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THREE-POINT INTEGRAL BOUNDARY-VALUE PROBLEMS FOR PIECEWISE FRACTIONAL IMPULSIVE DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN OPERATOR

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Abstract. We study the existence and uniqueness of solutions for three-point integral boundary-value problems of piecewise fractional impulsive differential equations with p-Laplacian operator and delay. We prove the existence of solutions, by using the ψ-Hadamard fractional definition, the Leray-Schauder nonlinear alternative, and the Krasnosel'skiis fixed point theorem. Subsequently, we prove uniqueness of the solution using the Banach fixed point theorem. To verify the feasibility of the main results, we give two examples.

1. Introduction

Fractional calculus, as an extension of classical calculus, originated from the conjectures of Leibniz and Euler, and it has evolved to the present day. In recent decades, boundary value problems (BVPs) associated with fractional ordinary differential equations (FODEs) have garnered significant attention, driving the rapid development and broad application of fractional calculus in various fields such as fractional physics, viscoelastic mechanics, stochastic processes, and reaction-diffusion equations. Additionally, as a crucial aspect of fractional calculus, fractional differential equations with boundary conditions have garnered significant attention and produced a wide range of research outcomes [8, 11, 19, 20, 30, 31]. In 1983, during studies on turbulence in porous media, Leibenson introduced the p-Laplacian operator. Since then, fractional differential equations involving the p-Laplacian operator have attracted considerable attention from scholars. However, most of the resulting research has been conducted under the standard definitions of Caputo and Riemann-Liouville fractional derivatives [7, 17, 22, 25, 28, 32]. In many continuous gradual processes, the state of a system undergoes sudden changes at certain moments due to disturbances or external influences. This phenomenon is known as impulsive effects. Because of the significant role of impulsive differential equations in describing such abrupt changes in electronic technology and communication engineering, they have become an important subject of research in recent years. Dynamical systems with impulsive phenomena have extensive applications in fields such as physics, biology, economics, and engineering. Differential equations with impulsive conditions are often used to simulate processes that exhibit discontinuous jumps and abrupt changes. The significance of studying fractional impulsive differential equations lies in their extension of classical differential equation theory, their ability to reveal new characteristics of complex systems, their capacity to provide precise models for practical problems, and their promotion of innovation in related mathematical methodologies. These theoretical achievements and application values make them an important frontier topic in interdisciplinary research across modern mathematics, physics, engineering, biology, and economics [2, 3, 14, 23, 24, 26].

In perfect conditions, state variables change over time following a consistent motion law, but nearly all systems experience delays due to limited motion speeds. Different time intervals cause variations in the motion laws of state variables, which interact through time lags τ and time leads

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$-\tau$, thereby making the dynamics more realistic. In the real world, piecewise differential systems serve as effective mathematical models, reflecting the differing motion laws of state variables across distinct time segments. Piecewise fractional differential systems can more accurately depict complex dynamical phenomena characterized by discontinuities or abrupt changes prevalent in reality. In application domains such as signal processing, image restoration, robotic path planning, and power system stabilization control, piecewise fractional differential systems offer more precise models and algorithms. They better accommodate nonlinearity, time-varying behavior, and discontinuities inherent in real data, thereby enhancing system performance and robustness.

Zhang et al. [34] studied the existence, uniqueness, and stability of solutions for piecewise continuous fractional impulsive differential systems with delay:

$$\begin{gathered} D^{\alpha}_{t_k}u(t)+g_k\big(t,u(t),u(t+(-1)^k\tau)\big)=0, \quad t\in(t_k,t_{k+1}),\ k=0,1,\dots,m,\\ \Delta u(t)|_{t=t_k}=I_k(u(t_k)),\ \Delta u'(t)|_{t=t_k}=\bar{I}_k(u(t_k)), \quad k=1,2,\dots,m,\\ u(0)=a, \quad u'(1)=b, \end{gathered}$$

where $1<\alpha<2$, $D^{\alpha}_{t_k}$ is conformable fractional derivative of order α starting from t_k ($k=0,1,2,\dots,m$).

Poovarasan et al. [27] investigated the existence and uniqueness of solutions to the three-point impulsive boundary value problem with ψ-Caputo fractional derivative

$$\begin{gathered} {}^{C}D^{\zeta,\psi}u(t)=g(t,u(t)),\ t\in J_0:=(0,1), \quad t\neq t_k,\ k\in\mathbb{N}_m,\\ \Delta u|_{t=t_k}=I_k(u(t_k)), \quad \Delta u'|_{t=t_k}=\bar{I}_k(x(t_k)), \quad k\in\mathbb{N}_m,\\ u(0)+\mathrm{u}'(0)=u(1)+u'(\vartheta)=0, \end{gathered}$$

where $1\le\zeta\le2$, ${}^{C}D^{\zeta,\psi}$ is the ψ-Caputo fractional derivative.

Ali et al. [4] studied the existence and uniqueness of solutions to a class of nonlinear implicit impulsive fractional differential equations with three-point boundary-value problems

$$\begin{gathered} {}^{c}D^{q}_{t_k}u(t)=f(t,u(t),{}^{c}D^{q}_{t_k}u(t)), \quad t\in(t_k,t_{k+1}],\ k=0,1,\dots,b,\ 1<q\le2,\\ u(t)|_{t=0}=0, \ \ u(t)|_{t=1}=\lambda u(t)|_{t=\eta}, \quad \lambda,\eta\in(0,1),\\ \Delta u(t)|_{t=t_k}=P_k(u(t_k)), \quad \Delta u'(t)|_{t=t_k}=Q_k(u(t_k)), \quad k=1,2,\dots,b, \end{gathered}$$

where ${}^{c}D^{q}_{t_k}$ is the Caputo fractional derivative of order q starting from t_k.

In 2024, Balachandran et al. [10] defined a new class of Hadamard fractional integrals and fractional derivatives for function ψ. From existing research results, it has been found that there are few achievements in piecewise fractional impulsive differential equation systems with delay and p-Laplacian operator, thus having great research prospects.

Inspired by the aforementioned work and the [18, 12, 1, 6, 15], this paper uses Leray-Schauder nonlinear alternative, Krasnosel'skiis fixed point theorem, and Banach fixed point theorem to discuss the existence and uniqueness of solutions for a class of piecewise impulsive fractional differential equations with p-Laplacian operator and delay under the definition of ψ-Hadamard fractional derivatives

$$\begin{gathered} \phi_p({}^{H}D^{\alpha,\psi}_{t_k}x(t))=\lambda h_k(t,x(t),x(t+(-1)^k\tau)),\\ \Delta x(t_k)=P_k(x(t_k)), \quad \Delta x'(t_k)=Q_k(x(t_k)),\\ x(0)=x'(0), \quad x'(1)=\int_0^{\eta}g(s,x(s))ds, \end{gathered} \tag{1.1}$$

where $1<\alpha\le2$, ${}^{H}D^{\alpha,\psi}_{t_k}$ is the ψ-Hadamard fractional derivative of order α starting from t_k ($k=0,1,2,\dots,m$), $\lambda>0$ is a parameter, $1<p<2$, $\eta\in(t_a,t_{a+1})$, a is a nonnegative integer, $v=\sup_{1\le i\le m+1}\{\ln\psi(t_i)-\ln\psi(t_{i-1})\}$. $0\le a\le m$, $J=[0,1]$, $0=t_0<t_1<t_2<\dots<t_m<t_{m+1}=1$, $h_k\in C([0,1]\times\mathbb{R}^2,\mathbb{R})(k=0,1,2,\dots,m)$, $g\in C(J\times\mathbb{R},\mathbb{R})$, $P_k,Q_k\in C(\mathbb{R},\mathbb{R})$ ($k=1,2,\dots,m$), $0\le\tau\le\min_{1\le k\le m+1}\{t_k-t_{k-1}\}$, and $\Delta x(t)|_{t=t_k}=x(t_k^+)-x(t_k^-)$, $\Delta x'(t)|_{t=t_k}=x'(t_k^+)-x'(t_k^-)$, $x(t_k^+)$, $x'(t_k^+)$ and $x(t_k^-),x'(t_k^-)$ represent the right and left limits of $x(t)$, $x'(t)$ at $t=t_k(k=1,2,\dots,m)$, respectively.

2. Preliminaries

This section includes crucial definitions and lemmas necessary for proving the primary results. Let $\varpi_0 = [0, t_1]$, $\varpi_1 = (t_1, t_2]$, $\varpi_2 = (t_2, t_3]$, $\dots$, $\varpi_{m-1} = (t_{m-1}, t_m], \varpi_m = (t_m, 1]$, $\varpi' = \varpi \backslash \{t_1, t_2, \dots, t_m\}$. We denote

$$\Im(\varpi, \mathbb{R}) = \{\varpi \to \mathbb{R} : x \in C(\varpi', \mathbb{R}),\ x(t_k^+),\ x(t_k^-) \text{ exist},\ x(t_k^-) = x(t_k^+),\ k = 1, 2, \dots, m\},$$

with norm $\|x\|_{\Im} = \sup_{t\in\varpi} |x(t)|$, then $\Im(\varpi, \mathbb{R})$ is a Banach space. $p \geq 1$, $p^{-1} + q^{-1} = 1$, $\phi_p(s) = |s|^{p-2}s$, $(\phi_p)^{-1} = \phi_q$, $\lambda > 0$ is a parameter, let $\psi(t)$ be an increasing differentiable function on $\Im(\varpi, \mathbb{R})$, then $\psi'(t) > 0$. Let $\psi^* = \min_{t\in\varpi} \psi'(t)$.

Definition 2.1 ([10]). Let $f \in C[a, b]$ and $\varphi \in C^n[a, b]$ be an increasing function, with $\varphi'(x) \neq 0$, for all $x \in [a, b]$. The fractional integral of the left-sided Hadamard functional of order $\alpha > 0$ is defined as

$${}^H I_{a^+}^{\alpha,\varphi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi'(t)}{\varphi(t)} (\ln \varphi(x) - \ln \varphi(t))^{\alpha-1} f(t) dt,$$

and the fractional integral of the right-sided Hadamard functional of order α is defined as

$${}^H I_{b^-}^{\alpha,\varphi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi'(t)}{\varphi(t)} (\ln \varphi(t) - \ln \varphi(x))^{\alpha-1} f(t) dt.$$

Definition 2.2 ([10]). Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $f \in C^{n-1}[a, b]$, and $\varphi \in C^n[a, b]$ be an increasing function, with $\varphi'(x) \neq 0$, for all $x \in [a, b]$. The left-sided Hadamard functional fractional derivative of f of order α is defined as

$$\begin{aligned} {}^H D_{a^+}^{\alpha,\varphi} f(x) &= \Big(\frac{\varphi(x)}{\varphi'(x)} \frac{d}{dx}\Big)^n {}^H I_{a^+}^{n-\alpha,\varphi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \Big(\frac{\varphi(x)}{\varphi'(x)} \frac{d}{dx}\Big)^n \int_a^x \frac{\varphi'(t)}{\varphi(t)} (\ln \varphi(x) - \ln \varphi(t))^{n-\alpha-1} f(t) dt, \end{aligned}$$

and the right-hand sided of Hadamard functional fractional derivative of f of order α is

$${}^H D_{b^-}^{\alpha,\varphi} f(x) = \frac{1}{\Gamma(n-\alpha)} \Big(-\frac{\varphi(x)}{\varphi'(x)} \frac{d}{dx}\Big)^n \int_x^b \frac{\varphi'(t)}{\varphi(t)} (\ln \varphi(t) - \ln \varphi(x))^{n-\alpha-1} f(t) dt.$$

Obviously, if $0 < \alpha < 1$, then we have

$$\begin{aligned} {}^H D_{a^+}^{\alpha,\varphi} f(x) &= (\frac{\varphi(x)}{\varphi'(x)} \frac{d}{dx})^n {}^H I_{a^+}^{1-\alpha,\varphi} f(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \Big(\frac{\varphi(x)}{\varphi'(x)} \frac{d}{dx}\Big) \int_a^x \frac{\varphi'(t)}{\varphi(t)} (\ln \varphi(x) - \ln \varphi(t))^{-\alpha} f(t) dt, \end{aligned}$$

and

$${}^H D_{b^-}^{\alpha,\varphi} f(x) = \frac{1}{\Gamma(1-\alpha)} \Big(-\frac{\varphi(x)}{\varphi'(x)} \frac{d}{dx}\Big) \int_x^b \frac{\varphi'(t)}{\varphi(t)} (\ln \varphi(t) - \ln \varphi(x))^{-\alpha} f(t) dt.$$

Lemma 2.3 ([10]). *Let $n-1 < \alpha, \beta < n, f \in C[a, b]$ and $\varphi \in C^n[a, b]$. Then*

$${}^H I_{a^+}^{\alpha,\varphi} {}^H I_{a^+}^{\beta,\varphi} f(x) = {}^H I_{a^+}^{\alpha+\beta,\varphi} f(x) \quad \text{and} \quad {}^H I_{b^-}^{\alpha,\varphi} {}^H I_{b^-}^{\beta,\varphi} f(x) = {}^H I_{b^-}^{\alpha+\beta,\varphi} f(x).$$

For functions $f \in C[a, b], \varphi \in C^n[a, b]$ and $n - 1 < \alpha, \beta < n$ with $n \in \mathbb{N}$, we obtain

$${}^H D_{a^+}^{\alpha,\varphi} {}^H I_{a^+}^{\alpha,\varphi} f(x) = f(x) \quad \text{and} \quad {}^H D_{b^-}^{\alpha,\varphi} {}^H I_{b^-}^{\alpha,\varphi} f(x) = f(x).$$

Lemma 2.4 ([10]). *For each a function $f \in C^{n-1}[a, b]$ and $n-1 < \alpha, \beta < n$ with $n \in \mathbb{N}$, we have*

$${}^H I_{a^+}^{\alpha,\varphi} {}^H D_{a^+}^{\alpha,\varphi} f(x) = f(x) - \sum_{k=0}^{n-1} \Big[(\frac{\varphi(t)}{\varphi'(t)} \frac{d}{dt})^k f(t)\Big]_{t=a} \Big(\frac{(\ln \varphi(x) - \ln \varphi(a))^k}{k!}\Big),$$

and

$${}^H I_{b^-}^{\alpha,\varphi} {}^H D_{b^-}^{\alpha,\varphi} f(x) = f(x) - \sum_{k=0}^{n-1} \Big[(-\frac{\varphi(t)}{\varphi'(t)} \frac{d}{dt})^k f(t)\Big]_{t=b} \Big(\frac{(\ln \varphi(b) - \ln \varphi(x))^k}{k!}\Big).$$

Lemma 2.5 ([29]). *Let ϕ_p be the p-Laplacian operator, then we have*

(i) *If $1 < p < 2$, $\zeta_1, \zeta_2 > 0$, and $|\zeta_1|, |\zeta_2| \geq \rho$, then $|\phi_p(\zeta_1) - \phi_q(\zeta_2)| \leq (p-1)\rho^{p-2}|\zeta_1 - \zeta_2|$;*
(ii) *If $p > 2$ and $|\zeta_1|, |\zeta_2| < \rho$, then $|\phi_p(\zeta_1) - \phi_q(\zeta_2)| \leq (p-1)\rho^{p-2}|\zeta_1 - \zeta_2|$.*

Now we have the Leray-Schauder nonlinear alternative.

Lemma 2.6 ([13]). *Let $\Im(\varpi, \mathbb{R})$ be the Banach space, $B_r \subset \Im(\varpi, \mathbb{R})$ is a convex closed set, B_{N_x} is an open set relative to B_r and $0 \in B_{N_x}$. If $\Lambda : \bar{B}_{N_x} \to \Im(\varpi, \mathbb{R})$ be a completely continuous operator and $\Lambda(\bar{U})$ is bounded, then it satisfies*

(i) *Λ has a fixed point in $\bar{B}_{N_x}$; or*
(ii) *there exists $x \in \partial B_{N_x}$, and $\gamma_1 \in (0, 1)$, such that $x = \gamma_1 \Lambda x$.*

Now we have the Banach fixed point theorem.

Lemma 2.7 ([16]). *Let $\Im(\varpi, \mathbb{R})$ be a Banach space, and mapping $\Lambda : \Im(\varpi, \mathbb{R}) \to \Im(\varpi, \mathbb{R})$ be a contraction on $\Im(\varpi, \mathbb{R})$. Then there is a unique $x^* \in \Im(\varpi, \mathbb{R})$ with $\Lambda x^* = x^*$.*

Now we have the Krasnosel'skiis fixed point theorem.

Lemma 2.8 ([9]). *Let B_R be a bounded closed convex non-empty subset on Banach space $\Im(\varpi, \mathbb{R})$, where operators Ψ, T satisfy*

(i) *$\Psi x_1 + T x_2 \in B_R$, where $x_1, x_2 \in B_R$;*
(ii) *operator Ψ is compact and continuous;*
(iii) *operator T is a contraction mapping,*

Then there exists $z \in B_R$, such that $z = Tz + \Psi z$.

Now we have the Arzela-Ascoli theorem.

Lemma 2.9 ([21]). *Let $\Im(\varpi, \mathbb{R})$ be the Banach space, If B_r is a compact set of $\Im(\varpi, \mathbb{R})$, and sequence $\{x_n\}$ is uniformly bounded and equicontinuous in B_r, then this sequence has uniformly continuous subsequences in B_r.*

Lemma 2.10. *Let $1 < \alpha \leq 2$, $y_k \in C(\varpi, \mathbb{R})$ $(k = 0, 1, 2, \dots, m)$, $p_k, q_k \in \mathbb{R}$ $(k = 1, 2, \dots, m)$. Then the linear piecewise fractional impulsive differential equation*

$$\begin{gathered} \phi_p\Big({}^H D_{t_k}^{\alpha,\psi} x(t)\Big) = \lambda y_k(t), \\ \Delta x(t_k) = p_k, \quad \Delta x'(t_k) = q_k, \\ x(0) = x'(0), \quad x'(1) = \int_0^\eta g(s, x(s)) ds, \end{gathered} \tag{2.1}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) = {}& \frac{\psi(1)}{\psi'(1)} \Big[\frac{\psi'(0)}{\psi(0)} + \ln\psi(t) - \ln\psi(0)\Big] \int_0^\eta g(s, x(s)) ds + \sum_{i=1}^{k} p_i - \sum_{i=1}^{m} \chi(t, t_i) q_i \frac{\psi(t_i)}{\psi'(t_i)} \\ & + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_i) - \ln\psi(s))^{\alpha-1} \phi_q(y_{i-1}(s)) ds \\ & + \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t) - \ln\psi(s))^{\alpha-1} \phi_q(y_k(s)) ds \\ & - \frac{\lambda}{\Gamma(\alpha-1)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \chi(t, t_i) \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_i) - \ln\psi(s))^{\alpha-2} \phi_q(y_{i-1}(s)) ds, \end{aligned} \tag{2.2}$$

where

$$\chi(t, s) = \begin{cases} \frac{\psi'(0)}{\psi(0)} + \ln\psi(s) - \ln\psi(0), & 0 \leq s \leq t \leq 1, \\ \frac{\psi'(0)}{\psi(0)} + \ln\psi(t) - \ln\psi(0), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Proof. Suppose $x = x(t)$ represents the solution to BVP(2.1),

$$
\begin{gathered}
{}^{H}D^{\alpha,\psi}_{t_k} x(t) = \lambda \phi_q(y_k(t)), \\
{}^{H}I^{\alpha,\psi}_{t_k} {}^{H}D^{\alpha,\psi}_{t_k} x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t) - \ln\psi(s))^{\alpha-1} \phi_q(y_k(s))ds.
\end{gathered}
$$

For each $t \in \varpi_0$, there are constants $c_{0,0}$ and $c_{0,1}$ that belong to the set of real numbers, we have

$$
\begin{gathered}
x(t) = c_{0,0} + c_{0,1}(\ln\psi(t) - \ln\psi(0)) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{\psi'(s)}{\psi(s)} (\ln\psi(t) - \ln\psi(s))^{\alpha-1} \phi_q(y_0(s))ds, \\
x'(t) = c_{0,1} \frac{\psi'(t)}{\psi(t)} + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t \frac{\psi'(s)\psi'(t)}{\psi(s)\psi(t)} (\ln\psi(t) - \ln\psi(s))^{\alpha-2} \phi_q(y_0(s))ds, \\
x(t_1^-) = c_{0,0} + c_{0,1}(\ln\psi(t_1) - \ln\psi(0)) + \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_1} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_1) - \ln\psi(s))^{\alpha-1} \phi_q(y_0(s))ds, \\
x'(t_1^-) = c_{0,1} \frac{\psi'(t_1)}{\psi(t_1)} + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{\psi'(s)\psi'(t_1)}{\psi(s)\psi(t_1)} (\ln\psi(t_1) - \ln\psi(s))^{\alpha-2} \phi_q(y_0(s))ds.
\end{gathered}
$$

For each $t \in \varpi_1$, there are real constants $c_{1,0}$ and $c_{1,1}$ such that

$$
\begin{gathered}
x(t) = c_{1,0} + c_{1,1}(\ln\psi(t) - \ln\psi(t_1)) + \frac{\lambda}{\Gamma(\alpha)} \int_{t_1}^t \frac{\psi'(s)}{\psi(s)} (\ln\psi(t) - \ln\psi(s))^{\alpha-1} \phi_q(y_1(s))ds, \\
x'(t) = c_{1,1} \frac{\psi'(t)}{\psi(t)} + \frac{\lambda}{\Gamma(\alpha-1)} \int_{t_1}^t \frac{\psi'(s)\psi'(t)}{\psi(s)\psi(t)} (\ln\psi(t) - \ln\psi(s))^{\alpha-2} \phi_q(y_1(s))ds, \\
x(t_1^+) = c_{1,0}, \quad x'(t_1^+) = c_{1,1} \frac{\psi'(t_1)}{\psi(t_1)}, \\
\begin{aligned}
x(t_2^-) &= c_{1,0} + c_{1,1}(\ln\psi(t_2) - \ln\psi(t_1)) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_2) - \ln\psi(s))^{\alpha-1} \phi_q(y_1(s))ds,
\end{aligned} \\
x'(t_2^-) = c_{1,1} \frac{\psi'(t_2)}{\psi(t_2)} + \frac{\lambda}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} \frac{\psi'(s)\psi'(t_2)}{\psi(s)\psi(t_2)} (\ln\psi(t_2) - \ln\psi(s))^{\alpha-2} \phi_q(y_1(s))ds.
\end{gathered}
$$

From the impulsive conditions $\Delta x(t_1) = p_1,\ \Delta x'(t_1) = q_1$, we have

$$
\begin{aligned}
p_1 &= \Delta x(t_1) = x(t_1^+) - x(t_1^-) \\
&= c_{1,0} - c_{0,0} - c_{0,1}(\ln\psi(t_1) - \ln\psi(0)) \\
&\quad - \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_1} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_1) - \ln\psi(s))^{\alpha-1} \phi_q(y_0(s))ds, \\
q_1 &= \Delta x'(t_1) = x'(t_1^+) - x'(t_1^-) \\
&= c_{1,1} \frac{\psi'(t_1)}{\psi(t_1)} - c_{0,1} \frac{\psi'(t_1)}{\psi(t_1)} \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{\psi'(s)\psi'(t_1)}{\psi(s)\psi(t_1)} (\ln\psi(t_1) - \ln\psi(s))^{\alpha-2} \phi_q(y_0(s))ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
c_{1,0} - c_{0,0} &= p_1 + c_{0,1}(\ln\psi(t_1) - \ln\psi(0)) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_1} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_1) - \ln\psi(s))^{\alpha-1} \phi_q(y_0(s))ds,
\end{aligned} \tag{2.4}
$$

$$
c_{1,1} - c_{0,1} = q_1 \frac{\psi(t_1)}{\psi'(t_1)} + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{\psi'(s)}{\psi(s)} (\ln\psi(t_1) - \ln\psi(s))^{\alpha-2} \phi_q(y_0(s))ds. \tag{2.5}
$$

For each $t \in \varpi_2$, there are real constants $c_{2,0}$ and $c_{2,1}$ such that

$$
\begin{gathered}
x(t) = c_{2,0} + c_{2,1}(\ln\psi(t) - \ln\psi(t_2)) + \frac{\lambda}{\Gamma(\alpha)}\int_{t_2}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}\phi_q(y_2(s))ds, \\
x'(t) = c_{2,1}\frac{\psi'(t)}{\psi(t)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_2}^{t}\frac{\psi'(s)\psi'(t)}{\psi(s)\psi(t)}(\ln\psi(t) - \ln\psi(s))^{\alpha-2}\phi_q(y_2(s))ds, \\
x(t_2^+) = c_{2,0}, \\
x'(t_2^+) = c_{2,1}\frac{\psi'(t_2)}{\psi(t_2)}.
\end{gathered}
$$

From the impulsive conditions $\Delta x(t_2) = p_2,\ \Delta x'(t_2) = q_2$, we have

$$
\begin{aligned}
p_2 &= \Delta x(t_2) = x(t_2^+) - x(t_2^-) \\
&= c_{2,0} - c_{1,0} - c_{1,1}(\ln\psi(t_2) - \ln\psi(t_1)) \\
&\quad - \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2) - \ln\psi(s))^{\alpha-1}\phi_q(y_1(s))ds, \\
q_2 &= \Delta x'(t_2) = x'(t_2^+) - x'(t_2^-) \\
&= c_{2,1}\frac{\psi'(t_2)}{\psi(t_2)} - c_{1,1}\frac{\psi'(t_2)}{\psi(t_2)} \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}\frac{\psi'(s)\psi'(t_2)}{\psi(s)\psi(t_2)}(\ln\psi(t_2) - \ln\psi(s))^{\alpha-2}\phi_q(y_1(s))ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
c_{2,0} - c_{1,0} &= p_2 + c_{1,1}(\ln\psi(t_2) - \ln\psi(t_1)) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2) - \ln\psi(s))^{\alpha-1}\phi_q(y_1(s))ds,
\end{aligned} \tag{2.6}
$$

$$
c_{2,1} - c_{1,1} = q_2\frac{\psi(t_2)}{\psi'(t_2)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2) - \ln\psi(s))^{\alpha-2}\phi_q(y_1(s))ds. \tag{2.7}
$$

In the same manner, for each $t \in \varpi_k$, there are constants $c_{k,0}$ and $c_{k,1}$ that belong to the set of real numbers, we have

$$
\begin{aligned}
x(t) &= c_{k,0} + c_{k,1}(\ln\psi(t) - \ln\psi(t_k)) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}\phi_q(y_k(s))ds, \\
x'(t) &= c_{k,1}\frac{\psi'(t)}{\psi(t)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_k}^{t}\frac{\psi'(s)\psi'(t)}{\psi(s)\psi(t)}(\ln\psi(t) - \ln\psi(s))^{\alpha-2}\phi_q(y_k(s))ds,
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
c_{k,0} - c_{k-1,0} &= p_k + c_{k-1,1}(\ln\psi(t_k) - \ln\psi(t_{k-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_k) - \ln\psi(s))^{\alpha-1}\phi_q(y_{k-1}(s))ds, \\
c_{k,1} - c_{k-1,1} &= q_k\frac{\psi(t_k)}{\psi'(t_k)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_{k-1}}^{t_k}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_k) - \ln\psi(s))^{\alpha-2}\phi_q(y_{k-1}(s))ds.
\end{aligned} \tag{2.9}
$$

In particular, for each $t \in \varpi_m$, there are constants $c_{m,0}$ and $c_{m,1}$ that belong to the set of real numbers, we have

$$
\begin{gathered}
x(t) = c_{m,0} + c_{m,1}(\ln\psi(t) - \ln\psi(t_m)) + \frac{\lambda}{\Gamma(\alpha)}\int_{t_m}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}\phi_q(y_m(s))ds, \\
x'(t) = c_{m,1}\frac{\psi'(t)}{\psi(t)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_m}^{t}\frac{\psi'(s)\psi'(t)}{\psi(s)\psi(t)}(\ln\psi(t) - \ln\psi(s))^{\alpha-2}\phi_q(y_m(s))ds,
\end{gathered} \tag{2.10}
$$

$$
\begin{aligned}
c_{m,0}-c_{m-1,0} &= p_m + c_{m-1,1}(\ln\psi(t_m)-\ln\psi(t_{m-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_m)-\ln\psi(s))^{\alpha-1}\phi_q(y_{m-1}(s))ds, \\
c_{m,1}-c_{m-1,1} & \\
&= q_m\frac{\psi(t_m)}{\psi'(t_m)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_{m-1}}^{t_m}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_m)-\ln\psi(s))^{\alpha-2}\phi_q(y_{m-1}(s))ds.
\end{aligned} \tag{2.11}
$$

From (2.4)-(2.11), we have

$$
\begin{aligned}
c_{1,0}-c_{0,0} &= p_1 + c_{0,1}(\ln\psi(t_1)-\ln\psi(0)) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_1)-\ln\psi(s))^{\alpha-1}\phi_q(y_0(s))ds, \\
c_{2,0}-c_{1,0} &= p_2 + c_{1,1}(\ln\psi(t_2)-\ln\psi(t_1)) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2)-\ln\psi(s))^{\alpha-1}\phi_q(y_1(s))ds, \\
&\dots \\
c_{k,0}-c_{k-1,0} &= p_k + c_{k-1,1}(\ln\psi(t_k)-\ln\psi(t_{k-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_k)-\ln\psi(s))^{\alpha-1}\phi_q(y_{k-1}(s))ds, \\
&\dots \\
c_{m,0}-c_{m-1,0} &= p_m + c_{m-1,1}(\ln\psi(t_m)-\ln\psi(t_{m-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_m)-\ln\psi(s))^{\alpha-1}\phi_q(y_{m-1}(s))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
c_{1,1}-c_{0,1} &= q_1\frac{\psi(t_1)}{\psi'(t_1)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_0^{t_1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_1)-\ln\psi(s))^{\alpha-2}\phi_q(y_0(s))ds, \\
c_{2,1}-c_{1,1} &= q_2\frac{\psi(t_2)}{\psi'(t_2)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2)-\ln\psi(s))^{\alpha-2}\phi_q(y_1(s))ds, \\
&\dots \\
c_{k,1}-c_{k-1,1} &= q_k\frac{\psi(t_k)}{\psi'(t_k)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_{k-1}}^{t_k}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_k)-\ln\psi(s))^{\alpha-2}\phi_q(y_{k-1}(s))ds, \\
&\dots \\
c_{m,1}-c_{m-1,1} &= q_m\frac{\psi(t_m)}{\psi'(t_m)} + \frac{\lambda}{\Gamma(\alpha-1)}\int_{t_{m-1}}^{t_m}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_m)-\ln\psi(s))^{\alpha-2}\phi_q(y_{m-1}(s))ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
c_{k,0}-c_{0,0} &= \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} c_{i-1,1}(\ln\psi(t_i)-\ln\psi(t_{i-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-1}\phi_q(y_{i-1}(s))ds,
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
c_{m,0}-c_{0,0} &= \sum_{i=1}^{m} p_i + \sum_{i=1}^{m} c_{i-1,1}(\ln\psi(t_i)-\ln\psi(t_{i-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-1}\phi_q(y_{i-1}(s))ds,
\end{aligned} \tag{2.13}
$$

and

$$\begin{aligned}
&c_{k,1}-c_{0,1}\\
&=\sum_{i=1}^{k} q_i\frac{\psi(t_i)}{\psi'(t_i)}+\frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds, \quad (2.14)\\
&c_{m,1}-c_{0,1}\\
&=\sum_{i=1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)}+\frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds. \quad (2.15)
\end{aligned}$$

Applying the boundary conditions $x(0)=x'(0)$, $x'(1)=\int_0^\eta g(s,x(s))ds$, we obtain

$$x'(0)=c_{0,1}\frac{\psi'(0)}{\psi(0)},\quad x(0)=c_{0,0},$$
$$x'(1)=c_{m,1}\frac{\psi'(1)}{\psi(1)}+\frac{\lambda}{\Gamma(\alpha-1)}\int_{t_m}^{1}\frac{\psi'(s)\psi'(1)}{\psi(s)\psi(1)}(\ln\psi(1)-\ln\psi(s))^{\alpha-2}\phi_q(y_m(s))ds.$$

Then

$$c_{0,0}=c_{0,1}\frac{\psi'(0)}{\psi(0)},$$
$$c_{m,1}=\frac{\psi(1)}{\psi'(1)}\int_0^\eta g(s,x(s))ds-\frac{\lambda}{\Gamma(\alpha-1)}\int_{t_m}^{1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(1)-\ln\psi(s))^{\alpha-2}\phi_q(y_m(s))ds.$$

From (2.15), we have

$$\begin{aligned}
c_{0,1}&=c_{m,1}-\sum_{i=1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)}-\frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds\\
&=\frac{\psi(1)}{\psi'(1)}\int_0^\eta g(s,x(s))ds-\frac{\lambda}{\Gamma(\alpha-1)}\int_{t_m}^{1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(1)-\ln\psi(s))^{\alpha-2}\phi_q(y_m(s))ds\\
&\quad-\sum_{i=1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)}-\frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds,\\
c_{0,0}&=\frac{\psi(1)}{\psi'(1)}\frac{\psi'(0)}{\psi(0)}\int_0^\eta g(s,x(s))ds-\frac{\psi'(0)}{\psi(0)}\sum_{i=1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)}\\
&\quad-\frac{\lambda}{\Gamma(\alpha-1)}\frac{\psi'(0)}{\psi(0)}\int_{t_m}^{1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(1)-\ln\psi(s))^{\alpha-2}\phi_q(y_m(s))ds\\
&\quad-\frac{\lambda}{\Gamma(\alpha-1)}\frac{\psi'(0)}{\psi(0)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds.
\end{aligned}$$

From (2.12) and (2.14), we have

$$\begin{aligned}
c_{k,0}&=\frac{\psi(1)}{\psi'(1)}\frac{\psi'(0)}{\psi(0)}\int_0^\eta g(s,x(s))ds-\frac{\psi'(0)}{\psi(0)}\sum_{i=1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)}\\
&\quad-\frac{\lambda}{\Gamma(\alpha-1)}\frac{\psi'(0)}{\psi(0)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds\\
&\quad+\sum_{i=1}^{k}p_i+\sum_{i=1}^{k}c_{i-1,1}\big(\ln\psi(t_i)-\ln\psi(t_{i-1})\big)\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-1}\phi_q(y_{i-1}(s))ds,
\end{aligned}$$

$$\begin{aligned}
c_{k,1} &= \frac{\psi(1)}{\psi'(1)}\int_0^\eta g(s,x(s))ds - \sum_{i=1}^{m} q_i \frac{\psi(t_i)}{\psi'(t_i)} \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds \\
&\quad + \sum_{i=1}^{k} q_i\frac{\psi(t_i)}{\psi'(t_i)} + \frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds \\
&= \frac{\psi(1)}{\psi'(1)}\int_0^\eta g(s,x(s))ds - \sum_{i=k+1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)} \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=k+1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds.
\end{aligned}$$

So, for each $t \in \varpi_k$,

$$\begin{aligned}
x(t) &= \frac{\psi(1)}{\psi'(1)}\frac{\psi'(0)}{\psi(0)}\int_0^\eta g(s,x(s))ds - \frac{\psi'(0)}{\psi(0)}\sum_{i=1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)} \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)}\frac{\psi'(0)}{\psi(0)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds \\
&\quad + \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} c_{i-1,1}(\ln\psi(t_i)-\ln\psi(t_{i-1})) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-1}\phi_q(y_{i-1}(s))ds \\
&\quad + \frac{\psi(1)}{\psi'(1)}(\ln\psi(t)-\ln\psi(t_k))\int_0^\eta g(s,x(s))ds - (\ln\psi(t)-\ln\psi(t_k))\sum_{i=k+1}^{m} q_i\frac{\psi(t_i)}{\psi'(t_i)} \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)}(\ln\psi(t)-\ln\psi(t_k))\sum_{i=k+1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t)-\ln\psi(s))^{\alpha-1}\phi_q(y_k(s))ds.
\end{aligned}$$

Thus

$$\begin{aligned}
x(t) &= \frac{\psi(1)}{\psi'(1)}\Big[\frac{\psi'(0)}{\psi(0)} + \ln\psi(t)-\ln\psi(0)\Big]\int_0^\eta g(s,x(s))ds + \sum_{i=1}^{k} p_i - \sum_{i=1}^{m}\chi(t,t_i)q_i\frac{\psi(t_i)}{\psi'(t_i)} \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-1}\phi_q(y_{i-1}(s))ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t)-\ln\psi(s))^{\alpha-1}\phi_q(y_k(s))ds \\
&\quad - \frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\chi(t,t_i)\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\phi_q(y_{i-1}(s))ds.
\end{aligned}$$

The proof is complete. □

From the definition of $\chi(t,s)$ we obtain

$$|\chi(t,s)| \le \Big|\frac{\psi'(0)}{\psi(0)} + \ln\psi(1)-\ln\psi(0)\Big|.$$

We define operator $\Lambda : \Im(\varpi, \mathbb{R}) \to \Im(\varpi, \mathbb{R})$ as

$$\begin{aligned}
&(\Lambda x)(t) \\
&= \frac{\psi(1)}{\psi'(1)}\Big[\frac{\psi'(0)}{\psi(0)} + \ln\psi(t) - \ln\psi(0)\Big]\int_0^\eta g(s,x(s))ds + \sum_{i=1}^{k} P_i(x(t_i)) \\
&\quad - \sum_{i=1}^{m} \chi(t,t_i)Q_i(x(t_i))\frac{\psi(t_i)}{\psi'(t_i)} + \frac{\lambda}{\Gamma(\alpha)} \\
&\quad \times \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-1}\phi_q\Big(h_{i-1}\Big(s, x(s), x(s+(-1)^{i-1}\tau)\Big)\Big)ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t} \frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}\phi_q\Big(h_k(s, x(s), x\Big(s+(-1)^{k}\tau\Big))\Big)ds - \frac{\lambda}{\Gamma(\alpha-1)} \\
&\quad \times \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i} \chi(t,t_i)\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-2}\phi_q\Big(h_{i-1}\Big(s, x(s), x(s+(-1)^{i-1}\tau)\Big)\Big)ds.
\end{aligned}$$

Therefore, the existence of a solution to BVP (1.1) can be transformed into the existence of a fixed point of the operator Λ on $\Im(\varpi, \mathbb{R})$. For convenience in calculations, we define

$$\begin{gathered}
\Theta_1 = \frac{\psi'(0)}{\psi(0)} + \ln\psi(1) - \ln\psi(0), \\
\Phi_1 = (q-1)(\sigma^*)^{q-2}(M_h^* + N_h^*), \\
\Phi_2 = \frac{\lambda(m+1)(v^\alpha + \Theta_1\alpha v^{\alpha-1})}{\Gamma(\alpha+1)}.
\end{gathered}$$

3. Existence and uniqueness of solutions

We obtain the existence and uniqueness of solutions for BVP (1.1) by applying three fixed point theorems. First, we state some assumptions needed.

(H1) There exist functions $\sigma_k \in C(\varpi_k, [0, +\infty))$ $(k = 0, 1, 2, \dots, m)$, such that for all $u_k, v_k \in \mathbb{R}$ $(k = 0, 1, 2, \dots, m)$ and $t \in \varpi_k$, we have $|h_k(t, u_k, v_k)| \le \sigma_k(t)$; and we denote

$$\sigma^* = \max\big\{\sup_{t\in\varpi_0}\sigma_0(t), \sup_{t\in\varpi_1}\sigma_1(t), \sup_{t\in\varpi_2}\sigma_2(t), \dots, \sup_{t\in\varpi_m}\sigma_m(t)\big\}.$$

(H2) There exist constants $M_1, N_1, M_2, N_2 > 0$, such that, for any $u \in \mathbb{R}$,

$$\begin{gathered}
|P_k(u)| \le M_1|u| + N_1, \\
|Q_k(u)| \le M_2|u| + N_2, \ k = 1, 2, \dots, m.
\end{gathered}$$

(H3) There exist constants $M_h^*, N_h^* > 0$, such that, for any $u_k, v_k, \bar{u}_k, \bar{v}_k \in \mathbb{R}$, $t \in \varpi_k$ $(k = 0, 1, 2, \dots, m)$, we have

$$|h_k(t, u_k, v_k) - h_k(t, \bar{u}_k, \bar{v}_k)| \le M_h^*|u_k - v_k| + N_h^*|\bar{u}_k - \bar{v}_k|, \quad k = 0, 1, 2, \dots, m.$$

(H4) There exist constants $l, l^* > 0$, such that, for any $u, \bar{u} \in \mathbb{R}$,

$$\begin{gathered}
|P_k(u) - P_k(\bar{u})| \le l|u - \bar{u}|, \\
|Q_k(u) - Q_k(\bar{u})| \le l^*|u - \bar{u}|, \quad k = 1, 2, \dots, m.
\end{gathered}$$

(H5) Function $g \in C(\varpi \times \mathbb{R}, \mathbb{R})$, and there exists a function $\varphi_1(t) \in L^{1/2}(\varpi, \mathbb{R}^+)$, such that, for all $u, \bar{u} \in \mathbb{R}$, $t \in \varpi$, we have

$$|g(t, u) - g(t, \bar{u})| \le \varphi_1(t)|u - \bar{u}|,$$

where $\|\varphi_1\| = (\int_0^1 {\varphi_1}^2(s)ds)^{1/2}$.

(H6) There exist constants $\varphi_2^*, \varphi_3^* > 0$, such that, for all $u \in \mathbb{R}, t \in \varpi$,

$$|g(t, u)| \le \varphi_2^* + \varphi_3^*|u|.$$

(H7) There exists a constant $N_x > 0$, such that

$$\frac{N_x}{\frac{\psi(1)}{\psi'(1)}\Theta_1(\varphi_2^* + \varphi_3^* N_x)\eta + m(M_1 N_x + N_1) + \Theta_1(M_2 N_x + N_2)\frac{m\psi(1)}{\psi^*} + [\Phi_1 N_x + \phi_q(K)]\Phi_2} > 1,$$

where $\sup_{t\in\varpi_k} |h_k(t, 0, 0)| = K < \infty$, $(k = 0, 1, 2, \dots, m)$.

Theorem 3.1. *Under assumptions* (H1)–(H7)*, BVP* (1.1) *has at least one solution.*

Proof. When $t \in \varpi_k$ $(k = 0, 1, 2, \dots, m)$, for any $x_1, x_2 \in \Im(\varpi, \mathbb{R})$, according to conditions (H1), (H3), and lemma 2.5, we have

$$\begin{aligned}
&\big|\phi_q(h_k(t, x_1(t), x_1(t + (-1)^k\tau))) - \phi_q(h_k(t, x_2(t), x_2(t + (-1)^k\tau)))\big| \\
&\le (q-1)(\sigma^*)^{q-2}(M_h^*|x_2(t) - x_1(t)| + N_h^*|x_1(t + (-1)^k\tau) - x_2(t + (-1)^k\tau)|) \\
&\le (q-1)(\sigma^*)^{q-2}(M_h^* + N_h^*)\|x_2 - x_1\|_{\Im},
\end{aligned}$$

according to the definition of operator Λ and the continuity of h_k $(k = 0, 1, 2, \dots, m)$, operator Λ is continuous. For any $r > 0$, denote $B_r = \{x \in \Im(\varpi, \mathbb{R}) : \|x\| \le r\}$, it is easy to check that B_r is a bounded closed ball in $\Im(\varpi, \mathbb{R})$.

Firstly, prove that operator Λ maps a bounded set in $\Im(\varpi, \mathbb{R})$ to a bounded set. There exists a constant ρ_1, for any $x \in B_r$, we have $\|\Lambda x\| \le \rho_1$.

Let $\sup_{t\in\varpi_k} |h_k(t, 0, 0)| = K < \infty$. For any $t \in \varpi_k(k = 0, 1, 2, \dots, m)$, $x \in B_r$, we have

$$\begin{aligned}
&|(\Lambda x)(t)| \\
&\le \frac{\psi(1)}{\psi'(1)}\Big[\frac{\psi'(0)}{\psi(0)} + \ln\psi(t) - \ln\psi(0)\Big]\int_0^\eta |g(s, x(s))|ds + \sum_{i=1}^{k}|P_i(x(t_i))| \\
&\quad + \sum_{i=1}^{m}|\chi(t, t_i)||Q_i(x(t_i))|\frac{\psi(t_i)}{\psi'(t_i)} + \frac{\lambda}{\Gamma(\alpha)} \\
&\quad \times \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) \\
&\quad - \ln\psi(s))^{\alpha-1}\big|\phi_q(h_{i-1}(s, x(s), x(s + (-1)^{i-1}\tau))) - \phi_q(h_{i-1}(s, 0, 0))\big|ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-1}|\phi_q(h_{i-1}(s, 0, 0))|ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t) \\
&\quad - \ln\psi(s))^{\alpha-1}\big|\phi_q(h_k(s, x(s), x(s + (-1)^k\tau))) - \phi_q(h_k(s, 0, 0))\big|ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}|\phi_q(h_k(s, 0, 0))|ds + \frac{\lambda}{\Gamma(\alpha-1)} \\
&\quad \times \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}|\chi(t, t_i)|\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) \\
&\quad - \ln\psi(s)^{\alpha-2}\big|\phi_q(h_{i-1}(s, x(s), x(s + (-1)^{i-1}\tau))) - \phi_q(h_{i-1}(s, 0, 0))\big|ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}|\chi(t, t_i)|\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-2}|\phi_q(h_{i-1}(s, 0, 0))|ds \\
&\le \frac{\psi(1)}{\psi'(1)}\Big(\frac{\psi'(0)}{\psi(0)} + \ln\psi(1) - \ln\psi(0)\Big)(\varphi_2^* + \varphi_3^*\|x\|)\eta + m(M_1\|x\| + N_1) \\
&\quad + (\frac{\psi'(0)}{\psi(0)} + \ln\psi(1) - \ln\psi(0))(M_2\|x\| + N_2)\frac{m\psi(1)}{\psi^*}
\end{aligned}$$

$$
\begin{aligned}
&+(q-1)(\sigma^*)^{q-2}(M_h^*+N_h^*)\|x\|\frac{\lambda(m+1)v^\alpha}{\Gamma(\alpha+1)}+\frac{\lambda(m+1)v^\alpha}{\Gamma(\alpha+1)}\phi_q(K)\\
&+\Big(\frac{\psi'(0)}{\psi(0)}+\ln\psi(1)-\ln\psi(0)\Big)(q-1)(\sigma^*)^{q-2}(M_h^*+N_h^*)\|x\|\frac{(m+1)\lambda v^{\alpha-1}}{\Gamma(\alpha)}\\
&+\Big(\frac{\psi'(0)}{\psi(0)}+\ln\psi(1)-\ln\psi(0)\Big)\frac{(m+1)\lambda v^{\alpha-1}}{\Gamma(\alpha)}\phi_q(K).
\end{aligned}
$$

Then

$$
\begin{aligned}
\|\Lambda x\| \le\ & \frac{\psi(1)}{\psi'(1)}\Theta_1(\varphi_2^*+\varphi_3^* r)\eta+m(M_1r+N_1)+\Theta_1(M_2r+N_2)\frac{m\psi(1)}{\psi^*}\\
&+\lambda(m+1)(q-1)(\sigma^*)^{q-2}(M_h^*+N_h^*)r\frac{v^\alpha+\Theta_1\alpha v^{\alpha-1}}{\Gamma(\alpha+1)}\\
&+\frac{\lambda(m+1)(v^\alpha+\alpha\Theta_1 v^{\alpha-1})}{\Gamma(\alpha+1)}\phi_q(K)=\rho_1.
\end{aligned}
$$

From this we infer that Λ maps the bounded set in $\Im(\varpi,\mathbb{R})$ to a bounded set.

Secondly, we show that Λ is equicontinuous. For any $x\in B_r$, $t_1,t_2\in\varpi_k$ $(k=0,1,2,\dots,m)$, where $t_1<t_2$, we have

$$
\begin{aligned}
&|(\Lambda x)(t_2)-(\Lambda x)(t_1)|\\
&\le \frac{\psi(1)}{\psi'(1)}[\ln\psi(t_2)-\ln\psi(t_1)]\int_0^\eta|g(s,x(s))|ds+\sum_{i=1}^m|\chi(t_2,t_i)-\chi(t_1,t_i)||Q_i(x(t_i))|\frac{\psi(t_i)}{\psi'(t_i)}\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k\Big(s,x(s),x(s+(-1)^k\tau)\Big)\Big)\Big|ds\\
&\quad-\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_1)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k\Big(s,x(s),x(s+(-1)^k\tau)\Big)\Big)\Big|ds+\frac{\lambda}{\Gamma(\alpha-1)}\\
&\quad\times\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}|\chi(t_2,t_i)-\chi(t_1,t_i)|\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)\\
&\quad-\ln\psi(s))^{\alpha-2}\Big|\phi_q\Big(h_{i-1}(s,x(s),x\Big(s+(-1)^{i-1}\tau\Big))\Big)\Big|ds\\
&\le \frac{\psi(1)}{\psi'(1)}[\ln\psi(t_2)-\ln\psi(t_1)]\int_0^\eta|g(s,x(s))|ds+[\ln\psi(t_2)-\ln\psi(t_1)]\sum_{i=1}^m|Q_i(x(t_i))|\frac{\psi(t_i)}{\psi'(t_i)}\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k(s,x(s),x(s+(-1)^k\tau))\Big)-\phi_q\Big(h_k(s,0,0)\Big)\Big|ds\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k(s,0,0)\Big)\Big|ds\\
&\quad-\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_1)-\ln\psi(s))^{\alpha-1}\Big|\phi_q(h_k(s,x(s),x(s+(-1)^k\tau)))-\phi_q(h_k(s,0,0))\Big|ds\\
&\quad-\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_1)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k(s,0,0)\Big)\Big|ds+\frac{\lambda}{\Gamma(\alpha-1)}[\ln\psi(t_2)-\ln\psi(t_1)]\\
&\quad\times\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}|\phi_q(h_{i-1}(s,x(s),x(s+(-1)^{i-1}\tau)))-\phi_q(h_{i-1}(s,0,0))|ds\\
&\quad+\frac{\lambda}{\Gamma(\alpha-1)}[\ln\psi(t_2)-\ln\psi(t_1)]\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}|\phi_q(h_{i-1}(s,0,0))|ds\\
&\le \frac{\psi(1)}{\psi'(1)}[\ln\psi(t_2)-\ln\psi(t_1)](\varphi_2^*+\varphi_3^*r)\eta+[\ln\psi(t_2)-\ln\psi(t_1)]m(M_2r+N_2)\frac{\psi(1)}{\psi^*}
\end{aligned}
$$

$$
\begin{aligned}
&+ (q-1)(\sigma^*)^{q-2}(M_h^*+N_h^*)r\frac{\lambda}{\Gamma(\alpha+1)}\{[\ln\psi(t_2)-\ln\psi(t_k)]^\alpha-[\ln\psi(t_1)-\ln\psi(t_k)]^\alpha\}\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\phi_q(K)\{[\ln\psi(t_2)-\ln\psi(t_k)]^\alpha-[\ln\psi(t_1)-\ln\psi(t_k)]^\alpha\}\\
&+(m+1)[\ln\psi(t_2)-\ln\psi(t_1)](q-1)(\sigma^*)^{q-2}(M_h^*+N_h^*)r\frac{\lambda v^{\alpha-1}}{\Gamma(\alpha)}\\
&+\frac{\lambda v^{\alpha-1}}{\Gamma(\alpha)}(m+1)[\ln\psi(t_2)-\ln\psi(t_1)]\phi_q(K).
\end{aligned}
$$

Therefore, $|(\Lambda x)(t_2)-(\Lambda x)(t_1)|\to 0$, as $t_1\to t_2$. By Arzela-Ascoli theorem, Λ is compact, thus $\Lambda:\Im(\varpi,\mathbb{R})\to\Im(\varpi,\mathbb{R})$ is a completely continuous operator.

If x is a solution to the BVP(1.1), for any $t\in\varpi_k(k=0,1,2,\dots,m)$, similar to the previous proof method, we have

$$
\begin{aligned}
\|x\|\le{}&\frac{\psi(1)}{\psi'(1)}\Theta_1(\varphi_2^*+\varphi_3^*\|x\|)\eta+m(M_1\|x\|+N_1)+\Theta_1(M_2\|x\|+N_2)\frac{m\psi(1)}{\psi^*}\\
&+\lambda(m+1)\Phi_1\|x\|\frac{v^\alpha+\Theta_1\alpha v^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{\lambda(m+1)(v^\alpha+\alpha\Theta_1 v^{\alpha-1})}{\Gamma(\alpha+1)}\phi_q(K);
\end{aligned}
$$

thus,

$$
\frac{\|x\|}{\frac{\psi(1)}{\psi'(1)}\Theta_1(\varphi_2^*+\varphi_3^*\|x\|)\eta+m(M_1\|x\|+N_1)+\Theta_1(M_2\|x\|+N_2)\frac{m\psi(1)}{\psi^*}+[\Phi_1\|x\|+\phi_q(K)]\Phi_2}\le 1,
$$

according to (H7), there exists a constant N_x, such that $\|x\|\neq N_x$.

Assume $B_{N_x}=\{x\in\Im(\varpi,\mathbb{R}):\|x\|<N_x\}$, because Λ is a completely continuous operator, considering the choice of B_{N_x}, with respect to a particular $\gamma_1\in(0,1)$. There is no $x\in\bar{B}_{N_x}$, such that $x=\gamma_1\Lambda x$. From Lemma 2.6, it can be inferred that Λ has at least one fixed point $x\in\bar{B}_{N_x}$, which means that the BVP(1.1) has at least one solution. The verification has been accomplished. □

Theorem 3.2. *Under assumptions* (H3)–(H5), *if*

$$
M_1=\frac{\psi(1)}{\psi'(1)}\Theta_1\sqrt{\eta}\|\varphi_1\|+ml+\frac{\psi(1)}{\psi^*}\Theta_1 ml^*+\frac{\lambda\alpha\Theta_1(m+1)v^{\alpha-1}+\lambda(m+1)v^\alpha}{\Gamma(\alpha+1)}\Phi_1<1,
$$

then BVP (1.1) *has a unique solution.*

Proof. For each $x_1,x_2\in\Im(\varpi,\mathbb{R})$ and $t\in\varpi_k(k=0,1,2,\dots,m)$, by (H5), we obtain

$$
\begin{aligned}
\int_0^\eta|g(s,x_2(s))-g(s,x_1(s))|ds&\le\|x_2-x_1\|\int_0^\eta\varphi_1(s)ds\\
&\le\|x_2-x_1\|\Big(\int_0^\eta 1^2ds\Big)^{1/2}\Big(\int_0^\eta{\varphi_1}^2(s)ds\Big)^{1/2}\\
&\le\|x_2-x_1\|\sqrt{\eta}\Big(\int_0^1{\varphi_1}^2(s)ds\Big)^{1/2}\\
&\le\|x_2-x_1\|\sqrt{\eta}\|\varphi_1\|.
\end{aligned}\tag{3.1}
$$

From this inequality we have

$$
\begin{aligned}
&|(\Lambda x_2)(t)-(\Lambda x_1)(t)|\\
&\le\frac{\psi(1)}{\psi'(1)}\Theta_1\int_0^\eta|g(s,x_2(s))-g(s,x_1(s))|ds+\sum_{i=1}^m|P_i(x_2(t_i))-P_i(x_1(t_i))|\\
&\quad+\Theta_1\sum_{i=1}^m|Q_i(x_2(t_i))-Q_i(x_1(t_i))|\frac{\psi(t_i)}{\psi'(t_i)}\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)
\end{aligned}
$$

$$
\begin{aligned}
&- \ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_{i-1}\Big(s, x_2(s), x_2(s+(-1)^{i-1}\tau)\Big)\Big) \\
&- \phi_q\Big(h_{i-1}\Big(s, x_1(s), x_1(s+(-1)^{i-1}\tau)\Big)\Big)\Big|ds + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}\ln\psi(t) \\
&- \ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k\Big(s, x_2(s), x_2(s+(-1)^{k}\tau)\Big)\Big) \\
&- \phi_q\Big(h_k\Big(s, x_1(s), x_1(s+(-1)^{k}\tau)\Big)\Big)\Big|ds \\
&+ \frac{\lambda\Theta_1}{\Gamma(\alpha-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) \\
&- \ln\psi(s))^{\alpha-2}\Big|\phi_q(h_{i-1}(s, x_2(s), x_2(s+(-1)^{i-1}\tau))) \\
&- \phi_q\Big(h_{i-1}\Big(s, x_1(s), x_1(s+(-1)^{i-1}\tau)\Big)\Big)\Big|ds \\
\le{}& \Big\{\frac{\psi(1)}{\psi'(1)}\Theta_1\sqrt{\eta}\|\varphi_1\| + ml + \frac{\psi(1)}{\psi^*}\Theta_1 ml^* + \frac{\lambda\alpha\Theta_1(m+1)v^{\alpha-1} + \lambda(m+1)v^{\alpha}}{\Gamma(\alpha+1)}\Phi_1\Big\}\|x_2 - x_1\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|(\Lambda x_2)(t) - (\Lambda x_1)(t)\| \\
&\le \{\frac{\psi(1)}{\psi'(1)}\Theta_1\sqrt{\eta}\|\varphi_1\| + ml + \frac{\psi(1)}{\psi^*}\Theta_1 ml^* + \frac{\lambda\alpha\Theta_1(m+1)v^{\alpha-1} + \lambda(m+1)v^{\alpha}}{\Gamma(\alpha+1)}\Phi_1\}\|x_2 - x_1\|.
\end{aligned}
$$

Thus,

$$
\|(\Lambda x_2)(t) - (\Lambda x_1)(t)\| < M_1\|x_2 - x_1\|.
$$

Therefore, by lemma 2.7, it can be inferred that Λ is compressed and has a unique fixed point on $\Im(\varpi, \mathbb{R})$, that is, the BVP (1.1) has a unique solution. □

For the next theorem we need the following assumptions;

- (H8) There exists a constant $\varsigma > 0$, such that for any $u_k, v_k \in \mathbb{R}$ $(k = 0, 1, 2, \dots, m)$ and $t \in \varpi_k$, we have
$$
|\phi_q(h_k(t, u_k, v_k))| < \varsigma, \quad k = 0, 1, 2, \dots, m;
$$
- (H9)
$$
\frac{\psi(1)}{\psi'(1)}\Theta_1\|\varphi_1\|\sqrt{\eta} + ml + m\Theta_1 l^*\frac{\psi(1)}{\psi^*} < 1\,.
$$

Theorem 3.3. *Under assumptions* (H2), (H4)–(H6), (H8), (H9)*, the BVP* (1.1) *has at least one solution.*

Proof. We denote $B_R = \{x \in \Im(\varpi, \mathbb{R}) : \|x\| \le R\}$, where

$$
R > \frac{\Phi_2\varsigma + \frac{\psi(1)}{\psi'(1)}\Theta_1\varphi_2^*\eta + mN_1 + \Theta_1 N_2\frac{m\psi(1)}{\psi^*}}{1 - \frac{\psi(1)}{\psi'(1)}\Theta_1\varphi_3^*\eta - mM_1 - M_2\Theta_1\frac{m\psi(1)}{\psi^*}}.
$$

Then B_R is a bounded closed convex non-empty subset on Banach space $\Im(\varpi, \mathbb{R})$.

We define the two operators Ψ, T on B_R as

$$
\begin{aligned}
&(\Psi x)(t) \\
&= \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-1}\phi_q\Big(h_{i-1}(s, x(s), x\Big(s+(-1)^{i-1}\tau\Big))\Big)ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}\phi_q\Big(h_k\Big(s, x(s), x\Big(s+(-1)^{k}\tau\Big)\Big)\Big)ds - \frac{\lambda}{\Gamma(\alpha-1)} \\
&\quad \times \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\chi(t, t_i)\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-2}\phi_q\Big(h_{i-1}\Big(s, x(s), x\Big(s+(-1)^{i-1}\tau\Big)\Big)\Big)ds,
\end{aligned}
$$

and

$$(Tx)(t) = \frac{\psi(1)}{\psi'(1)}[\frac{\psi'(0)}{\psi(0)} + \ln\psi(t) - \ln\psi(0)]\int_0^\eta g(s,x(s))ds + \sum_{i=1}^k P_i(x(t_i))$$
$$- \sum_{i=1}^m \chi(t,t_i)Q_i(x(t_i))\frac{\psi(t_i)}{\psi'(t_i)}.$$

Firstly, we verify that $(\Psi x_1)(t) + (Tx_2)(t) \in B_R$. When $t \in \varpi_k$ $(k = 0,1,2,\dots,m)$, for any $x_1, x_2 \in B_R$, we have

$$\begin{aligned}
&|(\Psi x_1)(t) + (Tx_2)(t)| \\
&\le \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_{i-1}\Big(s,x_1(s),x_1(s+(-1)^{i-1}\tau)\Big)\Big)\Big|ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^t \frac{\psi'(s)}{\psi(s)}(\ln\psi(t) - \ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k(s,x_1(s),x_1\Big(s+(-1)^k\tau\Big))\Big)\Big|ds + \frac{\lambda}{\Gamma(\alpha-1)} \\
&\quad \times \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i} |\chi(t,t_i)|\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-2}\Big|\phi_q\Big(h_{i-1}\Big(s,x_1(s),x_1(s+(-1)^{i-1}\tau)\Big)\Big)\Big|ds \\
&\quad + \frac{\psi(1)}{\psi'(1)}\Theta_1\int_0^\eta |g(s,x_2(s))|ds + \sum_{i=1}^k |P_i(x_2(t_i))| + \sum_{i=1}^m |\chi(t,t_i)||Q_i(x_2(t_i))|\frac{\psi(t_i)}{\psi'(t_i)} \\
&\le \Phi_2\varsigma + \frac{\psi(1)}{\psi'(1)}\Theta_1(\varphi_2^* + \varphi_3^*\|x\|)\eta + m(M_1\|x\| + N_1) + \Theta_1(M_2\|x\| + N_2)\frac{m\psi(1)}{\psi^*} \\
&\le \Phi_2\varsigma + \frac{\psi(1)}{\psi'(1)}\Theta_1\varphi_2^*\eta + mN_1 + \Theta_1 N_2\frac{m\psi(1)}{\psi^*} + \Big(\frac{\psi(1)}{\psi'(1)}\Theta_1\varphi_3^*\eta + mM_1 + M_2\Theta_1\frac{m\psi(1)}{\psi^*}\Big)\|x\| \\
&\le \Phi_2\varsigma + \frac{\psi(1)}{\psi'(1)}\Theta_1\varphi_2^*\eta + mN_1 + \Theta_1 N_2\frac{m\psi(1)}{\psi^*} + \Big(\frac{\psi(1)}{\psi'(1)}\Theta_1\varphi_3^*\eta + mM_1 + M_2\Theta_1\frac{m\psi(1)}{\psi^*}\Big)R.
\end{aligned}$$

Therefore, $|(\Psi x_1)(t) + (Tx_2)(t)| \le B_R$, and $(\Psi x_1)(t) + (Tx_2)(t) \in B_R$.

Secondly, we prove that operator T is a contraction mapping within B_R, when $t \in \varpi_k$, for any $x_1, x_2 \in B_R$, from (3.1), we have

$$\begin{aligned}
|(Tx_2)(t) - (Tx_1)(t)| &\le \frac{\psi(1)}{\psi'(1)}\Theta_1\int_0^\eta |g(s,x_2(s)) - g(s,x_1(s))|ds + \sum_{i=1}^k |P_i(x_2(t_i)) - P_i(x_1(t_i))| \\
&\quad + \sum_{i=1}^m |\chi(t,t_i)||Q_i(x_2(t_i)) - Q_i(x_1(t_i))|\frac{\psi(t_i)}{\psi'(t_i)} \\
&\le [\frac{\psi(1)}{\psi'(1)}\Theta_1\|\varphi_1\|\sqrt{\eta} + ml + m\Theta_1 l^*\frac{\psi(1)}{\psi^*}]\|x_2 - x_1\|.
\end{aligned}$$

By (H9), operator T is a contraction mapping within B_R.

Finally, verify that operator Ψ is a completely continuous operator. From the definition of operator Ψ and the continuity of function h, it can be inferred that operator Ψ is continuous, thus, we only need to prove that operator Ψ is compact. The process is divided into the following two steps.

Step 1. Ψ is uniformly bounded. When $t \in \varpi_k$ $(k = 0,1,2,\dots,m)$, for any $x \in B_R$, there exists a constant ξ, such that $|(\Psi x)t| \le \xi$. According to (H8), for any $t \in \varpi_k(k = 0,1,2,\dots,m)$, $x \in B_R$, we have

$$\begin{aligned}
&|(\Psi x)(t)| \\
&\le \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i) - \ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_{i-1}\Big(s,x(s),x(s+(-1)^{i-1}\tau)\Big)\Big)\Big|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k(s,x(s),x\Big(s+(-1)^k\tau\Big))\Big)\Big|ds+\frac{\lambda}{\Gamma(\alpha-1)}\\
&\times\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}|\chi(t,t_i)|\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_i)-\ln\psi(s))^{\alpha-2}\Big|\phi_q\Big(h_{i-1}\Big(s,x(s),x\Big(s+(-1)^{i-1}\tau\Big)\Big)\Big)\Big|ds\\
\le{}&\frac{\lambda(m+1)\varsigma(v^\alpha+\Theta_1\alpha v^{\alpha-1})}{\Gamma(\alpha+1)}:=\xi.
\end{aligned}
$$

From this, we infer that operator Ψ is uniformly bounded.

Step 2. Ψ is equicontinuous. For any $x\in B_R$, $t_1,t_2\in\varpi_k(k=0,1,2,\dots,m)$, where $t_1<t_2$, we obtain

$$
\begin{aligned}
&|(\Psi x)(t_2)-(\Psi x)(t_1)|\\
&\le\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_2}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_2)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k\Big(s,x(s),x\Big(s+(-1)^k\tau\Big)\Big)\Big)\Big|ds\\
&\quad-\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t_1}\frac{\psi'(s)}{\psi(s)}(\ln\psi(t_1)-\ln\psi(s))^{\alpha-1}\Big|\phi_q\Big(h_k\big(s,x(s),x(s+(-1)^k\tau)\big)\Big)\Big|ds\\
&\quad+\frac{\lambda}{\Gamma(\alpha-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}|\chi(t_2,t_i)-\chi(t_1,t_i)|\frac{\psi'(s)}{\psi(s)}\big(\ln\psi(t_i)\\
&\quad-\ln\psi(s)\big)^{\alpha-2}|\phi_q\Big(h_{i-1}\Big(s,x(s),x(s+(-1)^{i-1}\tau)\Big)\Big)\Big|ds\\
&\le\frac{\varsigma\lambda}{\Gamma(\alpha+1)}[(\ln\psi(t_2)-\ln\psi(t_k))^\alpha-(\ln\psi(t_1)-\ln\psi(t_k))^\alpha]\\
&\quad+\frac{\lambda\varsigma(m+1)v^{\alpha-1}}{\Gamma(\alpha)}[\ln\psi(t_2)-\ln\psi(t_1)].
\end{aligned}
$$

Therefore, $|(\Psi x)(t_2)-(\Psi x)(t_1)|\to0$ as $t_1\to t_2$. Thus, the operator Ψ is equicontinuous on B_R. In line with lemma 2.9, operator Ψ is a compact operator within B_R, satisfying the conditions required by Lemma 2.8. In conclusion, there is at least one solution to BVP (1.1). ☐

4. Examples

In this section we verify the main results through two examples.

Example 4.1. Consider the BVP

$$
\begin{gathered}
\phi_p\Big({}^HD_0^{3/2,t^2+t+1}x(t)\Big)=\frac{1}{10}\Big(\frac{t}{100}+\frac{t\sin x}{150}+\frac{t\sin x(t+\frac13)}{100}\Big),\quad t\in(0,\frac12)\\
\phi_p\Big({}^HD_{1/2}^{3/2,t^2+t+1}x(t)\Big)=\frac{1}{10}\Big(\frac{t}{100+t}+\frac{t^2\sin x}{150+t}+\frac{t\sin x(t-\frac13)}{100}\Big),\quad t\in(\frac12,1)\\
\Delta x\big(t_{1/2}\big)=\frac{|x(1/2)|}{300+|x(1/2)|},\quad \Delta x'(t_{1/2})=\frac{|x(1/2)|}{200+|x(1/2)|},\\
x(0)=x'(0),\quad x'(1)=\frac{1}{10}\int_0^{1/2}\frac{e^s\sin^{1/2}s}{1+2s}\frac{x(s)}{1+s}\,ds,
\end{gathered}\tag{4.1}
$$

where $m=1$, $\alpha=3/2$, $t_0=0$, $t_1=1/2$, $t_2=1$, $\tau=1/3$, $\lambda=1/10$, $\eta=1/2$, and $p=3/2$. From $p=3/2$ and $q^{-1}+p^{-1}=1$, we obtain $q=3$, $\psi(t)=t^2+t+1$ $\psi'(t)=2t+1$, and $\psi^*=1$. Then $\psi(t)=t^2+t+1$, is an increasing function in $t\in[0,1]$, $v=0.81$,

$$
\begin{gathered}
h_1(t,x(t),x(t+\tau))=\frac{t}{100}+\frac{t\sin x}{150}+\frac{t\sin x(t+\frac13)}{100}\le\frac{2t}{75},\\
h_2(t,x(t),x(t-\tau))=\frac{t}{100+t}+\frac{t^2\sin x}{150+t}+\frac{t\sin x(t-\frac13)}{100}\le\frac{t^2+3t}{150},
\end{gathered}
$$

where $\sigma_1(t) = \frac{2t}{75}, \sigma_2(t) = \frac{t^2+3t}{150}$, then

$$\sigma^* = \max\Big\{\sup_{t\in(0,1/2)}(\frac{2t}{75}), \sup_{t\in(1/2,1)}(\frac{t^2+3t}{150})\Big\} = \frac{2}{75},$$
$$P(x(1/2)) = \frac{|x(1/2)|}{300+|x(1/2)|}, \quad Q(x(1/2)) = \frac{|x(1/2)|}{200+|x(1/2)|},$$
$$g(t,x(t)) = \frac{1}{10}\Big(\frac{e^t \sin^{1/2} t}{1+2t}\frac{x(t)}{1+t}\Big).$$

Obviously, $h_k(k=1,2)$ is a continuous function. For each $u_1, v_1, \bar{u}_1, \bar{v}_1, u_2, v_2, \bar{u}_2, \bar{v}_2, u, \bar{u} \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\begin{gathered}
|h_1(t,u_1,v_1)| \le \frac{2t}{75}, \\
|h_2(t,u_2,v_2)| \le \frac{t^2+3t}{150}, \\
|P(u)| \le \frac{1}{300}|u| + \frac{1}{10}, \\
|Q(u)| \le \frac{1}{200}|u| + \frac{1}{10}, \\
|h_1(t,u_1,v_1) - h_1(t,\bar{u}_1,\bar{v}_1)| \le \frac{1}{150}|u_1-\bar{u}_1| + \frac{1}{100}|v_1-\bar{v}_1|, \\
|h_2(t,u_2,v_2) - h_2(t,\bar{u}_2,\bar{v}_2)| \le \frac{1}{150}|u_2-\bar{u}_2| + \frac{1}{100}|v_2-\bar{v}_2|, \\
|P(u) - P(\bar{u})| \le \frac{1}{300}|u-\bar{u}|, \\
|Q(u) - Q(\bar{u})| \le \frac{1}{200}|u-\bar{u}|, \\
|g(t,u)| \le \frac{1}{10}e^t|u| + \frac{1}{100}, \\
|g(t,u) - g(t,\bar{u})| \le \frac{1}{10}\frac{e^t \sin^{1/2} t}{1+2t}|u-\bar{u}| \le \frac{1}{10}e^t \sin^{1/2} t|u-\bar{u}|.
\end{gathered}$$

So that conditions (H1)–(H6) hold, where

$$M_1 = \frac{1}{300}, \quad N_1 = \frac{1}{10}, \quad M_2 = \frac{1}{200}, \quad N_2 = \frac{1}{10}, \quad M_h^* = \frac{1}{150}, \quad N_h^* = \frac{1}{100},$$
$$l = \frac{1}{300}, l^* = \frac{1}{200}, \quad \varphi_2^* = \frac{1}{100}, \quad \varphi_3^* = \frac{e}{10}.$$

After calculations, it can be concluded that

$$\begin{gathered}
\|\varphi_1\| = \Big(\int_0^1 (\frac{1}{10}e^s \sin^{1/2} s)^2 ds\Big)^{1/2} \approx 0.1265, \\
\Theta_1 = \frac{\psi'(0)}{\psi(0)} + \ln\psi(1) - \ln\psi(0) = 3.09, \\
\Phi_1 = (q-1)(\sigma^*)^{q-2}(M_h^* + N_h^*) \approx 0.00089, \\
\Phi_2 = \frac{\lambda(m+1)(v^\alpha + \Theta_1 \alpha v^{\alpha-1})}{\Gamma(\alpha+1)} \approx 0.7283.
\end{gathered}$$

From the definition of h_k, $k=1,2$, it can be inferred that $K = \frac{1}{100}$, then $\phi_q(K) = \frac{1}{10000}$. Then there exists a constant $N_x = 10 > 0$, such that

$$\begin{aligned}
&N_x/\Big(\frac{\psi(1)}{\psi'(1)}\Theta_1(\varphi_2^* + \varphi_3^* N_x)\eta + m(M_1N_x + N_1) + \Theta_1(M_2N_x + N_2)\frac{m\psi(1)}{\psi^*} + [\Phi_1 N_x + \phi_q(K)]\Phi_2\Big) \\
&\approx 1.5163 > 1.
\end{aligned}$$

Therefore, all the assumptions in Theorem 3.1 are satisfied, so there is at least one solution to the BVP (4.1). Because

$$\begin{aligned} M_1 &= \frac{\psi(1)}{\psi'(1)}\Theta_1\sqrt{\eta}\|\varphi_1\| + ml + \frac{\psi(1)}{\psi^*}\Theta_1 ml^* + \frac{\lambda\alpha\Theta_1(m+1)v^{\alpha-1} + \lambda(m+1)v^{\alpha}}{\Gamma(\alpha+1)}\Phi_1 \\ &\approx 0.3267 < 1, \end{aligned}$$

all the assumptions in Theorem 3.2 are satisfied, thus, BVP (4.1) has a unique solution.

Example 4.2. Consider another BVP

$$\begin{gathered} \phi_p\Big({}^H D_0^{3/2,t^2+2t+1}x(t)\Big) = \frac{1}{20}(\frac{t}{100} + \frac{t\sin x}{150} + \frac{t\sin x(t+\frac{1}{3})}{100}), \quad t\in(0,\frac{1}{3}) \\ \phi_p\Big({}^H D_{1/3}^{3/2,t^2+2t+1}x(t)\Big) = \frac{1}{20}(\frac{t}{100+t} + \frac{t^2\sin x}{150+t} + \frac{t\sin x(t-\frac{1}{3})}{100}), \quad t\in(\frac{1}{3},\frac{2}{3}) \\ \phi_p\Big({}^H D_{2/3}^{3/2,t^2+2t+1}x(t)\Big) = \frac{1}{20}\Big(\frac{t}{100} + \frac{\sin x}{150(1+t)} + \frac{t\sin x(t+\frac{1}{3})}{100}\Big), \quad t\in(\frac{2}{3},1) \\ \Delta x\big(t_{1/3}\big) = \frac{|x(1/3)|}{30+|x(1/3)|}, \quad \Delta x'(t_{1/3}) = \frac{|x(1/3)|}{40+|x(1/3)|}, \\ x(0) = x'(0), \quad x'(1) = \frac{1}{10}\int_0^{1/3} e^{2s}\sin^{1/2}s\frac{x(s)}{1+s}ds, \end{gathered} \tag{4.2}$$

where $m=1$, $\alpha=3/2$, $t_0=0$, $t_1=1/3$, $t_2=2/3$, $t_3=1$, $\tau=1/3$, $\lambda=1/20$, $\eta=1/3$, $p=3/2$, From $p=3/2$ and $q^{-1}+p^{-1}=1$, we obtain $q=3$, $\psi(t)=t^2+2t+1$, $\psi'(t)=2t+2, \psi^*=2$. Then $\psi(t)=t^2+2t+1$ is an increasing function in $t\in[0,1]$,

$$\begin{aligned} h_1(t,x(t),x(t+\tau)) &= \frac{t}{100} + \frac{t\sin x}{100} + \frac{t\sin x(t+\frac{1}{3})}{150} \le \frac{2t}{75}, \\ h_2(t,x(t),x(t-\tau)) &= \frac{t}{100+t} + \frac{t^2\sin x}{150+t} + \frac{t\sin x(t-\frac{1}{3})}{100} \le \frac{t^2+3t}{150}, \\ h_3(t,x(t),x(t+\tau)) &= \frac{t}{100} + \frac{\sin x}{150(1+t)} + \frac{t\sin x(t+\frac{1}{3})}{100} \le \frac{3t+1}{150}. \end{aligned}$$

Obviously, h_k $(k=1,2,3)$ is a continuous function, and

$$|\phi_q(h_k(t,x(t),x(t+(-1)^k\tau)))| < (\frac{2}{75})^2 = \frac{4}{5625},$$

let $\varsigma = \frac{4}{5625}$, then the inequality $|\phi_q(h_k(t,x(t),x(t+(-1)^k\tau)))| < \varsigma$ holds, that is, condition (H8) is satisfied.

$$\begin{aligned} P\Big(x(1/3)\Big) &= \frac{|x(1/3)|}{30+|x(1/3)|}, \\ Q(x(1/3)) &= \frac{|x(1/3)|}{40+|x(1/3)|}, \\ g(t,x(t)) &= \frac{1}{10}e^{2t}\sin^{1/2}t\frac{x(t)}{1+t}. \end{aligned}$$

For each $u,\bar{u}\in\mathbb{R}, t\in[0,1]$, we have

$$\begin{gathered} |P(u)-P(\bar{u})| \le \frac{1}{30}|u-\bar{u}|, \\ |Q(u)-Q(\bar{u})| \le \frac{1}{40}|u-\bar{u}|, \\ |g(t,u)| \le \frac{1}{10} + \frac{e^2}{10}|u|, \\ |g(t,u)-g(t,\bar{u})| \le \frac{1}{10}e^{2t}\sin^{1/2}t|u-\bar{u}|, \end{gathered}$$

therefore, conditions (H1)–(H7) are satisfied, where

$$l = \frac{1}{30}, l^* = \frac{1}{40}, \quad \varphi_2^* = \frac{1}{10}, \varphi_3^* = \frac{e^2}{10}.$$

After calculation, we concluded that

$$\|\varphi_1\| = \Big(\int_0^1 \Big(\frac{1}{10}e^{2s}\sin^{1/2} s\Big)^2 ds\Big)^{1/2} \approx 0.2844$$
$$\Theta_1 = \frac{\psi'(0)}{\psi(0)} + \ln\psi(1) - \ln\psi(0) = 3.39'$$

For assumption (H9), we have

$$\frac{\psi(1)}{\psi'(1)}\Theta_1\|\varphi_1\|\sqrt{\eta} + ml + m\Theta_1 l^* \frac{\psi(1)}{\psi^*} \approx 0.8442 < 1.$$

In conclusion, all assumptions of Theorem 3.3 are satisfied, hence BVP (4.2) has at least one solution.

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References

[1] W. Abdelhedi; *Fractional differential equations with a ψ-Hilfer fractional derivative*, Computational and Applied Mathematics., **40**(2021), 53.

[2] R. Agarwal, S. Hristova, D. ORegan; *Mittag-Leffler stability for impulsive Caputo fractional differential equations*, Differential Equations and Dynamical Systems., **29**(2021), 689-705.

[3] B. Ahmad, M. Alghanmi, A. Alsaedi, et al.; *On an impulsive hybrid system of conformable fractional differential equations with boundary conditions*, International Journal Of Systems Science., **51**(2020), 958-970.

[4] A. Ali, K. Shah, F. Jarad; *Ulam-Hyers stability analysis to a class of nonlinear implicit impulsive fractional differential equations with three-point boundary conditions*, Advances in Difference Equations., **2019**(2019), 1-27.

[5] L. Almaghamsi, S. Horrigue; *Existence results for some p-Laplacian Langevin problems with a Hilfer fractional derivative with antiperiodic boundary conditions*, Fractal and Fractional, **9**(2025), 194.

[6] R. Almeida; *Functional differential equations involving the ψ-Caputo fractional derivative*, Fractal and Fractional., **4**(2020), 29.

[7] M. Aslam, J.F. Gmez-Aguilar, G. ur-Rahman, et al.; *Existence, uniqueness, and Hyers-Ulam stability of solutions to nonlinear p-Laplacian singular delay fractional boundary value problems*, Mathematical Methods in the Applied Sciences., **46**(2023), 8193-8207.

[8] Z. Bai; *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear Analysis: Theory, Methods and Applications., **72**(2010), 916-924.

[9] Z. Bai; *Theory and application of fractional differential equation boundary value problem*, Beijing: China Science and Technology Press, 2012.

[10] K. Balachandran, M. Matar, N. Annapoorani, et al.; *Hadamard functional fractional integrals and derivatives and fractional differential equations*, Filomat., **38**(2024), 779-792.

[11] A. Batool, I. Talib, M. B. Riaz, et al.; *Extension of lower and upper solutions approach for generalized nonlinear fractional boundary value problems*, Arab Journal of Basic and Applied Sciences., **29**(2022), 249-256.

[12] C. V. Bose, R. Udhayakumar, V. Muthukumaran, et al.; *A study on approximate controllability of ψ-Caputo fractional differential equations with impulsive effects*, Contemporary Mathematics., (2024),175-198.

[13] F. E. Browder; *Nonlinear Functional Analysis and Its Applications, Part 1*: Proceedings of the Summer Research Institute: the Result of the Thirty-first Summer Research Institute of the American Mathematical Society; Berkeley-California, July 11-29, 1983, American Mathematical Society, 1986.

[14] T. Chen, Y. Zhang, N. Huang, et al.; *A new nonlocal impulsive fractional differential hemivariational inclusions with an application to a frictional contact problem*, Appl. Math. Comput., **490** (2025), 129211.

[15] X. Dong, Z. Bai, S. Zhang; *Positive solutions to boundary value problems of p-Laplacian with fractional derivative, Boundary Value Problems*, **2017**(2017), 1-15.

[16] A. Granas, J. Dugundji; *Elementary fixed point theorems*, Fixed point theory 2003; 9-84.

[17] Y. He; *Extremal solutions for p-Laplacian fractional differential systems involving the Riemann-Liouville integral boundary conditions*, Advances in Difference Equations., **2018**(2018), 1-11.

[18] A. Irguedi, K. Nisse, S. Hamani; *Functional impulsive fractional differential equations involving the Caputo-Hadamard derivative and integral boundary conditions*, International Journal of Analysis and Applications., **21**(2023), 15-15.

[19] F. Jiao, Y. Zhou; *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Computers and Mathematics with Applications, **62**(2011), 1181-1199.

[20] F. Jiao, Y. Zhou; *Existence results for fractional boundary value problem via critical point theory*, International Journal of Bifurcation and Chaos., **22**(2012), 1250086.

[21] U. N. Katugampola; *A new approach to generalized fractional derivatives*, Bulletin of Mathematical Analysis and Applications., **6**(2014), 1-15.

[22] K. Kaushik, A. Kumar, A. Khan, et al.; *Existence of solutions by fixed point theorem of general delay fractional differential equation with p-Laplacian operator*, AIMS Mathematics., **8**(2023), 10160-10176.

[23] J. Liang, Y. Mu, T. Xiao; *Impulsive differential equations involving general conformable fractional derivative in Banach spaces*, Revista De La Real Academia De Ciencias Exactas Fisicas Y Naturales Serie A-mate. Matemticas., **116**(2022), 114.

[24] M. Mahemuti, R. Rouzimaimaiti, A. Abdujelil, et al.; *Fixed/Preassigned time synchronization of impulsive fractional-order reaction- diffusion bidirectional associative memory (BAM) neural networks*, Fractal and Fractional, **9**(2025), 88.

[25] M. J. Mardanov, N. I. Mahmudov, Y. A. Sharifov; *Existence and uniqueness results for q-fractional difference equations with p-Laplacian operators*, Advances in Difference Equations, **2015**(2015), 1-13.

[26] A. Martynyuk, G. Stamov, I. Stamova, et al.; *Formulation of impulsive ecological systems using the conformable calculus approach: qualitative analysis*, Mathematics., **11**(2023), 2221.

[27] R. Poovarasan, M. E. Samei, V. Govindaraj; *Study of three-point impulsive boundary value problems governed by ψ-Caputo fractional derivative*, Journal of Applied Mathematics and Computing, **70**(2024), 3947-3983.

[28] J. Shao, B. Guo; *Existence of solutions and Hyers-Ulam stability for a coupled system of nonlinear fractional differential equations with p-Laplacian operator*, Symmetry, **13**(2021), 1160.

[29] T. Shen, W. Liu, X. Shen; *Existence and uniqueness of solutions for several BVPs of fractional differential equations with p-Laplacian operator*, Mediterranean Journal of Mathematic., **13**(2016), 4623-4637.

[30] K. Sheng, W. Zhang, Z. Bai; *Positive solutions to fractional boundary-value problems with p-Laplacian on time scales*, Boundary Value Problems., **2018**(2018), 1-15.

[31] C. E. Torres Ledesma, N. Nyamoradi; *Impulsive fractional boundary value problem with p-Laplace operator*, Journal of Applied Mathematics and Computing., **55**(2017), 257-278.

[32] J. Wu, X. Zhang, L. Liu, et al.; *The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity*, Boundary Value Problems, **2018**(2018), 1-15.

[33] W. Yao, H. Zhang; *Multiple solutions for p-Laplacian fractional differential equations with Caputo derivative and impulsive effects*, J. Appl. Anal. Comput., 15 (2025), 3480-3503.

[34] L. Zhang, X. Liu, M. Jia, et al.; *Piecewise conformable fractional impulsive differential system with delay: existence, uniqueness and Ulam stability*, Journal of Applied Mathematics and Computing, **70**(2024), 1543-1570.

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