-W(T,a))^{-1}\|\\
&\quad\times \Big\|e^{u\big[\big((T-\sigma_{n(a,T)})^\beta\big)^+-\big((\sigma-\sigma_{n(a,\sigma)})^\beta\big)^+ +\sum_{k=n(a,\sigma)}^{n(a,T)-1}(\iota_{k+1}-\sigma_k)^\beta\big]}\Big\|d\sigma.
\end{aligned}
$$

If $u>0$, then

$$
\begin{aligned}
&\int_a^T \|\chi(\iota)\phi(\iota,\sigma)\|d\sigma\\
&\le T^{\beta-1}\max\{\|Q\|^q,1\}J\Big[\|(E-W(T,a))^{-1}\|\int_a^T e^{uqT^\beta}d\sigma+\int_a^\iota e^{uqT^\beta}d\sigma\Big]\\
&\le T^\beta\max\{\|Q\|^q,1\}Je^{uqT^\beta}(\|(E-W(T,a))^{-1}\|+1).
\end{aligned}
$$

If $u\le 0$, then

$$
\begin{aligned}
&\int_a^T \|\chi(\iota)\phi(\iota,\sigma)\|d\sigma\\
&\le T^{\beta-1}\max\{\|Q\|^q,1\}J\Big[\|(E-W(T,a))^{-1}\|\int_a^T e^{uqT^\beta}d\sigma+\int_a^\iota e^{uqT^\beta}d\sigma\Big]\\
&\le T^\beta\max\{\|Q\|^q,1\}J(\|(E-W(T,a))^{-1}\|+1).
\end{aligned}
$$

The proof is complete. □

Theorem 4.3. *Suppose that* (A1), (A2), (A4), (A6), (A7) *hold. If* $0<L_hK_u<1$, *equation* (1.3) *has a unique T-periodic solution $z\in PC_T(I,\mathbb{R}^n)$ satisfying*

$$
\|y\|\le \frac{L_h\|z(a)\|\|K_u+\|h_a\|K_u+F_u}{1-L_hK_u},
$$

where $\|h_a\|=\max_{\iota\in[a,T]}|h(\iota,a)|$.

Proof. For each $z\in PC_T$, it holds $z(\iota+T)+z(\iota)$. By (A6),

$$
h(\iota+T,z(\iota+T))=h(\iota+T,z(T))=h(\iota,z),\ \ \iota\in\mathbb{R},
$$

so $h(\cdot,(\cdot))\in PC_T$.

According to Lemma 3.1, we consider the equation

$$
z(\iota,a,z_a)=\int_a^T \chi(\sigma)\phi(\iota,\sigma)h(\sigma,z(\sigma))d\sigma+\sum_{k=1}^q\phi(\iota,\sigma_k)d_k.
$$

By using the operator $H:PC([a,T],\mathbb{R}^n)\to PC([a,T],\mathbb{R}^n)$, we have

$$
Hz(\iota,a,z_a)=\int_a^T \chi(\sigma)\phi(\iota,\sigma)h(\sigma,z(\sigma))d\sigma+\sum_{k=1}^q\phi(\iota,\sigma_k)d_k. \tag{4.1}
$$

If each $y, z \in PC([a,T], \mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|Hz(\iota) - Hz(\iota)\| &\le \Big\| \int_a^T \chi(\sigma)\phi(\iota,\sigma)h(\sigma, z(\sigma)) - \int_a^T \chi(\sigma)\phi(\iota,\sigma)h(\sigma, z(\sigma))\Big\| d\sigma \\
&\le \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| \|h(\sigma, z(\sigma)) - h(\sigma, z(\sigma))\| d\sigma \\
&\le L_h\|y - z\| \int_a^T \chi(\sigma)\phi(\iota,\sigma) d\sigma \\
&\le L_h K_u \|y - z\|
\end{aligned}$$

From $0 < L_h K_u < 1$, we know that H is a contraction mapping and H has a unique fixed point. Then we obtain

$$\begin{aligned}
&\|y\| = \|Hy\| \\
&\le \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| \|h(\sigma, z(\sigma))\| d\sigma + \sum_{k=1}^q \|\phi(\iota,\sigma_k)\| d_k\| \\
&\le \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| \|h(\sigma, z(\sigma)) - h(\sigma, z(a)) + h(\sigma, z(a))\| d\sigma \\
&\quad + \sum_{k=1}^q \|\phi(\iota,\sigma_k)\| d_k\| \\
&\le L_h\|y - z(a)\| \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| d\sigma + \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| \|h(\sigma, z(a))\| d\sigma \\
&\quad + \sum_{k=1}^q \|\phi(\iota,\sigma_k)\| d_k\| \\
&\le L_h\|y - z(a)\| K_u + \|h_a\| K_u + F_u \\
&\le L_h(\|y\| + \|z(a)\|)\| K_u + \|h_a\| K_u + F_u,
\end{aligned}$$

so

$$\|y\| \le \frac{L_h\|z(a)\|\|K_u + \|h_a\|K_u + F_u}{1 - L_h K_u}.$$

The proof is complete. □

Theorem 4.4. *Suppose that* (A1), (A2), (A4), (A6), (A8) *hold. If* $0 < AK_u < 1$, *then* (1.3) *has a unique* T*-periodic solution* $z \in PC_T(I, \mathbb{R}^n)$.

Proof. We use the operator H in (4.1) defined on $C_\tau := \{z \in PC([a,T], \mathbb{R}^n) | \|y\| \le \tau, \tau \ge \frac{BK_u + F_u}{1 - AK_u}\}$. For any $a \le \iota \le T$, $z \in C_\tau$, by lemma 4.1 and lemma 4.2, we have

$$\begin{aligned}
&\|Hz(\iota)\| \\
&\le \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| \|h(\sigma, z(\sigma))\| d\sigma + \sum_{k=1}^q \|\phi(\iota,\sigma_k) d_k\| \\
&\le A \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| \|z(\sigma)\| d\sigma + B \int_a^T \|\chi(\sigma)\phi(\iota,\sigma)\| d\sigma + \sum_{k=1}^q \|\phi(\iota,\sigma_k) d_k\| \\
&\le AK_u\|y\| + BK_u F_u = \tau,
\end{aligned}$$

so $\|Hy\| \le \tau$ and $H(C_\tau) \subset C_\tau$. We can show that H is continuous and $H(C_\tau)$ is pre-compact. By Schauder's fixed-point Theorem, (1.3) has at least one T-periodic solution $z \in PC_T(I, \mathbb{R}^n)$. □

Example 4.5. We consider (1.3), with

$$z(\iota) = \begin{pmatrix} z_1(\iota) \\ z_2(\iota) \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$h(\iota, z(\iota)) = \sin t \cos z(\iota), \quad \iota_k = \frac{2k-1}{4}\pi, \quad \sigma_k = \frac{k}{2}\pi, \quad k = 1, 2, \dots, \quad \sigma_0 = 0.$$

Let $T = \pi$. Then $PQ = QP$,

$$\iota_{k+2} = \frac{2k+3}{4}\pi = \frac{2k-1}{4}\pi + \pi = \iota_k + \pi, \quad \sigma_{k+2} = \frac{k+2}{2}\pi = \frac{k}{2}\pi + \pi = \sigma_k + \pi,$$

$d_{k+2} = d_k$ for $k = 0, 1, 2, \dots$. Then we obtain $q = 2$, so (A1) holds. We obtain

$$e^{Pt/\beta} = \begin{pmatrix} 3e^{-2t} - 2e^{-4t} & -2e^{-2t} + 2e^{-4t} \\ 3e^{-2t} - 3e^{-4t} & -2e^{-2t} + 3e^{-4t} \end{pmatrix}$$

and

$$\begin{aligned} W(T, 0) = W(\pi, 0) &= Q^{n(0,\pi)} e^{\frac{P}{\beta}\left[\left((\pi - \sigma_{n(0,\pi)})^\beta\right)^+ + \sum_{k=0}^{n(0,\pi)-1} (\iota_{k+1} - \sigma_k)^\beta\right]} \\ &= Q^2 e^{\frac{P}{\beta}\left[(\iota_1 - \sigma_0)^\beta + (\iota_2 - \sigma_1)^\beta\right]} \\ &= Q^2 e^{\frac{P}{\beta}\pi^{1/2}} \\ &= \begin{pmatrix} 3e^{-2\pi^{1/2}} - 2e^{-4\pi^{1/2}} & -2e^{-2\pi^{1/2}} + 2e^{-4\pi^{1/2}} \\ 3e^{-2\pi^{1/2}} - 3e^{-4\pi^{1/2}} & -2e^{-2\pi^{1/2}} + 3e^{-4\pi^{1/2}} \end{pmatrix}, \end{aligned}$$

$\|W(\pi, a)\| = 0.169$. So $\det(E - W(\pi, 0)) \neq 0$ and (A2) holds. Then

$$\begin{aligned} (E - W(\pi, 0))^{-1} &= \frac{1}{(1 - e^{-2\pi^{1/2}})(1 - e^{-4\pi^{1/2}})} \\ &\quad \times \begin{pmatrix} 1 + 2e^{-2\pi^{1/2}} - 3e^{-4\pi^{1/2}} & -2e^{-2\pi^{1/2}} + 2e^{-4\pi^{1/2}} \\ 3e^{-2\pi^{1/2}} - 3e^{-4\pi^{1/2}} & 1 - 3e^{-2\pi^{1/2}} + 2e^{-4\pi^{1/2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{1-e^{-2\pi^{1/2}}} + \frac{-2}{1-e^{-4\pi^{1/2}}} & \frac{-2}{1-e^{-2\pi^{1/2}}} + \frac{2}{1-e^{-4\pi^{1/2}}} \\ \frac{3}{1-e^{-2\pi^{1/2}}} + \frac{-3}{1-e^{-4\pi^{1/2}}} & \frac{-2}{1-e^{-2\pi^{1/2}}} + \frac{3}{1-e^{-4\pi^{1/2}}} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\|(E - W(\pi, 0))^{-1}\| \\ &= \max\Big\{\Big|\frac{3}{1 - e^{-2\pi^{1/2}}} + \frac{-2}{1 - e^{-4\pi^{1/2}}}\Big| + \Big|\frac{3}{1 - e^{-2\pi^{1/2}}} + \frac{-3}{1 - e^{-4\pi^{1/2}}}\Big|, \\ &\quad \Big|\frac{-2}{1 - e^{-2\pi^{1/2}}} + \frac{2}{1 - e^{-4\pi^{1/2}}}\Big| + \Big|\frac{-2}{1 - e^{-2\pi^{1/2}}} + \frac{3}{1 - e^{-4\pi^{1/2}}}\Big|\Big\} \\ &= 1.9617. \end{aligned}$$

Next, $h(\iota + \pi, z) = b\sin(\iota + \pi)\cos(z) = bh(\iota, z)$ and (A6) holds. By $|h(t, x) - h(t, z)| \le |b||\cos x - \cos y| \le |b||x - y|$, it follows that $L_h = |b|$ and (A7) holds. Because $\sigma(\frac{P}{\beta}) = \{-2, -4\}$, (A5) satisfies with $u = -2$. Now

$$J = \sup_{\iota \ge 0} e^{2t} \|e^{\frac{P}{\beta}\iota}\|$$

$$= \sup_{\iota\geq 0} \max\{|3-2e^{-2t}|+|3-3e^{-2t}|, |-2+2e^{-2t}|+|-2+3e^{-2t}|\} = 6,$$

and $F_u = 35.5404$, $K_u = 31.4969$.

If $L_h = |b| < 0.0281$, then $0 < L_h K_u < 1$ and the conditions of Theorem 4.3 hold. Thus there is a unique π-periodic solution $z \in PC_\pi([0,\infty),\mathbb{R}^2)$.

If $A < 0.0317, B = 2|b|$, we know that (A8) and $0 < Ak_u < 1$ hold. Thus the conditions of Theorem 4.4 hold and there is a unique π-periodic solution $z \in PC_\pi([0,\infty),\mathbb{R}^2)$.

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References

[1] S. Abbas, M. Benchohra; *Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses*, Applied Mathematics and Computation, 257 (2015), 190–198.

[2] T. Abdeljawad; *On conformable fractional calculus*, Journal of Computational and Applied Mathematics, 279 (2015), 57–66.

[3] M. Abul-Ez, M. Zayed, A. Youssef, M. De la Sen; *On conformable fractional Legendre polynomials and their convergence properties with applications*, Alexandria Engineering Journal, 59 (2020), 5231–5245.

[4] R. Agarwal, D. O'Regan, S. Hristova; *Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses*, Applied Mathematics and Computation, 298 (2017), 45–56.

[5] M. Ayata, O. Ozkan; *A new application of conformable Laplace decomposition method for fractional Newell-Whitehead-Segel equation*, AIMS Mathematics, 5 (2020),7402–7412.

[6] L. Bai, J. J. Nieto; *Variational approach to differential equations with not instantaneous impulses*, Applied Mathematics Letters, 73 (2017), 44–48.

[7] M. Benchohra, S. Litimein, J. J. Nieto; *Semilinear fractional differential equations with infinite delay and non-instantaneous impulses*, Journal of Fixed Point Theory and Applications, 21 (2019), 21.

[8] Y. Ding, M. Fečkan, J. Wang; *Conformable linear and nonlinear non-instantaneous impulsive differential equations*, Electronic Journal Differential Equations, 2020 (2020), 1–19.

[9] M. Fečkan, J. Wang, Y. Zhou; *Periodic solutions for nonlinear evolution equations with non-instantaneous impulses*, Nonautonomous Dynamical Systems, 1 (2014), 93–101.

[10] G. R. Gautam, J. Dabas; *Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses*, Applied Mathematics and Computation, 259 (2015), 480–489.

[11] E. Hernández, D. O'Regan; *On a new class of abstract impulsive differential equations*, Proceedings of the American Mathematical Society, 141 (2013), 1641–1649.

[12] E. Hernández, M. Pierri, D. O'Regan; *On abstract differential equations with non-instantaneous impulses*, Topological Methods in Nonlinear Analysis, 46 (2015), 1067–1085.

[13] K. Liu, M. Fečkan, D. O'Regan, J. Wang; *(ω,c)-periodic solutions for time-varying non-instantaneous impulsive differential systems*, Applicable Analysis, (2021), https://doi.org/10.1080/00036811.2021.1895123.

[14] K. Liu, J. Wang, D. O'Regan, M. Fečkan; *A new class of (ω,c)-periodic non-instantaneous impulsive differential equations*, Mediterranean Journal of Mathematics, 17 (2020), 155.

[15] M. Malik, V. Kumar; *Existence, stability and controllability results of a Volterra integro-dynamic system with non-instantaneous impulses on time scales*, IMA Journal of Mathematical Control and Information, 37 (2020), 276–299.

[16] S. Mehmood, F. Zafar, N. Yasmin; *Hermite-Hadamard-Fejer inequalities for generalized conformable fractional integrals*, Mathematical Methods in the Applied Sciences, 44 (2020), 3746-3758.

[17] A. Meraj, D. N. Pandey; *Approximate controllability of non-autonomous Sobolev type integro-differential equations having nonlocal and non-instantaneous impulsive conditions*, Indian Journal of Pure & Applied Mathematics, 51 (2020), 501–518.

[18] M. Muslim, A. Kumar, M. Fečkan; *Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses*, Journal of King Saud University Science, 30 (2018), 204–213.

[19] M. Muslim, A. Kumar, M. Fečkan; *Periodic solutions to second order nonlinear differential equations with non-instantaneous impulses*, Dynamic Systems and Applications, 26 (2017), 197–210.

[20] M. Pierri, H. R. Henriquez, A. Prokopczyk; *Global solutions for abstract differential equations with non-instantaneous impulses*, Mediterranean Journal of Mathematics, 13 (2016), 1685–1708.

[21] A. G. Talafha, S. M. Alqaraleh, M. Al-Smadi, S. Hadid, S. Momani; *Analytic solutions for a modified fractional three wave interaction equations with conformable derivative by unified method*, Alexandria Engineering Journal, 59 (2020), 3731–3739.

[22] Y. Tian, J. Wang, Y. Zhou; *Almost periodic solutions for a class of non-instantaneous impulsive differential equations*, Quaestiones Mathematicae, 42 (2019), 885–905.

[23] Y. Tian, M. Zhang; *Variational method to differential equations with instantaneous and non-instantaneous impulses*, Applied Mathematics Letters, 94 (2019), 160–165.

[24] J. Wang, M. Fečkan; *A general class of impulsive evolution equations*, Topological Methods in Nonlinear Analysis, 46 (2015), 915–933.

[25] J. Wang, M. Fečkan, Y. Zhou; *Random non-instantaneous impulsive models for studying periodic evolution processes in pharmacotherapy*, Mathematical Modeling and Applications in Nonlinear Dynamics, Springer, Cham, 2016, 87–107.

[26] G. Xiao, J. Wang; *On the stability of solutions to conformable stochastic differential equations*, Miskolc Mathematical Notes, 21 (2020), 509–523.

[27] D. Yang, J. Wang; *Non-instantaneous impulsive fractional-order implicit differential equations with random effects*, Stochastic Analysis and Applications, 35 (2017), 719–741.

[28] P. Yang, J. Wang, M. Fečkan; *Boundedness, periodicity, and conditional stability of noninstantaneous impulsive evolution equations*, Mathematical Methods in the Applied Sciences, 43 (2020), 5905–5926.

[29] P. Yang, J. Wang, M. Fečkan; *Periodic nonautonomous differential equations with noninstantaneous impulsive effects*, Mathematical Methods in the Applied Sciences, 42 (2019), 3700–3720.

[30] P. Yang, J. Wang, D. O'Regan; *Periodicity of non-homogeneous trajectories for non-instantaneous impulsive heat equations*, Electronic Journal of Differential Equations, 2020 (2020), 1-7.

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