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NEW APPROACH TO THE LAGRANGE-BÜRMANN THEOREM VIA OMEGA CALCULUS AND APPLICATIONS

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Abstract. A novel approach to the ubiquitous multidimensional Lagrange-Bürmann Theorem is developed which uses the omega calculus (OC) developed long ago by MacMahon to study the partition of natural numbers. Several applications are given including the answer to open questions regarding the generalized Lambert function W as stated in [56]. More precisely, a master theorem is presented introducing a new generalized Lambert function for which several previously known representations arise as special cases including most Taylor series results of [56] and some other integral representations. Furthermore, the convergence radius of the aforementioned generalized Lambert function is explicitly determined for which even special cases were not known before. This work shows another instance where omega calculus is useful, this time, to address inverse problems of general interest related to functional equations.

1. Introduction

The Lagrange-Bürmann (L-B) formula including its multidimensional version is a distinguished example of an inverse problem. More precisely, it provides an expansion of the composition of a function defined by a power series with an inverse function (provided it exists) and has a long history with many proofs, variations, and widespread applications. We refer the reader to the non-exhaustive list [1, 11, 26, 27, 28, 29, 31, 37, 40, 44, 48, 55, 60, 67, 69] including the many facets involving in its proof and extensions to the noncommutative and q-contexts. The applications are so widespread that even if we limit ourselves to the physical context many scales are involved ranging from the micro, such as quantum mechanics as in, e.g., [70], to the macro, such as general relativity as in, e.g., [68].

Another inverse problem comprises the computation of the Lambert function, $W(\zeta)$ for short, which solves the functional equation

$$W(\zeta)e^{W(\zeta)} = \zeta \tag{1.1}$$

and extensions [57] sharing the same ubiquitous character of the L-B formula and appearing also in a wide range of scales from the micro to the macro [14, 57, 16, 61, 65, 66]. In this respect, for a general account we refer the reader to [57] which includes a detailed analysis of its multi-valued character and applications in the physical context (see also [59] for a pedagogical account). Although we focus on the scalar Lambert W function as defined in (1.1) we refer the reader to [72] for applications of the matrix-valued Lambert W function in order to solve time-delay systems.

Omega Calculus (OC) also known as MacMahon Partition Analysis (MPA) was originally introduced by MacMahon in order to describe the partition of natural numbers [51]. Since then many extensions [18, 19, 21, 23] and unexpected applications emerge including a new basis and integral-free approach [20, 24] to the Dyson series [17, 43, 63] associated with the time-ordering operator [25] with dynamics dictated by the Schrödinger equation. In this way, a new combinatorial approach to perturbation expansion which implies the divided-differences approach of [45, 46] if a basis is used for the time-independent part of the generator of the dynamics was developed. Aside

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from the OC approach to perturbation theory, there are plenty of others combinatorial/algebraic approaches to perturbation theory each one with distinct flavors [8, 9, 10, 15, 30, 49, 53, 50] and several relying on graphs [6, 7]. More recently, the inverse problem associated with the dynamics of non-autonomous systems of differential equations was considered; that is, given the generator of the dynamics the problem comprises computing perturbatively the associated dynamical system. More concretely, a version of the aforementioned inverse question was raised as an open problem in [4] and answered in [22] using OC. In this sense OC provides another operational approach besides others more well-known such as [43, 54].

In this work, we show that the aforementioned inverse problems; that is, the L-B formula and the Lambert W function and generalizations are amenable to be treated in the realm of OC with interesting and unexpected consequences. We will show, among other things, that a generalized Lambert W function can be constructed which implies several previously known results concerning representations of the Lambert W functions and associated extensions. This fact justifies our use of the name master theorem as it encapsulates in a single expression many known results including Taylor series and integral type representations. Furthermore, the convergence radius of the aforementioned generalized Lambert W function is explicitly determined for which even special cases were not known before. The results summarized above answer some of the open questions put forward in [56, Section 5]. Furthermore, we generalize a previous representation of a function related to configuration spaces in the context of algebraic geometry [52, Chapter 4] and a formula for higher order derivatives of the inverse function as described in [42].

This work is organized as follows. In Section 2 the necessary background is introduced and some central results are stated in Theorems 3.1, 3.5, and 3.7 in Section 3. Some applications are discussed in Sections 7, 8, 9, and 10. Sections 7 and 8 concern applications of Theorem 3.1, where we introduce a generalized Lambert W function along with its radius of convergence, respectively, which implies as special cases several previous cases addressed in the literature including the answer to some of the open questions raised in [56]. Other applications are discussed in Sections 9 and 10 concerning a sequence arising in algebraic geometry using Theorem 3.5 and higher order derivatives related to the inverse function using Theorems 3.1 and 3.7, respectively. We finish with some concluding remarks in Section 11. To make the text more fluid we include the proofs of Theorems 3.1, 3.5, and 3.7 in Sections 4, 5, and 6, respectively. In doing so, the versatility of the OC is shown by allowing us to construct multiple proofs of the aforementioned theorems and auxiliary results with several distinct flavors; that is, exploring different aspects which come along the OC representation.

2. Auxiliary results

2.1. **Notation.** Before we continue, we introduce the notation used throughout this work:

- $i,\, j,\, k,\, \ell,\, m,\, n,\, p,\, q \in \mathbb{N}$.
- $a,\, b,\, c \in \mathbb{Z}$.
- Greek letters such as α, β, ζ, and so on, stand for complex numbers with the Omega variables represented by the middle of the Greek alphabet such as λ, μ, and ν. $\Re(\alpha)$ ($\Im(\alpha)$) stands for the real (complex) part of the complex number α so that $\alpha = \Re(\alpha) + \imath\Im(\alpha)$.
- $[m, m+n] = \{m, \dots, m+n\}$, $[n] = [1, n]$, and $[n]_0 = [0, n]$.
- $\epsilon, \varepsilon = \pm 1$.
- $\mathbf{e}_k$ stands for the unit vector in $\mathbb{R}^n$ with all the coordinates zero except for the $k-$coordinate which is one and $\mathbf{e} = \sum_{k=1}^n \mathbf{e}_k$.
- $F^{\langle -1\rangle}$ means the inverse function of F or
$$F^{\langle -1\rangle} \circ F = I = F \circ F^{\langle -1\rangle},$$
where I stands for the identity function and $F^{\langle n\rangle}$ means function composition or
$$F^{\langle n\rangle} = F \circ \dots \circ F$$
with F appearing n times.
- If $F(\xi) = \sum_n \alpha_n \xi^n$, then $\langle \xi^n\rangle F(\xi) := \alpha_n$.
- $F(n) \approx G(n)$ means asymptotic equality as $n \to \infty$.

- $\delta_{a,b}$ stands for the Kronecker delta.
- S_n is the set of permutations of n elements.

2.2. **Basics of omega calculus.** The OC is rooted in the use of the Omega operators.

Definition 2.1. Let $\boldsymbol{\alpha}_{\mathbf{a}} \in \mathbb{C}^n$ for each $\mathbf{a} \in \mathbb{Z}^n$ and $\boldsymbol{\lambda}^{\mathbf{a}} = \lambda_1^{a_1} \dots \lambda_n^{a_n}$. We define the linear operators acting on absolutely convergent matrix valued expansions by

$$\begin{gathered}\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \sum \boldsymbol{\alpha}_{\mathbf{a}} \boldsymbol{\lambda}^{\mathbf{a}} := \boldsymbol{\alpha}_{\mathbf{0}_n}, \\ \underset{\geq}{\overset{\boldsymbol{\lambda}}{\Omega}} \sum \boldsymbol{\alpha}_{\mathbf{a}} \boldsymbol{\lambda}^{\mathbf{a}} := \sum_{\mathbf{a} \geq \mathbf{0}} \boldsymbol{\alpha}_{\mathbf{a}}\end{gathered} \tag{2.1}$$

in an open neighbourhood of the complex circles $|\lambda_i| = 1$ with

$$\sum := \sum_{a_1=-\infty}^{\infty} \cdots \sum_{a_n=-\infty}^{\infty}.$$

In other words, the Omega operator in (2.1) extracts in a given convergent expansion only powers of λ^n such that $n = 0$. We note that the Omega operators above are connected. Indeed, e.g., we have

$$\underset{=}{\overset{\lambda}{\Omega}} F(\lambda) = \underset{\geq}{\overset{\lambda}{\Omega}} \Big(1 - \frac{1}{\lambda}\Big) F(\lambda)$$

which follows from

$$\underset{=}{\overset{\lambda}{\Omega}} \lambda^a = \delta_{a,0} = \underset{\geq}{\overset{\lambda}{\Omega}} \Big(1 - \frac{1}{\lambda}\Big) \lambda^a.$$

Definition 2.1 is well-posed in the sense that we can ensure that all expressions considered here have no singularities in the λ_i variable in an open neighbourhood of the circle $|\lambda_i| = 1$. As remarked in [2], this is an important ingredient leading to unique Laurent expansions avoiding ambiguous results as discussed in [2, Introduction] and in more details below. In Definition 2.1 we can see an example of the elimination procedure which is the basic building block about which the OC is based. More precisely, the Omega operator in (2.1) selects only the terms in $\sum \boldsymbol{\alpha}_{\mathbf{a}} \boldsymbol{\lambda}^{\mathbf{a}}$ with $\mathbf{a} = \mathbf{0}_n$ resulting in an expression free of Omega variables or, in other words, the Omega variable $\boldsymbol{\lambda}$ is eliminated. From now on we refer to the expression containing an Omega variable a crude generating function. For other aspects of the elimination procedure, we refer the reader to [3, 33, 71, 23]. Two key observations of the elimination procedure in the context of the OC which follow directly from Definition 2.1 will be crucial for us here.

The elimination is independent of the order chosen to eliminate the variables; that is, we have

$$\underset{=}{\overset{\lambda}{\Omega}} \Big(\underset{=}{\overset{\mu}{\Omega}} \Big) = \underset{=}{\overset{\lambda,\mu}{\Omega}} = \underset{=}{\overset{\mu}{\Omega}} \Big(\underset{=}{\overset{\lambda}{\Omega}} \Big).$$

It follows as a consequence that the Omega operator commutes with derivatives of complex variables not to be eliminated. Indeed, we have the next lemma.

Lemma 2.2. *We have*

$$D_\alpha \Big(\underset{=}{\overset{\lambda}{\Omega}} F(\alpha, \lambda) \Big) = \underset{=}{\overset{\lambda}{\Omega}} \big(D_\alpha F(\alpha, \lambda) \big).$$

Proof. The proof follows the usual limiting procedure and the linearity of the Omega operator or by the simple observation that the derivative is the coefficient of the linear term α in $F(\alpha, \lambda)$ which can be rewritten as an Omega operator. In symbols, we have

$$D_\alpha \Big(\underset{=}{\overset{\lambda}{\Omega}} F(\alpha, \lambda) \Big) = \underset{=}{\overset{\mu}{\Omega}} \frac{1}{\mu} \Big(\underset{=}{\overset{\lambda}{\Omega}} F(\alpha + \mu, \lambda) \Big) = \underset{=}{\overset{\lambda}{\Omega}} \Big(\underset{=}{\overset{\mu}{\Omega}} \frac{1}{\mu} F(\alpha + \mu, \lambda) \Big) = \underset{=}{\overset{\lambda}{\Omega}} \big(D_\alpha F(\alpha, \lambda) \big).$$

∎

Care must be taken about the smaller parameter in a crude generating function containing Omega variables in order to avoid ambiguity as discussed before. Indeed, on the one hand, observe that

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda}{1-\alpha/\lambda}=\alpha$$

if $|\alpha|<1$. On the other hand, if $|\alpha|>1$, we have

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda}{1-\alpha/\lambda}=-\frac{1}{\alpha}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^2}{1-\lambda/\alpha}=0.$$

The next lemmas which comprise particular elimination procedures will be useful later on.

Lemma 2.3. *We have*

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^n}{(1-\alpha\lambda)(1-\beta/\lambda)}=\frac{\beta^n}{1-\alpha\beta}.$$

Proof. Using Definition 2.1 we obtain

$$\begin{aligned}
\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^n}{(1-\alpha\lambda)(1-\beta/\lambda)}&=\underset{=}{\overset{\lambda}{\Omega}}\big(\lambda^n+\alpha\lambda^{n+1}+\alpha^2\lambda^{n+2}+\dots\big)\Big(1+\frac{\beta}{\lambda}+\Big(\frac{\beta}{\lambda}\Big)^2+\dots\Big)\\
&=\beta^n+\alpha\beta^{n+1}+\alpha\beta^{n+2}+\dots\\
&=\frac{\beta^n}{1-\alpha\beta}.
\end{aligned}$$

□

We also have the important lemma.

Lemma 2.4.

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{F(\alpha\lambda)}{\lambda^n}\Big(1-\frac{\beta}{\lambda}\Big)^{-1}=\begin{cases}\frac{F^{(n)}(0)}{n!}\alpha^n, & \beta=0\\ \frac{F(\alpha\beta)-\sum_{m=0}^{n-1}F^{(m)}(0)(\alpha\beta)^m/m!}{\beta^n}, & \beta\neq 0\end{cases}$$

with the convention $\sum_{m=0}^{-1}\equiv 0$.

Proof. We first show

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{e^{\alpha\lambda}}{\lambda^n}\Big(1-\frac{\beta}{\lambda}\Big)^{-1}=\begin{cases}\frac{\alpha^n}{n!}, & \beta=0\\ \frac{e^{\alpha\beta}-\sum_{m=0}^{n-1}(\alpha\beta)^m/m!}{\beta^n}, & \beta\neq 0.\end{cases}$$

Indeed, note that

$$\begin{aligned}
\underset{=}{\overset{\lambda}{\Omega}}\frac{e^{\alpha\lambda}}{\lambda^n}\Big(1-\frac{\beta}{\lambda}\Big)^{-1}&=\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^n}\Big(1+\frac{\alpha\lambda}{1!}+\frac{\alpha^2\lambda^2}{2!}+\dots\Big)\Big(1+\frac{\beta}{\lambda}+\frac{\beta^2}{\lambda^2}+\dots\Big)\\
&=\underset{=}{\overset{\lambda}{\Omega}}\Big(1+\frac{\alpha\lambda}{1!}+\frac{\alpha^2\lambda^2}{2!}+\dots\Big)\Big(\frac{1}{\lambda^n}+\frac{\beta}{\lambda^{n+1}}+\frac{\beta^2}{\lambda^{n+2}}+\dots\Big)\\
&=\frac{\alpha^n}{n!}+\frac{\alpha^{n+1}\beta}{(n+1)!}+\frac{\alpha^{n+2}\beta^2}{(n+2)!}+\dots\\
&=\begin{cases}\frac{\alpha^n}{n!}, & \beta=0\\ \frac{e^{\alpha\beta}-\sum_{m=0}^{n-1}(\alpha\beta)^m/m!}{\beta^n}, & \beta\neq 0.\end{cases}
\end{aligned}$$

The result now follows using

$$F(\alpha\lambda)=\lim_{k\to\infty}\sum_{\ell\geq k}F^{(\ell)}(0)\mu^\ell\exp\Big(\frac{\alpha\lambda}{\mu}\Big).$$

□

We also have

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{F(\alpha\lambda)}{\lambda^n}\Big(1-\frac{\beta}{\lambda}\Big)^{-m-1}=\frac{D_\beta^m}{m!}\underset{=}{\overset{\lambda}{\Omega}}\frac{F(\alpha\lambda)}{\lambda^{n-m}}\Big(1-\frac{\beta}{\lambda}\Big)^{-1}$$

using that we can permute the symbols Ω and D_β^m.

Lemma 2.5.

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{F(\lambda)G(\lambda)}{\lambda^k} = \sum_{m+n=k} \underset{=}{\overset{\mu,\nu}{\Omega}} \frac{F(\mu)}{\mu^m}\frac{G(\nu)}{\nu^n}.$$

Proof. We have

$$\begin{aligned}
\underset{=}{\overset{\lambda}{\Omega}} \frac{F(\lambda)G(\lambda)}{\lambda^k} &= \underset{=}{\overset{\lambda,\mu,\nu}{\Omega}} \frac{1}{\lambda^k}\frac{F(\mu)}{1-\lambda/\mu}\frac{G(\nu)}{1-\lambda/\nu} \\
&= \underset{=}{\overset{\mu,\nu}{\Omega}} \Big(\underset{=}{\overset{\lambda}{\Omega}} \frac{1}{\lambda^k}\frac{1}{1-\lambda/\mu}\frac{1}{1-\lambda/\nu}\Big) F(\mu)G(\nu) \\
&= \sum_{m,n\ge 0} \underset{=}{\overset{\mu,\nu}{\Omega}} \underbrace{\Big(\underset{=}{\overset{\lambda}{\Omega}} \lambda^{m+n-k}\Big)}_{=\delta_{m+n,k}} \frac{F(\mu)}{\mu^m}\frac{G(\nu)}{\nu^n} \\
&= \sum_{m+n=k} \underset{=}{\overset{\mu,\nu}{\Omega}} \frac{F(\mu)}{\mu^m}\frac{G(\nu)}{\nu^n},
\end{aligned}$$

where the first equality follows using Lemma 2.4 with $n = 0$. □

Finally, we recall the connection between OC and standard integral representations which follows from a direct observation,

$$\int e^{\imath m\theta} d\theta = 2\pi\delta_{m,0} = 2\pi \underset{=}{\overset{\lambda}{\Omega}} \lambda^m = 2\pi \underset{\ge}{\overset{\lambda}{\Omega}} \Big(1 - \frac{1}{\lambda}\Big)\lambda^m,$$

where $\int = \int_{-\pi}^{\pi}, \int_0^{2\pi}$. In this way, we can write

$$\int f(\cos\theta, \sin\theta) d\theta = \int g\big(e^{\imath\theta}, e^{-\imath\theta}\big) d\theta = 2\pi \underset{=}{\overset{\lambda}{\Omega}} g\big(\lambda, \lambda^{-1}\big) = 2\pi \underset{\ge}{\overset{\lambda}{\Omega}} \Big(1 - \frac{1}{\lambda}\Big) g\big(\lambda, \lambda^{-1}\big) \tag{2.2}$$

provided g satisfies the conditions stated in Definition 2.1. This observation was used in [23] to solve a nontrivial integral using OC.

Example 2.6 ([64, Problem 2202])**.** We have

$$\begin{aligned}
I_+ &= \int_0^{2\pi} \cos(\cos(\theta))\cosh(\sin(\theta)) d\theta = 2\pi, \\
I_- &= \int_0^{2\pi} \sin(\cos(\theta))\cosh(\sin(\theta)) d\theta = 0.
\end{aligned}$$

Indeed, if $\lambda_\epsilon := \imath(\lambda + \epsilon\lambda^{-1})/2$, we note that

$$\begin{aligned}
I_\epsilon &= 2\pi \underset{=}{\overset{\lambda}{\Omega}} \frac{e^{\lambda_+} + \epsilon e^{-\lambda_+}}{2}\frac{e^{\lambda_-} + e^{-\lambda_-}}{2} \\
&= \frac{\pi}{2} \underset{=}{\overset{\lambda}{\Omega}} \Big(e^{\imath\lambda} + e^{\imath\lambda^{-1}} + \epsilon e^{-\imath\lambda} + \epsilon e^{-\imath\lambda^{-1}}\Big) = 2\pi\delta_{\epsilon,+}.
\end{aligned}$$

Example 2.7. We have

$$I = \int_0^{\pi} \frac{\cos(\theta)}{1 + \alpha\cos(\theta)} d\theta = \pi\frac{\sqrt{1-\alpha^2} - 1}{\alpha\sqrt{1-\alpha^2}}.$$

Indeed, we note that

$$I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(\theta)}{1 + \alpha\cos(\theta)} d\theta = \frac{\pi}{2} \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda + \lambda^{-1}}{1 + \alpha(\lambda + \lambda^{-1})/2}.$$

Next, we introduce $\beta_+^2 + \beta_-^2 = 1$, $\beta_+\beta_- = -\alpha/2$, and assume $|\beta_-| < |\beta_+|$. A simple calculation gives

$$\beta_\epsilon = \frac{\alpha_+ + \epsilon\alpha_-}{2}$$

such that $\alpha_\epsilon := \sqrt{1+\epsilon\alpha^2}$. In this way, we have

$$\begin{aligned} I &= \frac{\pi}{2}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda+\lambda^{-1}}{(\beta_+-\beta_-\lambda)(\beta_+-\beta_-\lambda^{-1})} \\ &= \pi\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda}{(\beta_+-\beta_-\lambda)(\beta_+-\beta_-\lambda^{-1})} \\ &= \frac{\pi\beta_-}{\beta_+^2-\beta_-^2} \\ &= \pi\frac{\sqrt{1-\alpha^2}-1}{\alpha\sqrt{1-\alpha^2}}, \end{aligned}$$

where the second equality follows from the invariance of the denominator under the replacement $\lambda \to \lambda^{-1}$ and, the last but one follows from Lemma 2.3. Although this example is of elementary nature it is sufficient to show another nice feature of OC; that is, the ability to obtain closed form expressions.

3. Lagrange-Bürmann expansion via OC

In all the expansions that follow care must be taken such that Definition 2.1 applies. For this purpose we take

$$|\zeta| < |\lambda|.$$

We let

$$F(\zeta) = \sum_{n\geq 0}\alpha_n\zeta^n, \tag{3.1}$$

$$G(\zeta) = \sum_{n\geq 1}\beta_n\zeta^n \tag{3.2}$$

with $\beta_1 \neq 0$. We can now state the main results of this section.

Theorem 3.1. *We have*

$$\big(F\circ G^{\langle -1\rangle}\big)(\zeta) = \alpha_0 - \underset{=}{\overset{\lambda}{\Omega}}\lambda F'(\lambda)\ln\Big(1-\frac{\zeta}{G(\lambda)}\Big). \tag{3.3}$$

If we set $F = I$ in Theorem 3.1 we immediately obtain the Lagrange inversion formula stated in the next corollary.

Corollary 3.2. *We have*

$$G^{\langle -1\rangle}(\zeta) = -\underset{=}{\overset{\lambda}{\Omega}}\lambda\ln\Big(1-\frac{\zeta}{G(\lambda)}\Big).$$

To get a flavor of what is going on we describe some simple examples before embarking into more involved calculations.

Example 3.3. We take

$$G(\zeta) = \zeta\exp(\zeta) \tag{3.4}$$

to obtain

$$G^{\langle -1\rangle}(\zeta) = -\underset{=}{\overset{\lambda}{\Omega}}\lambda\ln\Big(1-\frac{\zeta}{\lambda\exp(\lambda)}\Big) = \sum_{n\geq 1}\frac{\zeta^n}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{\exp(-n\lambda)}{\lambda^{n-1}} = \sum_{n\geq 1}\frac{(-n)^{n-1}}{n!}\zeta^n$$

with radius of convergence

$$R = \lim_{n\to\infty}\Big|\frac{\gamma_n}{\gamma_{n+1}}\Big| = \frac{1}{e},$$

where $\gamma_n = (-n)^{n-1}/n!$. From now on we write $G^{\langle -1\rangle}(\zeta) = W(\zeta)$ for G as in (3.4) using the standard notation for the Lambert W function.

Example 3.4. We take

$$G(\zeta) = \frac{\zeta}{(1-\zeta)^{1/3}}$$

to obtain

$$\begin{aligned} G^{\langle -1\rangle}(\zeta) &= -\underset{=}{\overset{\lambda}{\Omega}}\lambda \ln\Big(1 - \frac{\zeta(1-\lambda)^{1/3}}{\lambda}\Big) \\ &= \sum_{n\geq 1} \frac{\zeta^n}{n} \underset{=}{\overset{\lambda}{\Omega}} \frac{(1-\lambda)^{n/3}}{\lambda^{n-1}} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \binom{n/3}{n-1} \zeta^n \end{aligned}$$

with radius of convergence

$$R = \lim_{n\to\infty} \big|\frac{\gamma_n}{\gamma_{n+1}}\big| = \frac{3}{e},$$

where

$$\begin{aligned} \gamma_n &= \frac{(-1)^{n-1}}{n} \binom{n/3}{n-1} \\ &= \frac{(-1)^{n-1}}{n} \underset{=}{\overset{\lambda}{\Omega}} \frac{(1-\lambda)^{n/3}}{\lambda^{n-1}} \\ &= \frac{(-1)^{n-1} n^{n-1}}{n} \underset{=}{\overset{\lambda}{\Omega}} \frac{(1-\lambda/n)^{n/3}}{\lambda^{n-1}} \\ &\approx \frac{(-1)^{n-1} n^{n-1}}{n} \underset{=}{\overset{\lambda}{\Omega}} \frac{\exp(-\lambda/3)}{\lambda^{n-1}} \\ &\approx \frac{n^{n-1}}{n! 3^{n-1}} \end{aligned}$$

using rescaling $\lambda \to \lambda/n$ to obtain the third equality and observing

$$\Big(1 + \frac{\alpha}{n}\Big)^n \approx e^{\alpha}. \tag{3.5}$$

Alternatively, we can state the following Omega representation which avoids the direct computation of F'.

Theorem 3.5. *We have*

$$\big(F \circ G^{\langle -1\rangle}\big)(\zeta) = \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda F(\lambda) G'(\lambda)}{\big(1 - \zeta/G(\lambda)\big) G(\lambda)}. \tag{3.6}$$

Again, if we set $F = I$ in Theorem 3.5 we immediately obtain the next corollary.

Corollary 3.6. *We have*

$$G^{\langle -1\rangle}(\zeta) = \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^2 G'(\lambda)}{\big(1 - \zeta/G(\lambda)\big) G(\lambda)}.$$

We now turn to a multivariable extension of Theorem 3.5. We take

$$\begin{gathered} |\zeta_{k\in[n]}| < |\lambda_{k\in[n]}|, \\ G_i(\boldsymbol{\zeta}) = \beta_i \zeta_i (1 + H_i(\boldsymbol{\zeta})), \end{gathered}$$

where $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ and $H_i(\boldsymbol{\zeta}) = \mathcal{O}(\boldsymbol{\zeta}^{\mathbf{k}})$ with $|\mathbf{k}| \geq 1$ such that we write

$$H_i(\boldsymbol{\zeta}) = \sum_{\mathbf{j}} \beta_{i\mathbf{j}} \boldsymbol{\zeta}^{\mathbf{j}}.$$

Theorem 3.7. *We have*

$$\big(F \circ \mathbf{G}^{\langle -1\rangle}\big)(\boldsymbol{\zeta}) = \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \frac{\boldsymbol{\lambda}^{\mathbf{e}} F(\boldsymbol{\lambda}) \det\big(D_i G_j(\boldsymbol{\lambda})\big)}{\prod_{k=1}^n \big(1 - \zeta_k/G_k(\boldsymbol{\lambda})\big) G_k(\boldsymbol{\lambda})}. \tag{3.7}$$

Of course Theorem 3.5 follows from Theorem 3.7, but we prefer to state them separately for readability because the proof of Theorem 3.7 given in Section 6 uses heavier notation and it is motivated by the proof of Theorem 3.5 in Section 5.

We now establish contact with [34, Theorem 1] by observing that if

$$F(\lambda) = \sum_{a \in \mathbb{Z}} \alpha_a \lambda^a$$

is as in Definition 2.1, then

$$\alpha_{-1} =: \operatorname{Res}(F) = \underset{=}{\overset{\lambda}{\Omega}} \lambda F(\lambda).$$

Therefore, using Theorem 3.1 we have

$$\big(F \circ G^{\langle -1 \rangle}\big)(\zeta) = \alpha_0 + \sum_{n \geq 1} \frac{1}{n} \operatorname{Res}\Big(\frac{F'}{G^n}\Big) \zeta^n$$

which is nothing more than [34, Theorem 1] in disguise; that is, once the correspondence between notations is established the equivalence is clear. A similar consideration applies to the multidimensional version in Theorem 3.7; that is, Theorem 3.7 is equivalent to [35, Theorem 7]. More precisely, Theorem 3.7 is equivalent to

$$\big(F \circ \mathbf{G}^{\langle -1 \rangle}\big)(\boldsymbol{\zeta}) = \sum_{\mathbf{n} \geq \mathbf{0}} \operatorname{Res}\Big(\frac{F \det\big(D_i G_j\big)}{\mathbf{G}^{\mathbf{n}+\mathbf{e}}}\Big) \boldsymbol{\zeta}^{\mathbf{n}},$$

where we now identify

$$\alpha_{-\mathbf{e}} =: \operatorname{Res}(F) = \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \boldsymbol{\lambda}^{\mathbf{e}} F(\boldsymbol{\lambda})$$

with

$$F(\boldsymbol{\lambda}) = \sum_{\mathbf{a} \in \mathbb{Z}^n} \alpha_{\mathbf{a}} \boldsymbol{\lambda}^{\mathbf{a}}.$$

See also [36].

If $G(\zeta) = \zeta / H(\zeta)$ (a functional dependence compatible with (3.2)) with $H(0) \neq 0$, then we can use Theorem 3.1 to obtain well-known equivalent forms of the L-B theorem. Indeed, we have

$$\langle \zeta^n \rangle \big(F \circ G^{\langle -1 \rangle}\big)(\zeta) = \frac{1}{n} \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda F'(\lambda)}{(G(\lambda))^n} = \frac{1}{n} \underset{=}{\overset{\lambda}{\Omega}} \frac{F'(\lambda)(H(\lambda))^n}{\lambda^{n-1}} = \frac{1}{n} \langle \zeta^{n-1} \rangle F'(\zeta)(H(\zeta))^n$$

in agreement with [29, Equation (2.1.1)]. If instead Theorem 3.5 is used, we obtain

$$\begin{aligned} \langle \zeta^n \rangle \big(F \circ G^{\langle -1 \rangle}\big)(\zeta) &= \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda F(\lambda) G'(\lambda)}{(G(\lambda))^{n+1}} \\ &= \underset{=}{\overset{\lambda}{\Omega}} \frac{F(\lambda)(H(\lambda) - \lambda H'(\lambda))(H(\lambda))^{n-1}}{\lambda^n} \\ &= \langle \zeta^n \rangle F(\zeta)(H(\zeta) - \zeta H'(\zeta))(H(\zeta))^{n-1} \end{aligned}$$

in agreement with [29, Equation (2.1.2)].

4. Proof of Theorem 3.1

We first introduce the key auxiliary result

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda G'(\lambda)}{(G(\lambda))^{m+1}} = \delta_{m,0}. \tag{4.1}$$

First Proof of (4.1)**.** Indeed, if $m=0$, then

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{(G(\lambda))^{m+1}}=\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{G(\lambda)}=1.$$

If $m>0$, then

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{(G(\lambda))^{m+1}}=-\frac{1}{m}\underset{=}{\overset{\lambda}{\Omega}}\lambda D(G(\lambda))^{-m}=-\frac{1}{m}\underset{=}{\overset{\lambda,\mu}{\Omega}}\frac{\lambda}{\mu(G(\lambda+\mu))^m}$$

with $|\mu|<|\lambda|$ since

$$\underset{=}{\overset{\lambda,\mu}{\Omega}}\frac{\lambda}{\mu}(\lambda+\mu)^a=0.$$

Indeed, if $a\ge 0$ the result is direct since there is no term independent of λ and μ. If $a=-n<0$ we write

$$\underset{=}{\overset{\lambda,\mu}{\Omega}}\frac{\lambda}{\mu(\lambda+\mu)^n}=\underset{=}{\overset{\lambda,\mu}{\Omega}}\frac{1}{\mu\lambda^{n-1}(1+\mu/\lambda)^n}=\sum_{m\ge 0}\binom{m+n-1}{n-1}\underset{=}{\overset{\lambda,\mu}{\Omega}}\frac{\mu^{m-1}}{\lambda^{m+n-1}}=0$$

since we cannot have $m-1=0=m+n-1$, because $n>0$.

Second proof of (4.1)**.** An even more compact approach is possible showing another instance where the power of the OC based approach is manifested. For $m>0$ and $\alpha\in\mathbb{C}\setminus\{0\}$, we have

$$\begin{aligned}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{\alpha(G(\lambda))^{m+1}}&=\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\alpha\lambda)}{(G(\alpha\lambda))^{m+1}}\\&=-\frac{1}{m}\underset{=}{\overset{\lambda}{\Omega}}D_\alpha(G(\alpha\lambda))^{-m}\\&=-\frac{D_\alpha}{m}\underset{=}{\overset{\lambda}{\Omega}}(G(\alpha\lambda))^{-m}\\&=-\frac{D_\alpha}{m}\underset{=}{\overset{\lambda}{\Omega}}(G(\lambda))^{-m}=0\end{aligned}$$

using invariance of the crude generating function under rescaling $\lambda\to\alpha\lambda$ to obtain the first equality and observing that $(G(\lambda))^{-m}$ does not depend on α to obtain the last one. Since $\alpha\neq 0$ we obtain the desired result.

Third proof of (4.1)**.** Another compact proof is available using basic OC properties. For $m>0$, we write

$$G'(\lambda)=\lim_{\delta\to 0}\frac{G((1+\delta)\lambda)-G(\lambda)}{\delta\lambda}$$

to obtain

$$\begin{aligned}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{(G(\lambda))^{m+1}}&=\lim_{\delta\to 0}\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)-G(\lambda)}{\delta(G(\lambda))^{m+1}}\\&=\lim_{\delta\to 0}\Big(\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G(\lambda))^{m+1}}-\underset{=}{\overset{\lambda}{\Omega}}\frac{G(\lambda)}{\delta(G(\lambda))^{m+1}}\Big)\\&=\lim_{\delta\to 0}\Big(\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G(\lambda))^{m+1}}-\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G((1+\delta)\lambda))^{m+1}}\Big)\\&=\lim_{\delta\to 0}\Big(\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G(\lambda))^{m+1}}-\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G(\lambda)+\delta\lambda G'(\lambda))^{m+1}}\Big)\\&=\lim_{\delta\to 0}\Big(\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G(\lambda))^{m+1}}-\underset{=}{\overset{\lambda}{\Omega}}\frac{G((1+\delta)\lambda)}{\delta(G(\lambda))^{m+1}(1+\delta\lambda G'(\lambda)/G(\lambda))^{m+1}}\Big),\end{aligned}$$

where the third equality follows from invariance under the rescaling $\lambda\to(1+\delta)\lambda$. Next, we can write

$$\Big(1+\frac{\delta\lambda G'(\lambda)}{G(\lambda)}\Big)^{-m-1}=1-(m+1)\frac{\delta\lambda G'(\lambda)}{G(\lambda)}+\mathcal{O}\big(\delta^2\big)$$

taking into account only terms linear in δ. Therefore, we have

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{(G(\lambda))^{m+1}} = (m+1)\lim_{\delta\to 0}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G((1+\delta)\lambda)G'(\lambda)}{(G(\lambda))^{m+2}} = (m+1)\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{(G(\lambda))^{m+1}}$$

which implies (4.1) (recall that $m > 0$).

First proof of Theorem 3.1. We prove Theorem 3.1 by showing that

$$\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)(\zeta) = \zeta = \big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta)$$

or, equivalently, that

$$H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big) = I = \big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k,$$

where I is the identity operator and H_k given by the RHS of (3.3) with $F_k(\zeta) = \zeta^k$ so that $F_k^{\langle -1\rangle}(\zeta) = \zeta^{1/k}$. We first show that H_k is a left inverse of $G \circ F_k^{\langle -1\rangle}$. We will show an equivalent statement

$$\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)'(\zeta) = 1, \quad \big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)(0) = 1.$$

Therefore, we conclude that

$$H_k = \big(G \circ F_k^{\langle -1\rangle}\big)^{\langle -1\rangle} = F_k \circ G^{\langle -1\rangle}$$

using the uniqueness of the inverse function. We first observe that

$$\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)(\zeta) = H_k\big(G\big(\zeta^{1/k}\big)\big) = -k\underset{=}{\overset{\lambda}{\Omega}}\lambda^k \ln\Big(1 - \frac{G\big(\zeta^{1/k}\big)}{G(\lambda)}\Big)$$

by recalling that $\alpha_0 = 0$ in this case. Indeed, we have

$$\begin{aligned}
\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)'(\zeta) &= H_k'\big(G\big(\zeta^{1/k}\big)\big)G'\big(\zeta^{1/k}\big)\big(\zeta^{1/k}\big)' \\
&= \underset{=}{\overset{\lambda}{\Omega}}\frac{k\lambda^k G'\big(\zeta^{1/k}\big)\big(\zeta^{1/k}\big)'}{G(\lambda) - G\big(\zeta^{1/k}\big)} \\
&= \zeta^{(1-k)/k}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^k G'\big(\zeta^{1/k}\big)}{G(\lambda) - G\big(\zeta^{1/k}\big)} \\
&= \zeta^{(1-k)/k}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^k G'\big(\zeta^{1/k}\big)}{\big(\lambda - \zeta^{1/k}\big)\big(G(\lambda) - G\big(\zeta^{1/k}\big)\big)/\big(\lambda - \zeta^{1/k}\big)} \\
&= \zeta^{(1-k)/k}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^{k-1} G'\big(\zeta^{1/k}\big)}{\big(1 - \zeta^{1/k}/\lambda\big)\big(G(\lambda) - G\big(\zeta^{1/k}\big)\big)/\big(\lambda - \zeta^{1/k}\big)} \\
&= \underbrace{\zeta^{(1-k)/k}\zeta^{(k-1)/k}}_{=1}\underbrace{\frac{G'\big(\zeta^{1/k}\big)}{\lim_{\lambda\to\zeta^{1/k}}\big(G(\lambda) - G\big(\zeta^{1/k}\big)\big)/\big(\lambda - \zeta^{1/k}\big)}}_{=1} \\
&= 1.
\end{aligned}$$

Now we show that H_k is a right inverse of $G \circ F_k^{\langle -1\rangle}$. We have

$$\big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(0) = \big(G \circ F_k^{\langle -1\rangle}\big)\Big(-k\underset{=}{\overset{\lambda}{\Omega}}\lambda^k \ln(1)\Big) = G(F_k^{\langle -1\rangle}(0)) = G(0) = 0$$

which implies

$$\big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta) = \mathcal{O}(\zeta). \tag{4.2}$$

Next, observe that

$$H_k = I \circ H_k = \underbrace{\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)}_{=I} \circ H_k = H_k \circ \big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big) \tag{4.3}$$

using that H_k is a left inverse of $G \circ F_k^{\langle -1 \rangle}$ which implies

$$\underbrace{\underset{=}{\overset{\lambda}{\Omega}} \lambda^k \ln\Big(1 - \frac{\zeta}{G(\lambda)}\Big)}_{= -H_k(\zeta)} = \underbrace{\underset{=}{\overset{\lambda}{\Omega}} \lambda^k \ln\Big(1 - \frac{\big((G \circ F_k^{\langle -1 \rangle}) \circ H_k\big)(\zeta)}{G(\lambda)}\Big)}_{= -\big(H_k \circ \big((G \circ F_k^{\langle -1 \rangle}) \circ H_k\big)\big)(\zeta)}. \tag{4.4}$$

By using (4.2) we can equate terms with equal powers of ζ in (4.4) to obtain the first non-null contribution

$$\zeta^k \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^k}{(G(\lambda))^k} = \sum_{n=1}^{k} \big(\big((G \circ F_k^{\langle -1 \rangle}) \circ H_k\big)(\zeta)\big)^n \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^k}{(G(\lambda))^n}.$$

Next, we use

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^k}{(G(\lambda))^n} = \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k-n}}{\beta_1^n (G(\lambda)/\lambda)^n} = \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k-n}}{\beta_1^n \big(1 + \beta_2 \lambda/\beta_1 + \beta_3 \lambda^2/\beta_1 + \dots\big)^n} = 0$$

if $n < k$ and observe that

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^k}{(G(\lambda))^k} = \frac{1}{\beta_1^k} \neq 0$$

if $n = k$. We arrive at

$$\begin{aligned} \zeta^k \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^k}{(G(\lambda))^k} &= \big(\big((G \circ F_k^{\langle -1 \rangle}) \circ H_k\big)(\zeta)\big)^k \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^k}{(G(\lambda))^k} \\ &\implies \big((G \circ F_k^{\langle -1 \rangle}) \circ H_k\big)(\zeta) = \zeta. \end{aligned}$$

Therefore, $H_k = F_k \circ G^{\langle -1 \rangle}$ using the uniqueness of the inverse function. Finally, the result now follows by observing that F in (3.1) is a linear combination of the F_k's. Since $F_k \circ G^{\langle -1 \rangle}$, the general case follows straightforwardly recalling that

$$\big(F \circ G^{\langle -1 \rangle}\big)(\zeta) = \alpha_0 + \sum_{k \geq 0} \alpha_k \big(F_k \circ G^{\langle -1 \rangle}\big)(\zeta)$$

and using (3.1) with $F = \sum_{k \geq 0} \alpha_k F_k$. □

Second proof of Theorem 3.1. We first observe that

$$\big(F \circ G^{\langle -1 \rangle}\big)(\zeta) = \sum_{n \geq 0} \gamma_n \zeta^n$$

so that

$$F(\zeta) = \sum_{n \geq 0} \gamma_n (G(\zeta))^n \implies F'(\zeta) = \sum_{n \geq 1} n \gamma_n (G(\zeta))^{n-1} G'(\zeta)$$

to obtain

$$\frac{F'(\zeta)}{(G(\zeta))^{m+1}} = \sum_{n \geq 0} n \gamma_n \frac{G'(\zeta)}{(G(\zeta))^{m-n+1}}.$$

Using the linearity of the Omega operator along with (4.1) we obtain

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda F'(\lambda)}{(G(\lambda))^{m+1}} = \sum_{n \geq 0} n \gamma_n \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda G'(\lambda)}{(G(\lambda))^{m-n+1}} = m \gamma_m$$

which is equivalent to the required result. □

5. Proof of Theorem 3.5

First proof of Theorem 3.5. We prove Theorem 3.5 using the same strategy of the previous section. More precisely, we show that

$$\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)(\zeta) = \zeta = \big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta),$$

where H_k now stands for the right-hand side of (3.6) and $F_k(\zeta) = \zeta^k$ so that $F_k^{\langle -1\rangle}(\zeta) = \zeta^{1/k}$. We first show that

$$\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)(\zeta) = \zeta.$$

Indeed, we have

$$\begin{aligned}
\big(H_k \circ \big(G \circ F_k^{\langle -1\rangle}\big)\big)(\zeta) &= H_k\big(G\big(\zeta^{1/k}\big)\big)\\
&= \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda F_k(\lambda) G'(\lambda)}{\big(1 - G\big(\zeta^{1/k}\big)/G(\lambda)\big) G(\lambda)}\\
&= \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{\big(1 - G\big(\zeta^{1/k}\big)/G(\lambda)\big) G(\lambda)}\\
&= \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{\big(\lambda - \zeta^{1/k}\big)\big(G(\lambda) - G\big(\zeta^{1/k}\big)\big)/\big(\lambda - \zeta^{1/k}\big)}\\
&= \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k} G'(\lambda)}{\big(1 - \zeta^{1/k}/\lambda\big)\big(G(\lambda) - G\big(\zeta^{1/k}\big)\big)/\big(\lambda - \zeta^{1/k}\big)}\\
&= \lim_{\lambda \to \zeta^{1/k}} \frac{\lambda^{k} G'(\lambda)}{\big(G(\lambda) - G\big(\zeta^{1/k}\big)\big)/\big(\lambda - \zeta^{1/k}\big)}\\
&= \zeta.
\end{aligned}$$

Next, we observe that

$$\big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(0) = G\big(F_k^{\langle -1\rangle}(0)\big) = G(0) = 0$$

which implies

$$\big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta) = \mathcal{O}(\zeta)$$

and use (4.3) to obtain

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{\big(1 - \zeta/G(\lambda)\big) G(\lambda)} = \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{\Big(1 - \big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta)/G(\lambda)\Big) G(\lambda)}.$$

Expanding in ζ we obtain the first non-null contribution

$$\zeta^k \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{(G(\lambda))^{k+1}} = \Big(\big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta)\Big)^k \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{(G(\lambda))^{k+1}}$$

since

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^{k+1} G'(\lambda)}{(G(\lambda))^{k+1}} = \frac{1}{\beta_1^k} \neq 0.$$

It follows that

$$\big(\big(G \circ F_k^{\langle -1\rangle}\big) \circ H_k\big)(\zeta) = \zeta$$

and the proof is complete, again, after recalling the uniqueness of the inverse and observing that F in (3.1) is a linear combination of the F_k's. □

Second proof of Theorem 3.5. We first observe that

$$\big(F\circ G^{\langle -1\rangle}\big)(\zeta)=\sum_{n\geq 0}\gamma_n\zeta^n$$

so that

$$F(\zeta)=\sum_{n\geq 0}\gamma_n(G(\zeta))^n$$

to obtain

$$\frac{F(\zeta)G'(\zeta)}{(G(\zeta))^m}=\sum_{n\geq 0}\frac{\gamma_n G'(\zeta)}{(G(\zeta))^{m-n+1}}.$$

Using the linearity of the Omega operator and (4.1) we obtain

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda F(\lambda)G'(\lambda)}{(G(\lambda))^m}=\sum_{n\geq 0}\gamma_n\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda G'(\lambda)}{(G(\lambda))^{m-n+1}}=\gamma_m.$$

□

6. Proof of Theorem 3.7

The proof follows along the lines of the proof of Theorem 3.5 in Section 5, but somewhat more involved. Again, we first introduce the key auxiliary result that follows which is the multidimensional counterpart of (4.1)

$$\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}}=\delta_{\mathbf{m},\mathbf{0}}. \tag{6.1}$$

First Proof of (6.1)**.** The case $m=0$ is handled as follows

$$\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{e}}}=\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\det\big(\beta_i\delta_{ij}(1+H_i(\boldsymbol{\lambda}))+\beta_j\lambda_jD_iH_j(\boldsymbol{\lambda})\big)}{\prod_{k=1}^n\beta_k(1+H_k(\boldsymbol{\lambda}))}=\frac{\det(\beta_i\delta_{ij})}{\prod_{k=1}^n\beta_k}=1$$

using

$$D_iG_j(\boldsymbol{\lambda})=D_i\big(\beta_j\lambda_j(1+H_j(\boldsymbol{\lambda}))\big)=\beta_i\delta_{ij}(1+H_i(\boldsymbol{\lambda}))+\beta_j\lambda_jD_iH_j(\boldsymbol{\lambda})$$

and observing that in the numerator and denominator only positive powers of λ_i contribute due to the functional dependence of $H_i(\boldsymbol{\lambda})$.

We next consider the case $m>0$ since if some $m_i=0$ then we have a trivial cancelation of the omega variable λ_i resulting in an expression free of the Omega variable λ_i. We have

$$\begin{aligned}\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}}&=\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\sum_{\pi\in S_n}\operatorname{sign}(\pi)\prod_{i=1}^nD_iG_{\pi(i)}(\boldsymbol{\lambda})}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}}\\&=\frac{(-1)^n}{\prod_{i=1}^nm_i}\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\boldsymbol{\lambda}^{\mathbf{e}}\sum_{\pi\in S_n}\operatorname{sign}(\pi)\prod_{i=1}^nD_iG_{\pi(i)}^{-m_{\pi(i)}}(\boldsymbol{\lambda})\\&=\frac{(-1)^n}{\prod_{i=1}^nm_i}\underset{=}{\overset{\boldsymbol{\lambda},\mu}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}}{\mu^{\mathbf{e}}}\sum_{\pi\in S_n}\operatorname{sign}(\pi)\prod_{i=1}^nG_{\pi(i)}^{-m_{\pi(i)}}(\boldsymbol{\lambda}+\mu_i\mathbf{e}_i)=0\end{aligned}$$

if $\mathbf{m}>\mathbf{0}$. Note that each term of

$$\sum_{\pi\in S_n}\operatorname{sign}(\pi)\underset{=}{\overset{\boldsymbol{\lambda},\mu}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}}{\mu^{\mathbf{e}}}\prod_{i=1}^nG_{\pi(i)}^{-m_{\pi(i)}}(\boldsymbol{\lambda}+\mu_i\mathbf{e}_i)$$

is of the type

$$\begin{aligned}\sum_{\pi\in S_n}\operatorname{sign}(\pi)\underset{=}{\overset{\boldsymbol{\lambda},\mu}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}}{\mu^{\mathbf{e}}}\prod_{i=1}^n(\boldsymbol{\lambda}+\mu_i\mathbf{e}_i)^{\mathbf{a}_{\pi(i)}}&=\sum_{\pi\in S_n}\operatorname{sign}(\pi)\underset{=}{\overset{\boldsymbol{\lambda},\mu}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}}{\mu^{\mathbf{e}}}\prod_{i=1}^n\lambda_i^{\sum_j a_{ij}}\Big(1+\frac{\mu_i}{\lambda_i}\Big)^{a_{i\pi(i)}}\\&=\sum_{\pi\in S_n}\operatorname{sign}(\pi)\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\boldsymbol{\lambda}^{\mathbf{e}}\prod_{i=1}^n\lambda_i^{\sum_j a_{ij}}\frac{a_{i\pi(i)}}{\lambda_i}\end{aligned}$$

$$= \sum_{\pi\in S_n} \operatorname{sign}(\pi) a_{i\pi(i)} \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \prod_{i=1}^n \lambda_i^{\sum_j a_{ij}} = 0$$

using

$$\begin{aligned}(\boldsymbol{\lambda}+\mu_i\mathbf{e}_i)^{\mathbf{a}_{\pi(i)}} &= \lambda_1^{a_{1\pi(i)}}\dots(\lambda_i+\mu_i)^{a_{i\pi(i)}}\dots\lambda_n^{a_{n\pi(i)}}\\ &= \lambda_1^{a_{1\pi(i)}}\dots\lambda_i^{a_{i\pi(i)}}\Big(1+\frac{\mu_i}{\lambda_i}\Big)^{a_{i\pi(i)}}\dots\lambda_n^{a_{n\pi(i)}}\end{aligned}$$

to obtain

$$\begin{aligned}\prod_{i=1}^n(\boldsymbol{\lambda}+\mu_i\mathbf{e}_i)^{\mathbf{a}_{\pi(i)}} &= \lambda_1^{\sum_j a_{1\pi(j)}}\dots\lambda_i^{\sum_j a_{i\pi(j)}}\prod_{i=1}^n\Big(1+\frac{\mu_i}{\lambda_i}\Big)^{a_{i\pi(i)}}\dots\lambda_n^{\sum_j a_{n\pi(j)}}\\ &= \prod_{i=1}^n \lambda_i^{\sum_j a_{ij}}\Big(1+\frac{\mu_i}{\lambda_i}\Big)^{a_{i\pi(i)}}\end{aligned}$$

and

$$\underset{=}{\overset{\mu_i}{\Omega}}\frac{1}{\mu_i}\Big(1+\frac{\mu_i}{\lambda_i}\Big)^{a_{i\pi(i)}} = \frac{a_{i\pi(i)}}{\lambda_i},$$

where $\mathbf{a}_{\pi(i)}$ is the $\pi(i)\in[n]$ column of

$$\mathbf{A} := (\mathbf{a}_1,\dots,\mathbf{a}_n).$$

Finally, since the $\boldsymbol{\lambda}$ variable forces a linear dependence among the rows of $\mathbf{A}$, we obtain a zero determinant.

Second Proof of (6.1)**.** Another compact proof is available using basic OC properties following the proof of (4.1). For $m_1>0$, we write

$$D_kG_1(\boldsymbol{\lambda}) = \lim_{\delta_k\to 0}\frac{G_1(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k)-G_1(\boldsymbol{\lambda})}{\delta_k\lambda_k}$$

and by Laplace expanding $\det(D_iG_j(\boldsymbol{\lambda}))$ along the first column we get

$$\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}} = \sum_{k=1}^n(-1)^{k+1}\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\lambda_kD_kG_1(\boldsymbol{\lambda})F_k(\boldsymbol{\lambda})}{(G_1(\boldsymbol{\lambda}))^{m_1+1}},$$

where

$$F_k(\boldsymbol{\lambda}) := \frac{\boldsymbol{\lambda}^{\mathbf{e}-\mathbf{e}_k}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}-(m_1+1)\mathbf{e}_1}}\Big|_{(i,j)\neq(k,1)}.$$

In this way, we obtain

$$\begin{aligned}&\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\lambda_kD_kG_1(\boldsymbol{\lambda})F_k(\boldsymbol{\lambda})}{(G_1(\boldsymbol{\lambda}))^{m_1+1}}\\ &= \lim_{\delta_k\to0}\Big(\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{G_1(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k)F_k(\boldsymbol{\lambda})}{\delta_k(G_1(\boldsymbol{\lambda}))^{m_1+1}} - \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{G_1(\boldsymbol{\lambda})F_k(\boldsymbol{\lambda})}{\delta_k(G_1(\boldsymbol{\lambda}))^{m_1+1}}\Big)\\ &= \lim_{\delta_k\to0}\Big(\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{G_1(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k)F_k(\boldsymbol{\lambda})}{\delta_k(G_1(\boldsymbol{\lambda}))^{m_1+1}} - \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{G_1(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k)F_k(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k)}{\delta_k(G_1(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k))^{m_1+1}}\Big).\end{aligned}$$

Next, we can write

$$X(\boldsymbol{\lambda}+\delta_k\lambda_k\mathbf{e}_k) = X(\boldsymbol{\lambda})+\delta_k\lambda_kD_kX(\boldsymbol{\lambda})+\mathcal{O}\big(\delta_k^2\big)$$

with $X=F_k,G_1$ and

$$\Big(1+\frac{\delta_k\lambda_kD_kG_1(\boldsymbol{\lambda})}{G_1(\boldsymbol{\lambda})}\Big)^{-m_1-1} = 1-(m_1+1)\frac{\delta_k\lambda_kD_kG_1(\boldsymbol{\lambda})}{G_1(\boldsymbol{\lambda})}+\mathcal{O}\big(\delta_k^2\big)$$

taking into account only terms linear in δ. Therefore, we have

$$\sum_{k=1}^n(-1)^{k+1}D_k\underset{=}{\overset{\lambda_{i\in[n]\setminus\{k\}}}{\Omega}}\Big(\frac{\boldsymbol{\lambda}^{\mathbf{e}-\mathbf{e}_k}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}-(m_1+1)\mathbf{e}_1}}\Big)\Big|_{(i,j)\neq(k,1)} = 0.$$

Indeed, we use induction with the base case given by (4.1) and the induction hypothesis given by

$$\underset{=}{\overset{\lambda_{i\in[n]\setminus\{1\}}}{\Omega}}\Big(\frac{\boldsymbol{\lambda}^{\mathbf{e}-\mathbf{e}_1}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}-(m_1+1)\mathbf{e}_1}}\Big)\Big|_{(i,j)\neq(1,1)}=\delta_{\mathbf{m}-m_1\mathbf{e}_1,\mathbf{0}}$$

along with a simple observation

$$\underset{=}{\overset{\lambda_{i\in[n]\setminus\{k\}}}{\Omega}}\Big(\frac{\boldsymbol{\lambda}^{\mathbf{e}-\mathbf{e}_k}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}-(m_1+1)\mathbf{e}_1}}\Big)\Big|_{(i,j)\neq(k,1)}=0$$

for $k\neq 1$ because we will always have a prefactor λ_1 in $\boldsymbol{\lambda}^{\mathbf{e}-\mathbf{e}_k}$ and no term containing λ_1^{-1}. Finally, we obtain

$$\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}}=(m_1+1)\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}}$$

which implies (6.1) (recall that $m_1>0$).

Proof of Theorem 3.7. We can now prove Theorem 3.7. We have

$$\big(F\circ\mathbf{G}^{\langle-1\rangle}\big)(\boldsymbol{\zeta})=\sum_{\mathbf{n}\in\mathbb{N}_0^n}\gamma_{\mathbf{n}}\boldsymbol{\zeta}^{\mathbf{n}}$$

so that

$$F(\boldsymbol{\zeta})=\sum_{\mathbf{n}\in\mathbb{N}_0^n}\gamma_{\mathbf{n}}(\mathbf{G}(\boldsymbol{\zeta}))^{\mathbf{n}}$$

to obtain

$$\frac{F(\boldsymbol{\zeta})\det\big(D_iG_j(\boldsymbol{\zeta})\big)}{(\mathbf{G}(\boldsymbol{\zeta}))^{\mathbf{m}+\mathbf{e}}}=\sum_{\mathbf{n}\in\mathbb{N}_0^n}\frac{\gamma_{\mathbf{n}}\det\big(D_iG_j(\boldsymbol{\zeta})\big)}{(\mathbf{G}(\boldsymbol{\zeta}))^{\mathbf{m}-\mathbf{n}+\mathbf{e}}}.$$

Using the linearity of the Omega operator along with (6.1) we obtain

$$\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}F(\boldsymbol{\lambda})\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}+\mathbf{e}}}=\sum_{\mathbf{n}\in\mathbb{N}_0^n}\gamma_{\mathbf{n}}\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}}\frac{\boldsymbol{\lambda}^{\mathbf{e}}\det\big(D_iG_j(\boldsymbol{\lambda})\big)}{(\mathbf{G}(\boldsymbol{\lambda}))^{\mathbf{m}-\mathbf{n}+\mathbf{e}}}=\gamma_{\mathbf{m}}$$

which implies the desired result. □

Remark 6.1. It is instructive to compare our proof with the one given in [35, Theorem 7]. We note that our proof is much simpler and direct. In particular, the handling of the case $k_i=0$ is simpler in the context of our approach as one may compare the proof of [35, Theorem 7] with the proof above.

7. A new generalized Lambert W function

In this section we introduce a new generalized Lambert W function, and we show that some of the main results of [56, 58, 12] follows from our new representation. This section is motivated by the following question left open in [56] which is quoted verbatim below:

Q1: Could one carry out some general analysis for $W\Big(\begin{smallmatrix}\tau_1\dots\tau_r\\ \sigma_1\dots\sigma_s\end{smallmatrix};\zeta\Big)$?

An even more recent reference [57] refers to Q1 as "elusive".

7.1. **Main theorem.** The aforementioned function $W\Big(\begin{smallmatrix}\tau_1\dots\tau_r\\ \sigma_1\dots\sigma_s\end{smallmatrix};\zeta\Big)$ solves the equation

$$e^{\xi}\frac{\prod_{p=1}^{r}(\xi-\tau_p)}{\prod_{q=1}^{s}(\xi-\sigma_q)}=\zeta \tag{7.1}$$

in the variable ξ and we have

$$\xi=W\Big(\begin{matrix}\tau_1\dots\tau_r\\ \sigma_1\dots\sigma_s\end{matrix};\zeta\Big) \tag{7.2}$$

using the notation of [57]. Equation (7.1) arise, e.g., in the context of Bose-Femi mixtures [56] and delay differential equations [59]. More generally, we will consider

$$\exp\Big(\sum_{i=2}^{d}\alpha_i(\xi-\tau_1)^i\Big)(\exp(\beta(\xi-\tau_1))+\rho)e^{\gamma\xi}\frac{\prod_{p=1}^{r}(\xi-\tau_p)}{\prod_{q=1}^{s}(\xi-\sigma_q)}=\zeta$$

and we are interested in determining ξ that solves the equation above in the variable ζ. From now on regarding such ξ we use the notation

$$\xi=W_\rho\Big({\tau\atop\sigma};\boldsymbol{\alpha},\beta,\gamma,\zeta\Big),$$

where $\tau=(\tau_p)_{p\in[r]}$ and $\sigma=(\sigma_q)_{q\in[s]}$. If

$$G(\xi)=\exp\Big(\sum_{i=2}^{d}\alpha_i\xi^i\Big)(\exp(\beta\xi)+\rho)\exp(\gamma\xi+\delta)\frac{\xi\prod_{p=2}^{r}(\xi+T_p)}{\prod_{q=1}^{s}(\xi+S_q)},$$

where $\delta:=\gamma\tau_1$, $T_p:=\tau_1-\tau_p$, and $S_q:=\tau_1-\sigma_q$. We obtain

$$W_\rho\Big({\tau\atop\sigma};\boldsymbol{\alpha},\beta,\gamma,\zeta\Big)=\tau_1+G^{\langle-1\rangle}(\zeta) \tag{7.3}$$

with the convention

$$W_\rho\Big({-\atop\sigma};\boldsymbol{\alpha},\beta,\gamma,\zeta\Big)$$

in (7.3) setting

$$\prod_{p=2}^{r}(\xi+T_p)\to 1.$$

Likewise, we use

$$W_\rho\Big({\tau\atop -};\boldsymbol{\alpha},\beta,\gamma,\zeta\Big)$$

meaning (7.3) upon setting

$$\prod_{q=1}^{s}(\xi+S_q)\to 1.$$

The explicit determination of $G^{\langle-1\rangle}(\zeta)$ in (7.3) is the content of the next theorem. We highlight that it is possible to obtain an explicit expression for $W_\rho\big({\tau\atop\sigma};\boldsymbol{\alpha},\beta,\gamma,\zeta\big)$ in a way that several theorems emerge as special cases including

$$W_0\Big({-\atop -};\mathbf{0},0,1,\zeta\Big)=W(\zeta)$$

as in Example 3.3,

$$W_0\Big({\tau\atop\sigma};\mathbf{0},0,1,\zeta\Big)=W\Big({\tau_1\dots\tau_r\atop\sigma_1\dots\sigma_s};\zeta\Big)$$

as in (7.2), and

$$W_\rho\Big({-\atop -};\mathbf{0},0,1,\zeta\Big)=W_\rho(\zeta)$$

for the ρ-Lambert W function.

To introduce our next result we recall the definition of the Stirling numbers of the second kind (see, e.g., [13, Chapter 5, Sections 1 and 2]). The Stirling numbers of the second kind $\left\{ n\atop m\right\}$ count the number of ways a set of n elements can be partitioned into m nonempty disjoint subsets. They can be described via an exponential generating function

$$\sum_{n\ge m}\left\{ n\atop m\right\}\frac{\xi^n}{n!}=\frac{(e^\xi-1)^m}{m!} \tag{7.4}$$

and adopting the convention

$$\left\{ n\atop m\right\}=0$$

if $n < m$. We will also need later on the rising factorial

$$\xi^{\overline{n}} := \xi(\xi+1)\dots(\xi+n-1) \tag{7.5}$$

with the convention $\xi^{\overline{0}} = 1$ and

$$\binom{\ell}{\boldsymbol{\ell}} := \frac{\ell!}{\prod_{i=1}^{d} \ell_i!} \tag{7.6}$$

stands for the multinomial coefficient.

We are now ready to state the main result of this section giving our answer to Q1. In this section from now on and the next section, we assume $i \in [2, d]$, $p \in [2, r]$, and $q \in [s]$ unless otherwise stated.

Theorem 7.1. *We have the Omega representations*

$$W_\rho\binom{\tau}{\sigma}; \boldsymbol{\alpha}, \beta, \gamma, \zeta\Big) = \tau_1 - \underset{=}{\overset{\lambda}{\Omega}} \lambda \ln\Big(1 - \frac{\zeta}{G(\lambda)}\Big), \tag{7.7}$$

$$W_\rho\Big(\begin{matrix}\tau\\ \sigma\end{matrix}; \boldsymbol{\alpha}, \beta, \gamma, \zeta\Big) = \underset{=}{\overset{\lambda}{\Omega}} \frac{\lambda^2 G'(\lambda)}{\big(1 - \zeta/G(\lambda)\big)G(\lambda)}, \tag{7.8}$$

where

$$G^{-1}(\lambda) = \frac{\prod_q (\lambda + S_q)}{\lambda \exp\big(\sum_i \alpha_i \lambda^i + \gamma\lambda + \delta\big)(e^{\beta\lambda} + \rho)\prod_p(\lambda + T_p)}.$$

Equivalently, if $\rho \neq -1$, we have the Omega free Taylor series expansion

$$\begin{aligned} W_\rho\Big(\begin{matrix}\tau\\ \sigma\end{matrix}; \boldsymbol{\alpha}, \beta, \gamma, \zeta\Big) = \tau_1 + \sum \frac{\big(\zeta e^{-\delta}\big)^n}{n} \frac{(-n)^m}{m!} \binom{m}{\ell} \binom{\ell}{\boldsymbol{\ell}} \binom{k+n-1}{n-1} \\ \times (-1)^{k + \sum_p m_p} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \frac{k!}{j!} \frac{\boldsymbol{\alpha}^{\boldsymbol{\ell}} \beta^j \gamma^{m-\ell}}{(1+\rho)^{k+n}} a^{(n)}_{\mathbf{n}} b^{(n)}_{\mathbf{m}} \frac{\mathbf{S}^{\mathbf{n}}}{\big(\prod_p T_p^n\big)\mathbf{T}^{\mathbf{m}}}, \end{aligned} \tag{7.9}$$

where

$$j := n - m + \ell - \sum_i i\ell_i - sn - 1 - \sum_p m_p + \sum_q n_q,$$

$$\sum := \sum_{n \ge 1} \sum_{m \ge 0} \sum_{\ell=0}^{m} \sum_{\boldsymbol{\ell}} \sum_{k=0}^{j} \sum_{\mathbf{m}, \mathbf{n} \ge \mathbf{0}}$$

with

$$\sum_{\boldsymbol{\ell}} := \sum_{\ell_2 + \dots + \ell_d = \ell}, \qquad a^{(n)}_{\mathbf{n}} := \prod_q \binom{n}{n_q}$$

with $\mathbf{n} = (n_q)$ such that $\mathbf{S}^{\mathbf{n}} = \prod_q S_q^{n_q}$, and

$$b^{(n)}_{\mathbf{m}} := \prod_p \binom{m_p + n - 1}{n - 1}$$

with $\mathbf{m} = (m_p)$ such that $\mathbf{T}^{\mathbf{m}} = \prod_p T_p^{m_p}$.

Proof. We observe that (7.7) and (7.8) follow directly from Theorems 3.1 and 3.5 applied to (7.3), respectively.

We now turn to the proof of (7.9). Starting with (7.7) we have

$$\begin{aligned} W_\rho\Big(\begin{matrix}\tau\\ \sigma\end{matrix}; \boldsymbol{\alpha}, \beta, \gamma, \zeta\Big) &= -\underset{=}{\overset{\lambda}{\Omega}} \lambda \ln\Big(1 - \frac{\zeta \prod_q (\lambda + S_q)}{\lambda \exp\Big(\sum_i \alpha_i \lambda^i + \gamma\lambda + \delta\Big)(e^{\beta\lambda} + \rho)\prod_p(\lambda + T_p)}\Big) \\ &= \sum_{n \ge 1} \frac{\zeta^n}{n} \exp(-n\delta) \underset{=}{\overset{\lambda}{\Omega}} \frac{\exp\big(-n\big(\sum_i \alpha_i \lambda^i + \gamma\lambda\big)\big)}{\lambda^{n-1}(e^{\beta\lambda} + \rho)^n} \frac{\prod_q (\lambda + S_q)^n}{\prod_p (\lambda + T_p)^n} \end{aligned}$$

using

$$(\lambda+S_q)^n=\sum_{n_q=0}^{n}\binom{n}{n_q}\lambda^{n-n_q}S_q^{n_q},$$

we write

$$\frac{1}{(\lambda+T_p)^n}=\frac{1}{T_p^n(1+\lambda/T_p)^n}$$

such that

$$\frac{1}{(1+\lambda/T_p)^n}=\sum_{m_p\ge 0}\binom{m_p+n-1}{n-1}\Big(-\frac{\lambda}{T_p}\Big)^{m_p} \tag{7.10}$$

to obtain

$$\begin{aligned}
&\underset{=}{\overset{\lambda}{\Omega}}\frac{\exp\big(-n\big(\sum_i\alpha_i\lambda^i+\gamma\lambda\big)\big)\lambda^{sn-\sum_q n_q}}{\lambda^{n-1-\sum_p m_p}(e^{\beta\lambda}+\rho)^n}\\
&=\sum_{m\ge 0}\frac{(-n)^m}{m!}\underset{=}{\overset{\lambda}{\Omega}}\frac{\big(\sum_i\alpha_i\lambda^i+\gamma\lambda\big)^m}{\lambda^{n-sn-1-\sum_p m_p+\sum_q n_q}(e^{\beta\lambda}+\rho)^n}\\
&=\sum_{m\ge 0}\frac{(-n)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}\gamma^{m-\ell}\underset{=}{\overset{\lambda}{\Omega}}\frac{\Big(\sum_i\alpha_i\lambda^i\Big)^{\ell}\lambda^{m-\ell}}{\lambda^{n-sn-1-\sum_p m_p+\sum_q n_q}(e^{\beta\lambda}+\rho)^n}\\
&=\sum_{m\ge 0}\frac{(-n)^m}{m!}\sum_{\ell=0}^{m}\sum_{\boldsymbol{\ell}}\binom{m}{\ell}\binom{\ell}{\boldsymbol{\ell}}\boldsymbol{\alpha}^{\boldsymbol{\ell}}\gamma^{m-\ell}\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^j((1+\rho)+(e^{\beta\lambda}-1))^n}\\
&=\sum_{m\ge 0}\frac{(-n)^m}{m!}\sum_{\ell=0}^{m}\sum_{\boldsymbol{\ell}}\binom{m}{\ell}\binom{\ell}{\boldsymbol{\ell}}\frac{\boldsymbol{\alpha}^{\boldsymbol{\ell}}\gamma^{m-\ell}}{(1+\rho)^n}\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^j(1+(e^{\beta\lambda}-1)/(1+\rho))^n}\\
&=\sum_{m\ge 0}\frac{(-n)^m}{m!}\sum_{\ell=0}^{m}\sum_{\boldsymbol{\ell}}\binom{m}{\ell}\binom{\ell}{\boldsymbol{\ell}}\frac{\boldsymbol{\alpha}^{\boldsymbol{\ell}}\beta^j\gamma^{m-\ell}}{(1+\rho)^n}\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{(\beta\lambda)^j(1+(e^{\beta\lambda}-1)/(1+\rho))^n}
\end{aligned}$$

using

$$\frac{1}{(1+(e^{\beta\lambda}-1)/(1+\rho))^n}=\sum_{k\ge 0}\binom{k+n-1}{n-1}\Big(-\frac{e^{\beta\lambda}-1}{1+\rho}\Big)^k \tag{7.11}$$

and $e^{\beta\lambda}-1=\mathcal{O}(\lambda)$ so that

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{(e^{\beta\lambda}-1)^k}{\lambda^j}=0$$

for $k>j$. Finally, the result follows observing that

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{(e^{\beta\lambda}-1)^k}{(\beta\lambda)^j}=\Big\{{j \atop k}\Big\}\frac{k!}{j!}$$

so that the Omega variable λ is now eliminated. We additionally observe that the expansions used in the proof in (7.10) and (7.11) must hold,

$$\Big|\frac{\lambda}{T_p}\Big|,\Big|\frac{e^{\beta\lambda}-1}{1+\rho}\Big|<1$$

and recalling that e^{α} is entire. In the next section we will show that such conditions always hold true by a suitable rescaling. □

7.2. **Connection of Theorem 7.1 with previous work: Taylor series and integral representations.** We now show that several known results emerge as special cases of Theorem 7.1. In the next proposition, we will need the Rodrigues formula for the Laguerre polynomials

$$L'(\alpha)=\frac{e^{\alpha}}{n!}D_{\alpha}^{n}\Big(\frac{\alpha^n}{e^{\alpha}}\Big). \tag{7.12}$$

Proposition 7.2 ([56, Theorem 1])**.** *The solution of*

$$\zeta = e^{\xi}\frac{(\xi-\tau)}{(\xi-\sigma)}$$

is given by the Taylor series

$$W_0\Big(\begin{matrix}\tau\\ \sigma\end{matrix};\mathbf{0},0,1,\zeta\Big) = \tau - S\sum_{n\ge 1}\frac{(\zeta e^{-\tau})^n}{n}L'(nS),$$

where $S=\tau-\sigma$*.*

Proof. In this case $\boldsymbol{\alpha}=\mathbf{0}$, $\beta=0=\rho$, and $\gamma,r,s=1$. Since $\boldsymbol{\alpha}^{\boldsymbol{\ell}}=\mathbf{0}^{\boldsymbol{\ell}}$ so that $\boldsymbol{\ell}=\mathbf{0}$ which implies $\ell=0$ and $\beta^j=0^j$ so that $j=0$. These observations imply

$$\begin{aligned} j &= n-m+\ell-\sum_i i\ell_i - sn - 1 - \sum_p m_p + \sum_q n_q\\ &\implies 0 = n-m+0-0-n-1-0+n_1\\ &\implies m = n_1-1 \end{aligned}$$

and we obtain

$$\begin{aligned} W_0\Big(\begin{matrix}\tau\\ \sigma\end{matrix};\mathbf{0},0,1,\zeta\Big) &= \tau + \sum_{n\ge 1}\frac{(\zeta e^{-\tau})^n}{n}\sum_{n_1=1}^{n}\binom{n}{n_1}\frac{(-n)^{n_1-1}}{(n_1-1)!}S^{n_1}\\ &= \tau - \sum_{n\ge 1}\frac{(\zeta e^{-\tau})^n}{n^2}\sum_{n_1=1}^{n}\binom{n}{n_1}\frac{(-n)^{n_1}}{(n_1-1)!}S^{n_1}. \end{aligned}$$

In the expression above, we put m in place of n_1. The result now follows by showing that

$$L'(nS) = -\sum_{k=1}^{n}\binom{n}{k}\frac{(-nS)^{k-1}}{(k-1)!}.$$

Indeed, using Rodrigues formula for the Laguerre polynomials in (7.12) and recalling the Omega representation of the derivative in the proof of Lemma 2.2 we have

$$L'(\alpha) = \frac{e^{\alpha}}{n!}D_{\alpha}^{n}\Big(\frac{\alpha^n}{e^{\alpha}}\Big) = e^{\alpha}\underset{=}{\overset{\lambda}{\Omega}}\frac{(\alpha+\lambda)^n}{\lambda^n e^{\alpha+\lambda}} = \underset{=}{\overset{\lambda}{\Omega}}\frac{(\alpha+\lambda)^n}{\lambda^n e^{\lambda}}$$

which implies

$$\begin{aligned} L'(\alpha) &= n\underset{=}{\overset{\lambda}{\Omega}}\frac{(\alpha+\lambda)^{n-1}}{\lambda^n e^{\lambda}}\\ &= \sum_{m=0}^{n}\sum_{\ell=0}^{n-1}\underset{=}{\overset{\lambda}{\Omega}}\frac{(-\lambda)^m}{m!}\frac{n}{\lambda^n}\binom{n-1}{\ell}\alpha^{n-1-\ell}\lambda^{\ell}\\ &= \sum_{m=0}^{n}\sum_{\ell=0}^{n-1}\frac{(-1)^m}{m!}n\binom{n-1}{\ell}\alpha^{n-1-\ell}\underbrace{\underset{=}{\overset{\lambda}{\Omega}}\lambda^{\ell+m-n}}_{=\delta_{\ell,n-m}}\\ &= \sum_{m=1}^{n}\frac{(-1)^m}{m!}n\binom{n-1}{n-m}\alpha^{m-1}\\ &= -\sum_{m=1}^{n}\binom{n}{m}\frac{(-\alpha)^{m-1}}{(m-1)!} \end{aligned}$$

by noting that when $m=0$ we have $\delta_{\ell,n-m}=\delta_{\ell,n}=0$ since $\ell\in[n-1]_0$. □

If we compare the proof given above with the one given in [56, Theorem 1] we see that ours is much more direct in the sense of avoiding the search for a pattern to be confirmed later by the principle of induction. Indeed, see the proof of [56, Theorem 1].

Proposition 7.3 ([56, Theorem 6]). *The solution of $\zeta = \xi e^{\xi} + \rho\xi$ is given by*

$$W_\rho\Big({-\atop -};\mathbf{0},1,0,\zeta\Big) = \frac{\zeta}{1+\rho} + \sum_{n\ge 2} M_{n-1}^{(n)}\Big(\frac{1}{1+\rho}\Big)\frac{\zeta^n}{n!(1+\rho)^n},$$

where

$$M_n^{(m)}(\zeta) = \sum_{\ell=1}^{n} m^{\overline{\ell}}\Big\{{n \atop \ell}\Big\}(-\zeta)^{\ell}.$$

Proof. In this case $\boldsymbol{\alpha} = \mathbf{0}$, $\beta = 1$, and $\gamma = 0 = \tau_1$ (and hence $\delta = 0$). Since $\boldsymbol{\alpha}^{\boldsymbol{\ell}} = \mathbf{0}^{\boldsymbol{\ell}}$ so that $\boldsymbol{\ell} = \mathbf{0}$ which implies $\ell = 0$ and $\gamma^{m-\ell} = 0^{m-\ell}$ so that $m = \ell$. These observations imply

$$j = n - m + \ell - \sum_i i\ell_i - sn - 1 - \sum_p m_p + \sum_q n_q \tag{7.13}$$

$$\implies j = n - m + m - 0 - 0n - 1 - 0 + 0 \tag{7.14}$$

$$\implies j = n - 1. \tag{7.15}$$

Therefore,

$$\binom{k+n-1}{n-1}\Big\{{j \atop k}\Big\}\frac{k!}{j!} = \frac{(k+n-1)!}{k!(n-1)!}\Big\{{n-1 \atop k}\Big\}\frac{k!}{(n-1)!} = \frac{n^{\overline{k}}}{(n-1)!}\Big\{{n-1 \atop k}\Big\}$$

using (7.5), (7.6), and (7.13) so that the result follows by observing that

$$\begin{aligned} W_\rho\Big({-\atop -};\mathbf{0},1,0,\zeta\Big) &= \sum_{n\ge 1}\sum_{k=0}^{n-1}\frac{\zeta^n}{n}\binom{k+n-1}{n-1}(-1)^k\Big\{{n-1 \atop k}\Big\}\frac{k!}{(n-1)!}\frac{1}{(1+\rho)^{k+n}} \\ &= \frac{\zeta}{1+\rho} + \sum_{n\ge 2}\Big(\sum_{k=0}^{n-1}\frac{n^{\overline{k}}}{(n-1)!}\Big\{{n-1 \atop k}\Big\}\frac{(-1)^k}{(1+\rho)^k}\Big)\frac{\zeta^n}{n!(1+\rho)^n} \end{aligned}$$

which implies the claim. ☐

Proposition 7.4 ([56, Theorem 9]). *The solution of $\zeta = e^{\xi}(\xi - \tau_1)(\xi - \tau_2)$ is given by the Taylor series*

$$W_0\Big({\tau_1\ \tau_2 \atop -};\mathbf{0},0,1,\zeta\Big) = \tau_1 - \sum_{n\ge 1}\frac{\zeta^n}{n!}\Big(-\frac{e^{-\tau_1}}{T}\Big)^n\sum_{m\ge 0}\frac{n^{n-1}(n+m-1)!}{m!(n-m-1)!}\Big(\frac{1}{nT}\Big)^m,$$

where $T = \tau_1 - \tau_2$.

Proof. In this case $\boldsymbol{\alpha} = \mathbf{0}$, β, ρ, $s = 0$, $\gamma = 1$, and $r = 2$. Since $\boldsymbol{\alpha}^{\boldsymbol{\ell}} = \mathbf{0}^{\boldsymbol{\ell}}$ so that $\boldsymbol{\ell} = \mathbf{0}$ which implies $\ell = 0$ and $\beta^j = 0^j$ so that $j = 0$. In this way we have

$$\begin{aligned} &j = n - m + \ell - \sum_i i\ell_i - sn - 1 - \sum_p m_p + \sum_q n_q \\ &\implies 0 = n - m + 0 - 0 - 0n - 1 - m_2 + 0 \\ &\implies m = n - 1 - m_2 \end{aligned}$$

so that

$$\begin{aligned} W_0\Big({\tau_1\ \tau_2 \atop -};\mathbf{0},0,1,\zeta\Big) = \tau_1 + \sum_{n\ge 1}\sum_{m_2=0}^{n-1}&\frac{(-n)^{n-m_2-1}}{(n-m_2-1)!}\frac{(\zeta e^{-\tau_1})^n}{n} \\ &\times(-1)^{m_2}\binom{m_2+n-1}{n-1}\frac{1}{T_2^{m_2+n}} \end{aligned}$$

and the result follows. (In the proposition statement we write m (T) instead of m_2 (T_2).) ☐

Proposition 7.5 ([12, Theorem 1.1]). *The solution of*

$$\zeta = \xi e^{\xi}\prod_p(\xi + T_p)$$

is given by the Taylor series

$$W_0\Big(\begin{matrix}\tau\\ -\end{matrix};\mathbf{0},0,1,\zeta\Big) = \sum_{n\ge 1}\frac{(-n)^{n-1}}{n!}T_2^{-n}\dots T_r^{-n} \times \sum_{\mathbf{m}\ge\mathbf{0}_{r-1}} \frac{(1-n)^{\overline{\sum_p m_p}} n^{\overline{m_2}}\dots n^{\overline{m_r}}}{m_2!\dots m_r!(-nT_2)^{m_2}\dots(-nT_r)^{m_r}}.$$

Proof. In this case $\boldsymbol{\alpha}=\mathbf{0}$, β, ρ, $\tau_1=0$, $\gamma=1$, and $s=0$. Since $\boldsymbol{\alpha}^{\boldsymbol{\ell}}=\mathbf{0}^{\boldsymbol{\ell}}$ so that $\boldsymbol{\ell}=\mathbf{0}$ which implies $\ell=0$ and $\beta^j=0^j$ so that $j=0$. In this way we have

$$\begin{aligned} j &= n-m+\ell-\sum_i i\ell_i - sn - 1 - \sum_p m_p + \sum_q n_q \\ \implies&\ 0 = n-m+0-0-0n-1-\sum_p m_p + 0 \\ \implies&\ m = n-1-\sum_p m_p \end{aligned}$$

and the result follows from

$$W_0\Big(\begin{matrix}\tau\\ -\end{matrix};\mathbf{0},0,1,\zeta\Big) = \sum_{n\ge1}\sum_{\mathbf{m}\ge\mathbf{0}_{r-1}} \frac{(-n)^{n-1-\sum_p m_p}}{(n-1-\sum_p m_p)!}\frac{(\zeta e^{-\tau})^n}{n} \times (-1)^{\sum_p m_p}\prod_p\binom{m_p+n-1}{n-1}\frac{1}{T_p^{m_p+n}}$$

after observing that

$$\frac{(-1)^{\sum_p m_p}}{n\big(n-1-\sum_p m_p\big)!} = \frac{(n-1)!(-1)^{\sum_p m_p}}{n!\big(n-1-\sum_p m_p\big)!} = \frac{(1-n)^{\overline{\sum_p m_p}}}{n!}$$

and

$$\binom{m+n-1}{n-1} = \frac{n^{\overline{m}}}{m!}$$

using (7.5). □

The next proposition describes a generalized Lambert W function appearing in the context of plane-symmetric Einstein-Maxwell fields.

Proposition 7.6 ([58, Theorem 3])**.** *The solution of $\zeta=\xi e^{\alpha\xi^2+\xi}$ is given by the Taylor series*

$$W_0\Big(\begin{matrix}-\\ -\end{matrix};(\alpha,\mathbf{0}_{d-2}),0,1,\zeta\Big) = \frac{1}{\alpha}\sum_{n\ge1}\frac{(\alpha\zeta)^n}{n}\sum_{m=\lfloor(n-1)/2\rfloor}^{n-1}\binom{m}{n-m-1}\frac{(-n/\alpha)^m}{m!}.$$

Proof. In this case β, ρ, $\tau_1=0$, $\gamma=1$, and $r=0=s$. Since $\beta^j=0^j$ we have $j=0$. It follows that

$$\begin{aligned} j &= n-m+\ell-\sum_i i\ell_i - sn - 1 - \sum_p m_p + \sum_q n_q \\ \implies&\ 0 = n-m+\ell-2\ell_2-0n-1-0+0 \\ \implies&\ 0 = n-m-\ell-0n-1-0+0 \\ \implies&\ \ell = n-m-1 \end{aligned}$$

since $\ell=\ell_2$, because $\ell_i=0$, $i\in[3,d]$, and the result follows by observing that

$$W_0\Big(\begin{matrix}-\\ -\end{matrix};(\alpha,\mathbf{0}_{d-2}),0,1,\zeta\Big) = \sum_{n\ge1}\sum_{m\ge0}\sum_{\ell=0}^{m}\frac{\zeta^n}{n}\frac{(-n)^m}{m!}\binom{m}{\ell}\alpha^\ell.$$

Finally, if $\ell=0$, then $m=n-1$ and if $\ell=m$, then $m=n-m-1$ so that $m=\lfloor(n-1)/2\rfloor$. □

Finally, we turn to obtaining previous integral representations. In contrast to Example 3.3, there is a right-side enlargement in the interval of validity because of the integral representation as discussed in [47].

Proposition 7.7 ([47, Equations (86) and (89)]). *For $\zeta \in (-1/e, e)$ we have*

$$W(\zeta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\imath\theta}\ln\Big(1-\zeta e^{-\imath\theta}\exp\big(-e^{\imath\theta}\big)\Big)d\theta.$$

Proof. Using observation (2.2) in (7.7) we obtain

$$W(\zeta) = -\underset{=}{\overset{\lambda}{\Omega}}\lambda\ln\big(1-\zeta\lambda^{-1}e^{-\lambda}\big) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\imath\theta}\ln\Big(1-\zeta e^{-\imath\theta}\exp\big(-e^{\imath\theta}\big)\Big)d\theta.$$

□

Proposition 7.8 ([47, Equation (81)]). *For $\zeta \in (-1/e, e)$ we have*

$$W(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{\imath\theta}\big(e^{\imath\theta}+1\big)}{1-\zeta e^{-\imath\theta}\exp\big(-e^{\imath\theta}\big)}d\theta.$$

Proof. Using observation (2.2) in (7.8) we obtain

$$W(\zeta) = \underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda(\lambda+1)}{1-\zeta\lambda^{-1}e^{-\lambda}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{\imath\theta}\big(e^{\imath\theta}+1\big)}{1-\zeta e^{-\imath\theta}\exp\big(-e^{\imath\theta}\big)}d\theta.$$

□

Proposition 7.9 ([47, Theorem 8.1]). *If $-1/e \le \zeta \le e$ we have*

$$W(\zeta) = \frac{1}{\pi}\Re\int_{0}^{\pi}\ln\Big(\frac{\exp\big(e^{\imath\theta}\big)-\zeta e^{-\imath\theta}}{\exp\big(e^{\imath\theta}\big)-\zeta e^{\imath\theta}}\Big)d\theta.$$

Proof. We first prove that

$$\frac{m^n}{n!} = \frac{2}{\pi}\Im\int_0^{\pi}\exp\big(me^{\imath\theta}\big)\sin(n\theta)d\theta.$$

Note that

$$\begin{aligned}
\frac{m^n}{n!}\delta_{\epsilon,\varepsilon} = \underset{=}{\overset{\lambda}{\Omega}}\frac{e^{m\lambda^{\epsilon}}}{\lambda^{\varepsilon n}} \quad &\Longrightarrow \underset{=}{\overset{\lambda}{\Omega}}\Big(e^{m\lambda}-e^{m/\lambda}\Big)\frac{\lambda^n-\lambda^{-n}}{2} = -\frac{m^n}{n!}\\
&\Longrightarrow \frac{1}{\imath}\underset{=}{\overset{\lambda}{\Omega}}\Big(e^{m\lambda}-e^{m/\lambda}\Big)\frac{\lambda^n-\lambda^{-n}}{2\imath} = \frac{m^n}{n!}\\
&\Longrightarrow \frac{1}{2\pi\imath}\int_{-\pi}^{\pi}\Big(\exp\big(me^{\imath\theta}\big)-\exp\big(me^{-\imath\theta}\big)\Big)\sin(n\theta)d\theta = \frac{m^n}{n!}
\end{aligned}$$

which is equivalent to the desired result. The result now follows using (2.2) in (7.7) and using the argument in [57, Section 1.7.3] showing that the representation is not valid when $\zeta < -1/e$ and $\zeta > e$. □

Remark 7.10. There are other cases that fit the general representation given in Theorem 3.7 although this may not appear to be the case at first sight. The reason is because under simple transformations they are related to $W(\zeta)$. Indeed, see [38, 39] and also the application discussed in Section 6, more precisely, Proposition 9.2.

8. Radius of convergence for the Taylor series of the generalized Lambert W function

We show that we can improve upon certain aspects the radius of convergence mentioned in [56, Theorems 2 and 9]. In particular, we aim to address some of the open questions left open in [56]. For the convenience of the reader the open questions are quoted verbatim below:

- Q2: Theorems 1 and 9 contain Taylor series including derivatives of Laguerre polynomials, and the Bessel polynomials. Could we say more on these Taylor series, especially how to find the radius of convergence in general apart from Theorem 2?

Q3: We could say nothing about the radius of convergence of (14) for the r-Lambert function except the only case $r=-2$. What can we say about the convergence radius ρ_r in general?

Our strategy consists in using as a starting point Theorem 7.1.

8.1. **Main theorem.** We can take $|\beta| << 1$ and $|\tau_p| >> 1$ so that $|T_p| >> 1$ by using a suitable rescaling $\xi \to \xi/\alpha$ with α sufficiently large so that

$$\exp\Big(\sum_i \alpha_i \xi^i/\alpha^i\Big)(\exp(\beta\xi/\alpha)+\rho)\exp(\gamma\xi/\alpha+\delta)\frac{\xi\prod_p(\xi+\alpha T_p)}{\prod_q(\xi+\alpha S_q)} = \alpha^{r-s}\zeta$$

and in all the expressions that follow Definition 2.1 applies. We are now ready to state the main result of this section.

Theorem 8.1. *The radius of convergence of the power series representation for*

$$W_\rho\Big({\tau \atop \sigma}; \boldsymbol{\alpha}, \beta, \gamma, \zeta\Big)$$

is

$$R = e^{\delta-1}|1+\rho|\frac{\prod_p |T_p|}{\prod_q |S_q|}\Big|\frac{\beta}{1+\rho}+\gamma-\sum_q \frac{1}{S_q}+\sum_p \frac{1}{T_p}\Big|^{-1}.$$

Proof. We write

$$\langle \zeta^n \rangle W_\rho\Big({\tau \atop \sigma}; \boldsymbol{\alpha}, \beta, \gamma, \zeta\Big) = \gamma_n.$$

Next, we define

$$A(\lambda) := \frac{\exp\big(-n\sum_i \alpha_i\lambda^i - n\gamma\lambda\big)\prod_q S_q^n(1+\lambda/S_q)^n}{\prod_p T_p^n(1+\lambda/T_p)^n}$$

so that

$$\begin{aligned}
\gamma_n &= \frac{e^{-n\delta}}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{A(\lambda)}{\lambda^{n-1}(1+\rho+\exp(\beta\lambda)-1)^n}\\
&= \frac{e^{-n\delta}}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{A(\lambda)}{\lambda^{n-1}(1+\rho)^n(1+(\exp(\beta\lambda)-1)/(1+\rho))^n}\\
&= \frac{e^{-n\delta}}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{n^{n-1}A(\lambda/n)}{\lambda^{n-1}(1+\rho)^n(1+(\exp(\beta\lambda/n)-1)/(1+\rho))^n}\\
&\approx \frac{e^{-n\delta}}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{n^{n-1}A(\lambda/n)}{\lambda^{n-1}(1+\rho)^n(1+\beta\lambda/(n(1+\rho)))^n}\\
&\approx \frac{e^{-n\delta}}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{n^{n-1}\exp(-\beta\lambda/(1+\rho))A(\lambda/n)}{\lambda^{n-1}(1+\rho)^n},
\end{aligned}$$

where the third equality follows from the invariance under rescaling $\lambda \to \lambda/n$ and the last one follows using (3.5). Next, we observe that

$$\begin{aligned}
A(\lambda/n) &= \frac{\exp\big(-n\sum_i \alpha_i\lambda^i/n^i - \gamma\lambda\big)\prod_q S_q^n\big(1+\lambda/(nS_q)\big)^n}{\prod_p T_p^n(1+\lambda/(nT_p))^n}\\
&\approx \frac{\exp(-\gamma\lambda)\prod_q S_q^n\big(1+\lambda/(nS_q)\big)^n}{\prod_p T_p^n\big(1+\lambda/(nT_p)\big)^n}\\
&\approx \frac{\prod_q S_q^n}{\prod_p T_p^n}\exp\Big(-\Big(\gamma-\sum_q\frac{1}{S_q}+\sum_p\frac{1}{T_p}\Big)\lambda\Big)
\end{aligned}$$

using again invariance under rescaling $\lambda \to \lambda/n$ and observing that

$$\exp\Big(-n\sum_i \alpha_i\lambda^i/n^i\Big) \approx 1$$

along with (3.5). Finally, we have

$$\begin{aligned}\gamma_n &\approx \frac{e^{-n\delta}n^{n-1}\prod_q S_q^n}{n\prod_p T_p^n(1+\rho)^n} \underset{=}{\overset{\lambda}{\Omega}} \frac{\exp\big(-(\beta/(1+\rho)+\gamma-\sum_q 1/S_q+\sum_p 1/T_p)\lambda\big)}{\lambda^{n-1}}\\ &\approx \frac{e^{-n\delta}n^{n-1}\prod_q S_q^n}{n\prod_p T_p^n(1+\rho)^n}\frac{(-1)^{n-1}}{(n-1)!}\Big(\frac{\beta}{1+\rho}+\gamma-\sum_q\frac{1}{S_q}+\sum_p\frac{1}{T_p}\Big)^{n-1}\\ &\approx \frac{e^{-n\delta}n^{n-1}\prod_q S_q^n}{\prod_p T_p^n(1+\rho)^n}\frac{(-1)^{n-1}}{n!}\Big(\frac{\beta}{1+\rho}+\gamma-\sum_q\frac{1}{S_q}+\sum_p\frac{1}{T_p}\Big)^{n-1}\end{aligned}$$

to obtain

$$R=\lim_{n\to\infty}\big|\frac{\gamma_n}{\gamma_{n+1}}\big| = e^{\delta-1}|1+\rho|\frac{\prod_p|T_p|}{\prod_q|S_q|}\Big|\frac{\beta}{1+\rho}+\gamma-\sum_q\frac{1}{S_q}+\sum_p\frac{1}{T_p}\Big|^{-1}$$

using Example 3.3. □

8.2. **Connection of Theorem 8.1 with previous work: convergence radius of certain Taylor series.** We first observe that if we take $\tau_1 = 0 = \rho$, $\gamma = 1 = s$, $r = 2$, and $S_1 = T_2$ in Theorem 8.1 we obtain the well-known radius of convergence of the usual W function as discussed in Example 3.3. Our answer to the query Q2 is the content of the next corollary.

Corollary 8.2. *The radius of convergence of the power series for* $W_0\Big(\substack{\tau\\ \sigma};\mathbf{0},0,\gamma,\zeta\Big)$ *is*

$$R = e^{\delta-1}\frac{\prod_p|T_p|}{\prod_q|S_q|}\Big|\gamma-\sum_q\frac{1}{S_q}+\sum_p\frac{1}{T_p}\Big|^{-1}.$$

Remark 8.3. The radius of convergence R stated in Theorem 8.1 improves the previous result [56, Theorem 2] in the sense that it is valid for all $S = \tau - \sigma \neq 0$ and not only if $S < 0$, but the radius of convergence in [56, Theorem 2] improves ours if $S < 0$. As a way of comparison, by adapting to our notation, [56, Theorem 2] reads

$$R^{(MB)} = \exp\Big(\frac{\tau+\sigma}{2}-2\sqrt{-S}\Big),$$

where the superscript (MB) stands for Mező-Baricz and note that $-S > 0$ while our result reads

$$R^{(N)} = \frac{e^{\tau-1}}{|S|}\Big|1-\frac{1}{S}\Big|^{-1} = \frac{e^{\tau-1}}{|1-S|},$$

where the superscript (N) stands for Neto. Furthermore, note also that if $T = -S > 0$ we have

$$R^{(MB)} = \exp\Big(-\frac{T}{2}-2\sqrt{T}+\sigma\Big)$$

and

$$R^{(N)} = \frac{e^{-T-1+\sigma}}{|1+T|}$$

such that $R^{(N)} < R^{(MB)}$ if $T > 0$.

Remark 8.4. The radius of convergence R stated in Theorem 8.1 improves the previous result [56, Theorem 9] in the sense that no radius of convergence was obtained in [56, Theorem 9]. Indeed, we recall the verbatim commentary taken from [56]: "Neither Mugnaini nor the authors could calculate the radius of convergence of this series."

We finally address the open problem quoted verbatim above in Q3. For easy of reference we have the correspondence between notations

$$r^{(MB)} = \rho^{(N)} \quad\text{and}\quad \rho_r^{(MB)} = R^{(N)}.$$

The ρ-Lambert function is the solution $\xi = W_\rho(\zeta)$ of the equation

$$\xi e^{\xi} + \rho\xi = \zeta.$$

Our answer is the content of the next corollary.

Corollary 8.5. *The radius of convergence of the Taylor series of $W_\rho(\zeta)$ around 0 is*

$$R = \Big|\frac{(1+\rho)^2}{e}\Big|.$$

9. A sequence arising in algebraic geometry

The problem comprises when solving the functional equation

$$a(1+\chi_a)\ln(1+\chi_a) = (a+1)\chi_a - \zeta \tag{9.1}$$

which arises in the study of configuration spaces [52, Chapter 4, Equation (4.25)] and the exact determination of χ_a is related to WDVV equations in physics [62]. We write

$$\chi_a(\zeta) = \zeta + \sum_{n\ge 2} m_n(a)\frac{\zeta^n}{n!}.$$

We consider first the simpler problem of solving

$$(1+\chi_-)\ln(1+\chi_-) = \zeta. \tag{9.2}$$

It is easy to show that (9.2) is equivalent to

$$W(\zeta)e^{W(\zeta)} = \zeta.$$

under the observation $\chi_-(\zeta) = e^{W(\zeta)} - 1$. In words, the solution of our original problem; that is, determine the solution of (9.2) is now connected to the Lambert W function. Therefore, we can write

$$\chi_-(\zeta) = e^{W(\zeta)} - 1 = (F\circ W)(\zeta)$$

with $F(\zeta) = e^\zeta - 1$ so that Theorem 3.1 can be applied as the next proposition shows.

Proposition 9.1 ([32, Theorem 2.4])**.** *We have*

$$m_n(-1) = (1-n)^{n-1}.$$

Proof. Note that

$$\begin{aligned}
\big(F\circ G^{\langle -1\rangle}\big)(\zeta) &= \alpha_0 - \underset{=}{\overset{\lambda}{\Omega}}\lambda F'(\lambda)\ln\Big(1-\frac{\zeta}{G(\lambda)}\Big)\\
&= -\underset{=}{\overset{\lambda}{\Omega}}\lambda e^\lambda \ln\Big(1-\frac{\zeta}{\lambda e^\lambda}\Big)\\
&= \sum_{n\ge 1}\frac{\zeta^n}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{e^{(1-n)\lambda}}{\lambda^{n-1}}\\
&= \sum_{n\ge 1}\zeta^n\frac{(1-n)^{n-1}}{n!}
\end{aligned}$$

using Theorem 3.1 with $F(\zeta) = e^\zeta - 1$ so that $\alpha_0 = 0$ and $G^{\langle -1\rangle} = W$ so that $G(\zeta) = \zeta e^\zeta$. □

More generally, we have the next result.

Proposition 9.2. *We have*

$$\chi_a(\zeta) = \exp\big(\alpha + W(-\beta\zeta - \gamma)\big) - 1,$$

where

$$\alpha := \frac{a+1}{a},\quad \beta := \frac{1}{ae^\alpha},\quad \text{and}\quad \gamma := \frac{\alpha}{e^\alpha}.$$

Proof. We introduce the change of variables

$$\chi_a = e^{\alpha+\xi} - 1$$

to obtain

$$-(a+1) - a\xi e^{\alpha+\xi} = \zeta \implies \xi e^\xi = -\beta\zeta - \gamma$$

using (9.1) and the result follows by recalling the definition of the Lambert W function. □

We next recall the definition of the generalized Stirling numbers of the second kind (see, e.g., [13, Chapter 5, Page 57])

$$\sum_{m,n\geq 0} S_r(m,n)\frac{\xi^n\tau^m}{m!} = \exp\Big(\xi\Big(e^\tau - \sum_{k=0}^{r-1}\frac{\tau^k}{k!}\Big)\Big). \tag{9.3}$$

In combinatorial terms $S_r(m,n)$ counts the number of partitions of $[m]$ in n blocks with at least r elements. Note that

$$S_1(m,n) = \begin{Bmatrix} m \\ n \end{Bmatrix}$$

in (7.4). For later use we also recall the fundamental triangular recurrence relation (see, e.g., [13, Chapter 5, Page 57])

$$S_r(m+1,n) = nS_r(m,n) + \binom{m}{r-1} S_r(m-r+1,n-1). \tag{9.4}$$

We can now state a generalization of a result of Koganov as stated in [32].

Theorem 9.3. *We have*

$$\begin{aligned} m_n(a) = 1 + (n-1)! \sum_{j=1}^{n_a} n(n+1)\dots(n+j-1) \sum_{\ell=0}^{n-1}\sum_{k=0}^{\min(j,\ell)} \frac{S_1(\ell+1,k+1)}{\ell!} \\ \times a^k(-1)^{j-k}(a+1)^{j-k}\frac{S_2(n-\ell-1+j-k,j-k)}{(n-\ell-1+j-k)!}, \end{aligned}$$

where

$$n_a := \lfloor\frac{n-1}{2}\rfloor\delta_{a,1} + (n-1)(1-\delta_{a,1}).$$

Proof. Note that

$$\begin{aligned} \big(F\circ G^{\langle -1\rangle}\big)(\zeta) &= \alpha_0 - \underset{=}{\overset{\lambda}{\Omega}}\lambda F'(\lambda)\ln\Big(1-\frac{\zeta}{G(\lambda)}\Big) \\ &= -\underset{=}{\overset{\lambda}{\Omega}}\lambda e^\lambda \ln\Big(1-\frac{\zeta}{(a+1)(e^\lambda-1)+a\lambda e^\lambda}\Big) \\ &= -\underset{=}{\overset{\lambda}{\Omega}}\lambda e^\lambda \ln\Big(1-\frac{\zeta/\lambda}{1-D_a(\lambda)}\Big), \end{aligned}$$

where

$$D_a(\lambda) := a(e^\lambda-1) - (a+1)\frac{e^\lambda-1-\lambda}{\lambda}$$

and using Theorem 3.1 with $F(\zeta)=e^\zeta-1$ as in Proposition 9.2 so that we have again $\alpha_0=0$, but now we take $G(\zeta)=(a+1)(e^\zeta-1)-a\zeta e^\zeta$. In this way we obtain

$$\big(F\circ G^{\langle -1\rangle}\big)(\zeta) = \sum_{n\geq 1}\frac{\zeta^n}{n}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda e^\lambda}{\lambda^n(1-D_a(\lambda))^n}.$$

Next, we use

$$\frac{1}{(1-D_a(\lambda))^n} = \sum_{j\geq 0}\frac{n(n+1)\dots(n+j-1)}{j!}D_a^j(\lambda). \tag{9.5}$$

We next observe that $D_a(\lambda)=\mathcal{O}\big(\lambda^{1+\delta_{a,1}}\big)$ so that we have $j\leq n_a$, because

$$j(1+\delta_{a,1}) \leq n-1 \implies j \leq \frac{n-1}{1+\delta_{a,1}}$$

and

$$\frac{1}{1+\delta_{a,1}} = \frac{\delta_{a,1}}{2} + (1-\delta_{a,1}).$$

We next note that

$$D_a^j(\lambda) = \sum_{k=0}^{j} \binom{j}{k} a^k (e^\lambda - 1)^k (-1)^{j-k} (a+1)^{j-k} \frac{(e^\lambda - 1 - \lambda)^{j-k}}{\lambda^{j-k}}$$

to obtain

$$\underset{=}{\overset{\lambda}{\Omega}} \frac{e^\lambda}{\lambda^{n-1}} \frac{D_a^j(\lambda)}{j!} = \sum_{k=0}^{j} \frac{1}{j!} \binom{j}{k} a^k (-1)^{j-k} (a+1)^{j-k} \underset{=}{\overset{\lambda}{\Omega}} \frac{e^\lambda}{\lambda^{n-1+j-k}} \frac{(e^\lambda - 1)^k}{k!} \frac{(e^\lambda - 1 - \lambda)^{j-k}}{(j-k)!}.$$

Finally, we have

$$\frac{e^\lambda (e^\lambda - 1)^k}{k!} = (k+1) \frac{(e^\lambda - 1)^{k+1}}{(k+1)!} + \frac{(e^\lambda - 1)^k}{k!}$$

to obtain

$$\begin{aligned}
&\underset{=}{\overset{\lambda}{\Omega}} \frac{e^\lambda}{\lambda^{n-1+j-k}} \frac{(e^\lambda - 1)^k}{k!} \frac{(e^\lambda - 1 - \lambda)^{j-k}}{(j-k)!} \\
&= \underset{=}{\overset{\lambda}{\Omega}} \frac{1}{\lambda^{n-1+j-k}} \Big((k+1) \frac{(e^\lambda - 1)^{k+1}}{(k+1)!} + \frac{(e^\lambda - 1)^k}{k!} \Big) \frac{(e^\lambda - 1 - \lambda)^{j-k}}{(j-k)!} \\
&= \sum_{\ell + m = n-1+j-k} \frac{(k+1) S_1(\ell, k+1) + S_1(\ell, k)}{\ell!} \frac{S_2(m, j-k)}{m!} \\
&= \sum_{\ell + m = n-1+j-k} \frac{S_1(\ell+1, k+1)}{\ell!} \frac{S_2(m, j-k)}{m!}
\end{aligned}$$

using Lemma 2.5 with $k \to n-1+j-k$,

$$F(\lambda) = \frac{(e^\lambda - 1)^\ell}{\ell!}$$

such that $\ell = k, k+1$, and

$$G(\lambda) = \frac{(e^\lambda - 1 - \lambda)^{j-k}}{(j-k)!}$$

along with

$$\frac{S_r(m,n)}{m!} = \frac{1}{n!} \underset{=}{\overset{\lambda}{\Omega}} \frac{1}{\lambda^m} \Big(e^\lambda - \sum_{k=1}^{r-1} \frac{\lambda^k}{k!} \Big)$$

which follows from (9.3) to obtain the second equality and (9.4) to obtain the last one. Collecting the results above and observing that $\ell \ge k$ and $m \ge j-k$ so that $0 \le \ell \le n-1$ and $0 \le k \le p$ with $p = j, k$ so that $0 \le k \le \min(j, \ell)$ we arrive at the desired result by going back to (9.5). □

Corollary 9.4. *We have*

$$\begin{aligned}
m_n(1) = 1 + (n-1)! \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} n(n+1)\dots(n+j-1) \sum_{\ell=0}^{n-1} \sum_{k=0}^{\min(j,\ell)} \frac{S_1(\ell+1, k+1)}{\ell!} \\
\times (-2)^{j-k} \frac{S_2(n-\ell-1+j-k, j-k)}{(n-\ell-1+j-k)!}.
\end{aligned}$$

This is nothing more than Koganov's aforementioned expression apart from a minor sign correction: $j-k$ should replace $k-j$ in

$$\frac{S_2(n-\ell-1+k-j, j-k)}{(n-\ell-1+k-j)!}$$

as stated in [32, Page 3].

10. Higher order derivatives of $F \circ G^{\langle -1\rangle}$

In this section we generalize the main result in [42]. The exact determination of higher order derivatives has some interesting consequences for the Langevin function [41]. In all that follows we let $F_k(\zeta) = \zeta^k$.

Theorem 10.1. *We have*

$$D_\zeta^m\big(F_k \circ G^{\langle -1\rangle}\big)(\zeta) = k\sum_{\ell=0}^{m-k}(\beta_1)^{-m-\ell}(-1)^\ell R_{m-k,\ell}\big(\beta_2,\dots,\beta_{m-k-\ell+2}\big),$$

where

$$R_{m-k,\ell}\big(\beta_2,\dots,\beta_{m-k-\ell+2}\big) = \frac{1}{\ell!}\sum_{k_1+\dots+k_\ell=m-k+\ell}(m+\ell-1)!\beta_{k_1}\dots\beta_{k_\ell}$$

with $k_i \in [2, m-k-\ell+2]$.

Proof. Using Theorem 3.1 we obtain

$$\begin{aligned}\langle\zeta^m\rangle\big(F_k \circ G^{\langle -1\rangle}\big)(\zeta) &= \frac{k}{m}\underset{=}{\overset{\lambda}{\Omega}}\frac{\lambda^k}{G^m(\lambda)}\\ &= \frac{k}{m\beta_1^m}\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^{m-k}}\,\frac{1}{(1+(G(\lambda)/(\beta_1\lambda)-1))^m}\\ &= \frac{k}{m\beta_1^m}\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^{m-k}}\sum_{\ell=0}^{m-k}(-1)^\ell\binom{\ell+m-1}{m-1}\Big(\frac{\beta_2\lambda}{\beta_1}+\dots+\frac{\beta_{m-k+1}\lambda^{m-k}}{\beta_1}\Big)^\ell.\end{aligned}$$

The result follows by using

$$\begin{aligned}\underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^{m-k}}\big(\beta_2\lambda+\dots\beta_{m-k+1}\lambda^{m-k}\big)^\ell &= \underset{=}{\overset{\lambda}{\Omega}}\frac{1}{\lambda^{m-k}}\big(\beta_2\lambda+\dots+\beta_{m-k-\ell+2}\lambda^{m-k-\ell+1}\big)^\ell\\ &= \sum_{(k_1-1)+\dots+(k_\ell-1)=m-k}\beta_{k_1}\dots\beta_{k_\ell}\\ &= \sum_{k_1+\dots+k_\ell=m-k+\ell}\beta_{k_1}\dots\beta_{k_\ell}\end{aligned}$$

with $k_i \in [2, m-k-\ell+2]$. By considering

$$\big(\beta_2\lambda+\dots\beta_{m-k+1}\lambda^{m-k}\big)^\ell \equiv (\dots)^\ell$$

we have λ^a with $a \le m-k$ if $\ell = 1$, $a \le m-k-1$ if $\ell = 2$ and so on to arrive at $(\dots)^\ell$ so that $a \le m-k-(\ell-1)$ to obtain the first equality. The general term in

$$\big(\beta_2\lambda+\dots+\beta_{m-k-\ell+2}\lambda^{m-k-\ell+1}\big)^\ell$$

is of the form $\prod_{i=1}^\ell \beta_{k_i}\lambda^{k_i-1}$ such that the Omega operator selects only terms satisfying

$$\underset{=}{\overset{\lambda}{\Omega}}\frac{\prod_{i=1}^\ell\lambda^{k_i-1}}{\lambda^{m-k}} = \delta_{\sum_{i=1}^\ell(k_i-1),m-k}$$

and the second equality follows. $\square$

By setting $k = 1$ we obtain the corollary that follows. To connect with [42, Theorem 2] observe that $\beta_k = G^{(k)}(0)/k!$.

Corollary 10.2 ([42, Theorem 2])**.** *We have*

$$D_\zeta^m G^{\langle -1\rangle}(\zeta) = \sum_{\ell=0}^{m-1}(\beta_1)^{-m-\ell}(-1)^\ell R_{m-1,\ell}\big(\beta_2,\dots,\beta_{m-\ell+1}\big),$$

where

$$R_{m-1,\ell}\big(\beta_2,\dots,\beta_{m-\ell+1}\big) = \frac{1}{\ell!}\sum_{k_1+\dots+k_\ell=m-1+\ell}(m+\ell-1)!\beta_{k_1}\dots\beta_{k_\ell}$$

with $k_i \ge 2$.

We can also compute higher order derivatives using Theorem 3.7. Indeed, we first observe that the Omega representation of $F \circ \mathbf{G}^{\langle -1 \rangle}$ carries the term

$$\boldsymbol{\lambda}^{\mathbf{e}} F(\boldsymbol{\lambda}) \det\big(D_i G_j(\boldsymbol{\lambda})\big) \in \mathbb{C}[[\boldsymbol{\lambda}]].$$

(See the right-hand side of (3.7).) Therefore, if we consider $\boldsymbol{\lambda}^{\mathbf{k}}$ with $\mathbf{k} = (k_1, \dots, k_n)$ and use the notation $F_{\mathbf{k}}(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^{\mathbf{k}}$, then it is possible to determine $D^{\mathbf{m}}_{\boldsymbol{\zeta}}\big(F \circ \mathbf{G}^{\langle -1 \rangle}\big)(\boldsymbol{\zeta})$ by computing the simpler expression $D^{\mathbf{m}}_{\boldsymbol{\zeta}}\big(F_{\mathbf{k}} \circ \mathbf{G}^{\langle -1 \rangle}\big)(\boldsymbol{\zeta})$.

Theorem 10.3. *We have*

$$D^{\mathbf{m}}_{\boldsymbol{\zeta}}\big(F_{\mathbf{k}} \circ \mathbf{G}^{\langle -1 \rangle}\big)(\boldsymbol{\zeta}) = \frac{1}{\boldsymbol{\beta}^{\mathbf{m}+\mathbf{e}}} \prod_{k=1}^{n} \sum_{\ell_k} (-1)^{\ell_k} \binom{\ell_k + m_k}{m_k} \sum \beta_{1\vec{\mathbf{k}}_1} \dots \beta_{n\vec{\mathbf{k}}_n},$$

where $\vec{\mathbf{k}}_j = (\mathbf{k}_{j1}, \dots, \mathbf{k}_{j\ell_j})$ and the sum is taken subject to

$$\sum_{i_1=1}^{\ell_1} \mathbf{k}_{1i_1} + \dots + \sum_{i_n=1}^{\ell_n} \mathbf{k}_{ni_n} = \mathbf{m} - \mathbf{k} + \mathbf{e}.$$

Proof. Using Theorem 3.7 we have

$$\begin{aligned}
&\langle \boldsymbol{\zeta}^{\mathbf{m}} \rangle \big(F_{\mathbf{k}} \circ \mathbf{G}^{\langle -1 \rangle}\big)(\boldsymbol{\zeta}) \\
&= \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \frac{\boldsymbol{\lambda}^{\mathbf{k}}}{\prod_{k=1}^{n} (G_k(\boldsymbol{\lambda}))^{m_k+1}} \\
&= \frac{1}{\boldsymbol{\beta}^{\mathbf{m}+\mathbf{e}}} \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \frac{1}{\boldsymbol{\lambda}^{\mathbf{m}-\mathbf{k}+\mathbf{e}} \prod_{k=1}^{n} (1 + H_k(\boldsymbol{\lambda}))^{m_k+1}} \\
&= \frac{1}{\boldsymbol{\beta}^{\mathbf{m}+\mathbf{e}}} \prod_{k=1}^{n} \sum_{\ell_k} (-1)^{\ell_k} \binom{\ell_k + m_k}{m_k} \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \frac{(H_k(\boldsymbol{\lambda}))^{\ell_k}}{\boldsymbol{\lambda}^{\mathbf{m}-\mathbf{k}+\mathbf{e}}} \\
&= \frac{1}{\boldsymbol{\beta}^{\mathbf{m}+\mathbf{e}}} \prod_{k=1}^{n} \sum_{\ell_k} (-1)^{\ell_k} \binom{\ell_k + m_k}{m_k} \underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \frac{\big(\sum_{\mathbf{k}_1} \beta_{1\mathbf{k}_1} \boldsymbol{\lambda}^{\mathbf{k}_1}\big)^{\ell_1} \dots \big(\sum_{\mathbf{k}_n} \beta_{n\mathbf{k}_n} \boldsymbol{\lambda}^{\mathbf{k}_n}\big)^{\ell_n}}{\boldsymbol{\lambda}^{\mathbf{m}-\mathbf{k}+\mathbf{e}}} \\
&= \frac{1}{\boldsymbol{\beta}^{\mathbf{m}+\mathbf{e}}} \prod_{k=1}^{n} \sum_{\ell_k} (-1)^{\ell_k} \binom{\ell_k + m_k}{m_k} \sum \beta_{1\vec{\mathbf{k}}_1} \dots \beta_{n\vec{\mathbf{k}}_n},
\end{aligned}$$

where $\vec{\mathbf{k}}_j = (\mathbf{k}_{j1}, \dots, \mathbf{k}_{j\ell_j})$ such that $\beta_{j\vec{\mathbf{k}}_j} = \beta_{j\mathbf{k}_{j1}} \dots \beta_{j\mathbf{k}_{j\ell_j}}$ with $j \in [n]$ and we write

$$\Big(\sum_{\mathbf{k}_j} \beta_{j\mathbf{k}_j} \boldsymbol{\lambda}^{\mathbf{k}_j}\Big)^{\ell_j} = \Big(\sum_{\mathbf{k}_{j1}} \beta_{j\mathbf{k}_{j1}} \boldsymbol{\lambda}^{\mathbf{k}_{j1}}\Big) \dots \Big(\sum_{\mathbf{k}_{j\ell_j}} \beta_{j\mathbf{k}_{j\ell_j}} \boldsymbol{\lambda}^{\mathbf{k}_{j\ell_j}}\Big)$$

for each j. The action of the Omega operator results in the constraints: the sum $\sum_{\ell_k}$ is limited to $\max_{j\in[n]}(m_j - k_j + 1)$ and the $\sum$ is taken subject to

$$\sum_{i_1=1}^{\ell_1} \mathbf{k}_{1i_1} + \dots + \sum_{i_n=1}^{\ell_n} \mathbf{k}_{ni_n} = \mathbf{m} - \mathbf{k} + \mathbf{e}.$$

□

11. Concluding Remarks

Up to now, there are several known proofs of the L-B inversion formula with several different flavors such as combinatorial, algebraic, analytic, and so on, and this work adds another perspective on this topic. We showed that an OC based proof of the L-B inversion formula is available in Theorems 3.1, 3.5, and 3.7 stated in Section 3. In a certain sense, this work comprises another instance of computing the continuous discretely [5]. By this we mean recasting the L-B theorem to compute inverse functions (hence a continuous tool) via combinatorial analysis using the OC which, as already mentioned, was originally used to study diophantine equations (hence a discrete tool).

This line of approach already made its appearance before and this work provides another instance where tools developed to address discrete problems are also useful in the continuous scenario. One may therefore ask what is the point of our OC based representation of the L-B formula? We have several answers to the query. First, our proofs are simple and our representation has the nice feature of requiring simple elimination rules in the context of the OC to obtain free Omega generating functions because all the expressions involved contain factored polynomials. Second, as one may see by carefully reading Sections 4, 5, and 6 we have at least two available proofs of most results regarding the aforementioned versions of the L-B inversion formula showing the versatility of the method in a way that the Omega elimination rules play and essential role. Third, some applications are advanced to show the versatility and usefulness of the aforementioned approach. In particular, we introduced a new generalized Lambert W function in Theorem 7.1 which implies several previous known theorems scattered in the literature and stated as propositions in Section 7. Furthermore, it provides insight into some of the open problems stated in [56] which are answered using our approach in Sections 7 and 8 dealing with the Omega free expansion of the generalized W function and the radius of convergence of some subclasses of the aforementioned generalized W function, respectively. We highlight also that the OC representation was crucial in obtaining by simple means the convergence radius of some generalized Lambert W functions in Section 8 pointing out another instance outside the original scope where OC can be useful. In Section 9 a generalization of a representation of a sequence arising in algebraic geometry was determined which implies Koganov's representation as a special case [32]. Finally, in Section 10 we showed that higher order derivatives of $F \circ \mathbf{G}^{\langle -1 \rangle}$ can be easily handled using OC based methods generalizing some previous results in the literature [42] corresponding to the case $F \equiv I$ and $\mathbf{G} \equiv G$.

Recently OC has shown to be extremely useful not only to describe the partition of natural numbers, but in a number of other problems including inverse problems such as the inverse problem for non-autonomous dynamical systems (answering a question left open in [4]) and the L-B inversion formula along with the generalized Lambert W function with this work (answering some of the questions left open in [56]). Therefore, this work reinforces the ubiquitous nature and usefulness character of OC at the interplay of the discrete-continuous settings by exploring the computation of inverse functions and associated consequences, most notably the case of the generalized Lambert W function.

References

[1] A. Abdesselam; A physicist's proof of the Lagrange-Good multivariable inversion formula. *J. Phys. A*, **36** (2003), 9471–9478.

[2] G. E. Andrews, P. Paule, A. Riese; MacMahon's partition analysis: the omega package. *European J. Combin.*, **22** (2001), no. 7, 887–904

[3] G. E. Andrews, P. Paule, A. Riese; MacMahon's partition analysis VI: a new reduction algorithm. *Ann. Comb.*, **5** (2001), no. 3-4, 251–270.

[4] A. Bassom, P. Howlett, P. Taylor; An explicit Maclaurin series solution to a classic non-autonomous abstract evolution equation. *Appl. Math. Lett.*, **139** (2023), 108537.

[5] M. Beck, S. Robins; *Computing the continuous discretely: integer-point enumeration in polyhedra.* Vol. 2, Springer, New York, 2007.

[6] J. A. Bondy, U. S. R. Murty; *Graph theory.* Springer, 2008.

[7] M. Borinsky; *Graphs in perturbation theory: algebraic structure and asymptotics.* Springer, Switzerland, 2018.

[8] C. Brouder, G. H. E. Duchamp, F. Patras, G. Z. Toth; The Rayleigh-Schrödinger perturbation series of quasi-degenerate systems. *Int. J. Quantum Chem.*, **112** (2012), no. 10, 2256–2266.

[9] C. Brouder, A. Mestre, F. Patras; Tree expansion in time-dependent perturbation theory. *J. Math. Phys.*, **51** (2010), 072104.

[10] F. Carbonell, J. Jimenez, L. Pedroso; Computing multiple integrals involving matrix exponentials. *J. Comput. Appl. Math.*, **213** (2008), 300–305.

[11] T. Carletti; The Lagrange inversion formula on non-Archimedean fields, non-analytical form of differential and finite difference equations. *Discrete Contin. Dyn. Syst.*, **9** (2003), no. 4, 835–858.

[12] P. Castle; Taylor series for generalized Lambert W functions. arXiv:1801.09904 (2018).

[13] L. Comtet; *Analyse Combinatorie*, Vol. II, Presses Universitaires de France, Paris, 1970.

[14] F. Constantinescu; Perturbation expansion for distributions on surfaces, Lagrange-Bürmann theorem and applications to mechanics. *Generalized Functions and Their Applications*, (1993), 51–52.

[15] H. D. Dahmen, W. Krzyzanowski, M. L. Larsen; Graph rules for functions of the time-evolution operator. *Phys. Rev. D*, **33** (1986), 1726–1744.

[16] R. M. Digilov; Gravity discharge vessel revisited: an explicit Lambert W function solution. *American Journal of Physics*, **85** (2017), no. 7, 510–514

[17] F. J. Dyson; The radiation theories of Tomonaga, Schwinger, and Feynman. *Phys. Rev.*, **75** (1949), no. 3, 486–502.

[18] A. Francisco Neto; Matrix analysis and omega calculus. *SIAM Rev.*, **62** (2020), no. 1, 264–280.

[19] A. Francisco Neto; Matrix computations with the omega calculus. *Linear Multilinear Algebra*, **70** (2022), no. 20, 5075–5106.

[20] A. Francisco Neto; A basis- and integral-free representation of time-dependent perturbation theory via the Omega matrix calculus. *Ann. Inst. Henri Poincaré Comb. Phys. Interact.*, **11** (2024), 383–407.

[21] A. Francisco Neto; A new approach to multi-delay matrix valued fractional linear differential equations with constant coefficients. *Fract. Calc. Appl. Anal.*, **26** (2023), no. 5, 2202–2236.

[22] A. Francisco Neto; An explicit Maclaurin series solution to non-autonomous and non-homogeneous evolution equation, Omega Calculus, and associated applications, *IMA J. Appl. Math.*, **89** (2024), no. 3, 533–573.

[23] A. Francisco Neto; A new approach to MacMahon's partition analysis and associated applications. *Linear Multilinear Algebra*, **73** (2025), no. 6, 1216–1256.

[24] A. Francisco Neto; A new basis- and integral-free approach to perturbation theory: the Schrodinger dynamics of N trapped ions in the high-intensity regime. *Ann. Physics*, **474** (2025), 16997.

[25] H. M. Fried; *Green's functions and ordered exponentials*. Cambridge University Press, Cambridge, 2002.

[26] A. M. Garsia; A q-analogue of the Lagrange inversion formula. *Houston J. Math*, **7** (1981), no. 2, 205–237.

[27] I. M. Gessel; A combinatorial proof of the multivariable Lagrange inversion formula. *J. Combin. Theory Ser. A*, **45** (1987), no. 2, 178–195.

[28] I. M. Gessel; A noncommutative generalization and q-analog of the Lagrange inversion formula. *Trans. Amer. Math. Soc.*, **257** (1980), no. 2, 455–482

[29] I. M. Gessel; Lagrange inversion. *J. Combin. Theory Ser. A*, **144** (2016), 212–249.

[30] P.-L. Giscard, K. Lui, S. J. Thwaite, D. Jaksch; An exact formulation of the time-ordered exponential using path-sums. *J. Math. Phys.* **56** (2015), no. 5, 053503.

[31] I. J. Good; Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes. *Math. Proc. Cambridge Philos. Soc.*, **56** (1960), no. 4, 367–380.

[32] I. P. Goulden, S. Litsyn, V. Shevelev; On a sequence arising in algebraic geometry. *J. Integer Seq.*, **8.05.4** (2005), 7.

[33] G.-H. Han; A general algorithm for the MacMahon omega operator. *Ann. Comb.* **7** (2003), 467–480.

[34] P. Henrici; An algebraic proof of the Lagrange-Bürmann formula. *J. Math. Anal. Appl.*, **8** (1964), no. 2, 218–224.

[35] P. Henrici; Topics in Computational Complex Analysis: IV. The Lagrange-Bürmann Formula for Systems of Formal Power Series. Computational Aspects of Complex Analysis: Proceedings of the NATO Advanced Study Institute held at Braunlage, Harz, Germany, July 26–August 6, 1982. Dordrecht: Springer Netherlands, 1983.

[36] P. Henrici; *Applied and computational complex analysis*, Vol. 1. Wiley, New York, 1974.

[37] J. Hofbauer; A short proof of the Lagrange-Good formula. *Discrete Math.* **25** (1979), no. 2, 135–139.

[38] A. Houari; New analytical results in solid state physics using the Lambert W function. *European Journal of Physics* **44** (2023), no. 6, 065502.

[39] A. Houari; Additional applications of the Lambert W function to solid state physics. *European Journal of Physics*, **45** (2024), no. 5, 055501.

[40] I-C. Huang; Residue methods in combinatorial analysis. In: *Local cohomology and its applications, edited by G. Lyubeznik*, pp. 255–352, Marcel Dekker Inc., 2002.

[41] M. Itskov, R. Dargazany, K. Hörnes; Taylor expansion of the inverse function with application to the Langevin function. *Math. Mec. Solids* **17** (2012), no. 7, 693–701.

[42] W. P. Johnson; Combinatorics of higher derivatives of inverses. *Amer. Math. Monthly* **109** (2002), no. 3, 273–277.

[43] G. W. Johnson, M. L. Lapidus; *The Feynman integral and Feynman's operational calculus*. Oxford University Press, Oxford, 2002.

[44] S. A. Joni; Lagrange inversion in higher dimensions and umbral operators. *Linear Multilinear Algebra* **6** (1978), no. 2, 111–121.

[45] A. Kalev, I. Hen; An integral-free representation of the Dyson series using divided differences. *New J. Phys.*, **23** (2021), 103035.

[46] A. Kalev, I. Hen; Feynman path integrals for discrete-variable systems: walks on Hamiltonian graphs. *Phys. Rev. Res.* **7** (2025), 013220.

[47] G. A. Kalugin, D. J. Jeffrey, R. M. Corless; Stieltjes, Poisson and other integral representations for functions of Lambert W. arXiv preprint arXiv:1103.5640 (2011).

[48] Ch. Krattenthaler; Operator methods and Lagrange inversion: a unified approach to Lagrange formulas. *Trans. Amer. Math. Soc.*, 305 (1988), no. 2, 431–465.

[49] I. Lindgren, S. Salomonson, B. Åsén; The covariant-evolution-operator method in bound-state QED. *Phys. Rep.*, **389** (2004), no. 4, 161–261.

[50] H. Maassen, D. Botvich; A Galton–Watson estimate for Dyson series. *Ann. Henri Poincaré*, **10** (2009), 1141–1158

[51] P. A. MacMahon; *Combinatory Analysis.* Volumes I and II, 137, AMS Chelsea Publishing, Providence, 2001

[52] Y. I. Manin; *Frobenius manifolds, quantum cohomology, and moduli spaces.* Vol. 47. American Mathematical Soc., 1999.

[53] J. Martínez-Carranza, F. Soto-Eguibar, H. Moya-Cessa; Alternative analysis to perturbation theory in quantum mechanics: Dyson series in matrix form. *The European Physical Journal D*, **66** (2012), 1–6.

[54] V. P. Maslov et al.; *Operational methods.* Mir Publishers, Moscou, 1976.

[55] D. Merlini, R. Sprugnoli, M. Cecilia Verri; Lagrange inversion: when and how. *Acta Appl. Math.*, **94** (2006), no. 3, 233–249.

[56] I. Mező, A. Baricz; On the generalization of the Lambert W function. *Trans. Amer. Math. Soc.* **369** (2017), no. 11, 7917–7934.

[57] I. Mező; *The Lambert W function: its generalizations and applications.* Chapman and Hall/CRC, 2022.

[58] I. Mező, C. B. Corcino, R. B. Corcino; Resolution of the plane-symmetric Einstein-Maxwell fields with a generalization of the Lambert W function. *Journal of Physics Communications*, **4** (2020), no. 8, 085008.

[59] I. Mező, G. Keady; Some physical applications of generalized Lambert functions. *European Journal of Physics* **37** (2016), no. 6, 065802.

[60] J.-C. Novelli, J.-Y. Thibon; Noncommutative symmetric functions and Lagrange inversion. *Adv. Appl. Math.* **40** (2008), no. 1, 8–35.

[61] E. Panzer, R. Wulkenhaar; Lambert-W solves the noncommutative Φ_4-model. *Comm. Math. Phys.* **374** (2020), no. 3, 1935–1961.

[62] M. A. Readdy; The pre-WDVV ring of physics and its topology. *Ramanujan J.* **10** (2005), 269–281.

[63] M. Reed, B. Simon; *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness.* Academic Press, San Diego, 1975.

[64] J. Santmayer; Problems and solutions, *Math. Mag.*, **97**, no. 4, 433–443.

[65] A. Schulze-Halberg, A. M. Ishkhanyan; Exactly-solvable quantum systems in terms of Lambert-W functions. *Few-Body Systems* **61** (2020), 1–15.

[66] T. C. Scott, R. Mann, R. E. Martinez II; General relativity and quantum mechanics: towards a generalization of the Lambert W function. *Appl. Algebra Engrg. Comm. Comput.*, **17** (2006), 41–47.

[67] A. Sokal; A ridiculously simple and explicit implicit function theorem. *Sém. Lothar. Combin.*, **61A** (2009/2011), Article B61Ad

[68] A. Tavlayan, B. Tekin; Exact formulas for spherical photon orbits around Kerr black holes. *Phys. Rev. D*, **102** (2020), no. 10, 104036.

[69] L. Verde-Star; Dual operators and Lagrange inversion in several variables. *Adv. Math.* **58** (1985), no. 1, 89–108.

[70] M. Vittot; Perturbation theory and control in classical or quantum mechanics by an inversion formula. *J. Phys. A*, **37** (2004), no. 24, 6337–6358.

[71] G. Xin; A Euclid style algorithm for MacMahon's partition analysis. *J. Combin. Theory Ser. A*, **131** (2015), 32–60.

[72] S. Yi, P. W. Nelson, A. G. Ulsoy; *Time-delay systems: analysis and control using the Lambert W function.* World Scientific, Singapore, 2010.

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