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COMPACT ALMOST AUTOMORPHIC DYNAMICS OF LINEAR NON-AUTONOMOUS DIFFERENTIAL EQUATIONS WITH EXPONENTIAL DICHOTOMY AND OF DELAYED BIOLOGICAL MODELS

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Abstract. In this work, we study the dynamics of linear non-autonomous differential equations with exponential dichotomy and compact almost automorphic perturbations. First, we prove that if the homogeneous system is exponentially dichotomous and the coefficient matrix is compact almost automorphic, then the associated Green's function is compact bi-almost automorphic and uniformly continuous relative to the principal diagonal of the two-dimensional Euclidean space. Next, we demonstrate the invariance of the compact almost automorphic function space under convolution products with Green's function as the kernel. These results ensures that the unique bounded solution of a linear non-autonomous differential equation, under exponential dichotomy and with compact almost automorphic perturbation, is itself compact almost automorphic. Finally, we investigate the existence and the global exponential stability of a unique positive compact almost automorphic solution for a nonlinear non-autonomous delayed biological model with nonlinear harvesting or immigration terms and mixed delays.

1. Introduction

The qualitative theory of differential equations, initiated with the pioneering works of Poincaré and Lyapunov [27], is a well established area of research with significant developments in pure and applied mathematics. This paper focuses on the qualitative theory of linear non-autonomous ordinary differential equations under exponential dichotomy and with compact almost automorphic perturbations.

A central notion in the qualitative theory of differential equations is that of *exponential dichotomy* (see [15] or Definition 2.7 in this work). Consider the non-autonomous differential equation

$$x'(t) = A(t)x(t) + f(t)\,, \tag{1.1}$$

and suppose that the associated system

$$x'(t) = A(t)x(t) \tag{1.2}$$

possesses an (α, K, P)-exponential dichotomy with Green's function $G(\cdot,\cdot)$ (Definition 2.7). Then, it is well known (or see for instance [15]) that the unique bounded solution to equation (1.1) is

$$x(t) = \int_{\mathbb{R}} G(t,s)f(s)ds\,. \tag{1.3}$$

It is also well established that if A is an almost periodic matrix and f is an almost periodic function then, the solution to (1.1) given by (1.3) is also almost periodic [15, 21]. The almost periodicity of the solution x is rooted in the bi-almost periodicity of Green's function G and in the almost periodicity of f; the bi-almost periodicity of G arises from the almost periodicity of A.

The concepts of bi-almost periodicity and bi-almost automorphy of a continuous function of two variables were introduced in the work of Xiao et al. [33], and have proven to be useful in

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the study of almost periodic/automorphic solutions of non-autonomous differential and integral equations as shown in [3, 4, 14, 18, 29] and the references cited therein.

The concept of compact almost automorphic function (or motion) was introduced by Bender in his Ph.D. thesis [7] and has been further explored by Fink [19, 20]; this is a motion of a recurrent nature related to periodic, almost periodic, and almost automorphic types. A key relationship exists between these spaces: for a Banach space $\mathbb{X}$, denoting by $AP(\mathbb{R};\mathbb{X})$, $\mathcal{K}AA(\mathbb{R};\mathbb{X})$, and $AA(\mathbb{R};\mathbb{X})$ the spaces of almost periodic, compact almost automorphic, and almost automorphic functions, we have the strict inclusions:

$$AP(\mathbb{R};\mathbb{X}) \subset \mathcal{K}AA(\mathbb{R};\mathbb{X}) \subset AA(\mathbb{R};\mathbb{X}).$$

Moreover, while almost periodic and compact almost automorphic functions are uniformly continuous, general almost automorphic functions need not be.

Regarding the almost periodic dynamics of differential equations, Johnson showed that there exist linear almost periodic differential equations with a unique almost automorphic solution [24], highlighting the intriguing chaotic behavior of almost periodic systems.

Delay differential equations play a crucial role in various applications, as they model different real life phenomena, particularly in biological systems [30]. The study of almost periodic, compact almost automorphic, almost automorphic, and other recurrent solutions of delay differential equations has been extensively developed by various researchers, as seen in [1, 6, 16, 29, 34]. For example in [6], the authors studied the existence of a unique positive pseudo almost periodic solution for the generalized Mackey-Glass model with mixed delays,

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{N} \frac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))} - H(t, x(t-\sigma(t))), \tag{1.4}$$

where $1 < m \le n$ and $t \in \mathbb{R}$. Also, in the work [1], Abbas and his collaborators have studied the existence of a unique pseudo compact almost automorphic solution that is exponentially attractive of the abstract model with mixed delays,

$$\dot{u}(t) = -\alpha(t)u(t) + \sum_{i=1}^{n} \beta_i(t) f_i(\lambda_i(t)u(t-\tau_i(t))) + b(t)H(u(t)), \tag{1.5}$$

It is important to emphasize that, depending on the non-linearities encoded in f_i, equation (1.5) represent at least the following models:

(a) The Nicholson blowflies model: $f_i(z) = ze^{-z}$ (describe population dynamics of blowflies [22, 9]).
(b) The Lasota-Wazewska model: $f_i(z) = e^{-z}$ (describe the survival of red blood cells in animals [32]).
(c) The Mackey-Glass model: $f_{i,m}(z) = \dfrac{z}{1+z^m}$ (describe white blood cells production - hematopoiesis dynamics - [8]).

Thus, in the abstract model (1.5) it is considered a combination of the biological models listed above. Also note that, in the model (1.5) the nonlinear term H does not present a delay, while in equation (1.4) H does.

More recently, Zheng and Li [34] investigated the existence and stability of pseudo compact almost automorphic solutions for a family of differential equations with mixed delays, involving a nonlinear term H with a constant delay of the form $H(u(t-\sigma))$, where $\sigma > 0$. In their study, they introduced the interesting concepts of bi-pseudo almost automorphy and bi-uniformly continuous functions to achieve their results. Their primary focus was the analysis of the family of differential equations with mixed delays proposed in their work, and they did not conduct a complete analysis of the abstract differential equations (1.1) and (1.2).

The introduction of a time delay in the harvesting term, which occurs when harvesting decisions are based on outdated population data, presents mathematical challenges, particularly in the study of the existence of positive solutions. As noted in [25], since harvesting policies depend on population data, any delay in the data can lead to delayed responses in the model's harvesting term. This delay significantly impacts population management, making it essential to assess its effects for informed decision-making.

Despite the well understood theory for almost periodic systems, several significant gaps motivate the present work when considering the broader class of compact almost automorphic and almost automorphic motions.

The primary motivation stems from the goal of proving that if system (1.2) possesses an (α, K, P)-exponential dichotomy, A is compact almost automorphic (almost automorphic) and f is compact almost automorphic (almost automorphic), then the solution x given by (1.3) to equation (1.1) is also compact almost automorphic (almost automorphic). Achieving this goal faces several obstacles:

(1) The Green's function $G(\cdot,\cdot)$ is discontinuous along the principal diagonal of $\mathbb{R}^2$, though continuous almost everywhere. This means it is measurable but does not fit the notion of bi-almost periodicity/automorphy from [33], which applies only to continuous functions.
(2) Establishing that $G(\cdot,\cdot)$ is bi-almost automorphic is a complex task. For example, in infinite-dimensional settings, it is crucial to include as hypothesis that the resolvent operator is also almost automorphic; moreover, as discussed in [3, 4, 18, 26], a detailed analysis of the behavior of the Yosida approximations of the evolution family $A(\cdot)$ is necessary. A key question is whether this machinery is needed in finite dimensions. Fortunately, we will see that it is not.
(3) Previous approaches, such as that of Coronel et al. [16], introduced the concept of *integrable bi-almost automorphy.* In their work, the authors demonstrated that if f is almost automorphic and $G(\cdot,\cdot)$ is integrable bi-almost automorphic, then x in (1.3) is almost automorphic. Furthermore, they showed that the Green function $G(\cdot,\cdot)$ is integrable bi-almost automorphic if the projection matrix P commutes with the fundamental matrix solution of (1.2). In this way, a natural motivation was to avoid the notion of integrable bi-almost automorphy and the commutation relation described above.

The second fundamental motivation arises from the context of biological models. Specifically, we are interested in studying the abstract biological model (1.5) with a nonlinear term of the form $b(t)H(u(t-\sigma(t)))$ rather than $b(t)H(u(t))$ $(H > 0)$. This scenario, motivated by real world situations where harvesting decisions rely on outdated data [25], was not fully addressed in [1] and therefore merits further investigation.

This work provides the following specific contributions:

(C1) *Introduction of new concepts for measurable functions:* Since $G(\cdot,\cdot)$ is discontinuous, we introduce the notion of *(compact) bi-almost automorphy for measurable functions.* This extends the existing theory and provides the necessary framework to handle the Green's function.

(C2) *Stability of exponential dichotomy under Hull:* We prove that if $A(\cdot)$ is a compact almost automorphic matrix and the system (1.2) possesses an exponential dichotomy, then the system

$$y'(t) = B(t)y(t), \tag{1.6}$$

also possesses an exponential dichotomy for any $B(\cdot)$ in the hull of $A(\cdot)$, $\mathcal{H}(A)$. This result is the compact almost automorphic analogue of the classical theorem for almost periodic systems [21, Theorem 7.6].

(C3) *Compact Bi-almost automorphy of the Green's function:* As a consequence of the previous point, we demonstrate that the associated Green's function $G(\cdot,\cdot)$ is *compact bi-almost automorphic* (Theorem 3.1 in Section 3).

(C4) *Analysis of the Green's function's continuity:* Since $G(\cdot,\cdot)$ is not continuous and therefore it is not uniformly continuous in $\mathbb{R}^2$, we establish that $G(\cdot,\cdot)$ is Δ_2*-like uniformly continuous* (Lemma 3.4), where $\Delta_2 := \{(t,t) \in \mathbb{R}^2 : t \in \mathbb{R}\} \subset \mathbb{R}^2$ denotes the principal diagonal of $\mathbb{R}^2$. We perceive this notion as a weaker version of uniform continuity and, as we will see in section 4, it is very useful in showing the invariance of the compact almost automorhic function space under convolution products, see also the counterexample in Remark 4.3 and also Remark 4.4.

Based on the preceding contributions, we show that for the finite-dimensional system (1.1) and under exponential dichotomy, the solution x is compact almost automorphic (almost automorphic) whenever A and f are. This is achieved without the notion of integrable bi-almost automorphy and the commutation relation proposed in [16] or the machinery typically required in infinite dimensional analysis.

The last contribution in this work is the following:

(C5) *Application to abstract biological models:* We apply the abstract results to prove the *existence and global exponential stability of a unique positive compact almost automorphic solution* of the following delayed biological model with nonlinear harvesting or immigration term and mixed delays

$$\dot{u}(t) = -\alpha(t)u(t) + \sum_{i=1}^{n} \beta_i(t) f_i(\lambda_i(t)u(t-\tau_i(t)) + b(t)H(u(t-\sigma(t)))\,, \tag{1.7}$$

where, for instance, $\alpha(t)$, $\sigma(t)$, $\beta_i(t)$, $\lambda_i(t)$, $\tau_i(t)$ for $i = 1, 2, \cdots, n$ are positive compact almost automorphic functions on $\mathbb{R}$, $b(t)$ is compact almost automorphic and H is a non-negative Lipschitz function.

Although equation (1.7) is quite similar to equation (1.5), our study is conducted, in some instances, under different assumptions than those in [1] (see Remark 5.2 for details). In addition, the model is adapted to clarify certain aspects of previous investigations (see, for instance, Remark 5.8).

This work is organized as follows: in section 2, we revisit some preliminary facts on compact almost automorphic functions and introduce the notion of compact Bi-almost automorphy for measurable functions. In section 3 we prove the main results of this work, which are Theorem 3.1 and Lemma 3.4. In section 4, we use the Δ_2-like uniform continuity of $G(\cdot,\cdot)$ to demonstrate the invariance of the almost automorphic function space under convolution products whose kernel is the Green function $G(\cdot,\cdot)$ and we show that the unique bounded solution of a linear non-autonomous differential equation, under exponential dichotomy and with compact almost automorphic perturbation, is itself compact almost automorphic. Finally, in section 5 we prove the existence and global exponential stability of a unique positive compact almost automorphic solution of the delayed biological model (1.7).

2. Preliminaries

Before presenting the main results of this work, some preliminary statements are necessary. We begin by noting that throughout this work, $\mathbb{X}$ denotes a real or complex Banach space with norm $\|\cdot\|_{\mathbb{X}}$, $\|\cdot\|$ represents the matrix norm, $\|\cdot\|_\infty$ denotes the supremum norm, and $|\cdot|$ stands for the absolute value. $\Delta_2 := \{(t,t) \in \mathbb{R}^2 : t \in \mathbb{R}\} \subset \mathbb{R}^2$ denotes the principal diagonal of $\mathbb{R}^2$.

The following is the definition of a Bochner almost automorphic function [10, 11].

Definition 2.1. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be *almost automorphic* if, for every sequence $\{s'_n\} \subset \mathbb{R}$, there exist a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a function $\tilde{f} : \mathbb{R} \to \mathbb{X}$ such that the following pointwise limits are satisfied:

$$\lim_{n\to+\infty} f(t+s_n) = \tilde{f}(t), \quad \lim_{n\to+\infty} \tilde{f}(t-s_n) = f(t). \tag{2.1}$$

Definition 2.2. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be *compact almost automorphic* if, for every sequence $\{s'_n\} \subset \mathbb{R}$, there exist a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a function $\tilde{f} : \mathbb{R} \to \mathbb{X}$ such that the following limits hold:

$$\lim_{n\to+\infty} \sup_{t\in\mathcal{C}} \|f(t+s_n) - \tilde{f}(t)\|_{\mathbb{X}} = 0, \quad \lim_{n\to+\infty} \sup_{t\in\mathcal{C}} \|\tilde{f}(t-s_n) - f(t)\|_{\mathbb{X}} = 0, \tag{2.2}$$

where $\mathcal{C} \subset \mathbb{R}$ is compact.

Note that, the limit function $\tilde{f}$ in Definition 2.1 is not necessarily continuous, but is measurable; whereas, in Definition 2.2 $\tilde{f}$ is continuous. We denote by $\mathcal{K}AA(\mathbb{R},\mathbb{X})$ the space of compact almost automorphic functions and by $AA(\mathbb{R},\mathbb{X})$ the space of almost automorphic functions. Both $AA(\mathbb{R},\mathbb{X})$ and $\mathcal{K}AA(\mathbb{R},\mathbb{X})$ are Banach spaces under the supremum norm, and $\mathcal{K}AA(\mathbb{R},\mathbb{X}) \subset AA(\mathbb{R},\mathbb{X})$ holds; see [13].

For an almost automorphic function f, its *Hull*, denoted by $\mathcal{H}(f)$, is the set of all functions $\tilde{f}$ satisfying Definition 2.2. The Hull is an important notion in the analysis of almost periodic and almost automorphic dynamics of differential equations [21, 24].

The following result provides a useful characterization of compact almost automorphic functions on the real line. For the multidimensional Euclidean space, see [13].

Proposition 2.3. *A continuous function $f : \mathbb{R} \to \mathbb{X}$ is compact almost automorphic if and only if it is almost automorphic and uniformly continuous.*

The next proposition summarizes some key properties of compact almost automorphic functions. For further details, see [12, 13] and the books [17, 28].

Proposition 2.4. *We have:*

(1) *If $\mathbb{X}$ is a Banach algebra with norm $\|\cdot\|_{\mathbb{X}}$, addition $+_{\mathbb{X}}$, and multiplication $\times_{\mathbb{X}}$, then $\mathcal{K}AA(\mathbb{R},\mathbb{X})$ is also a Banach algebra with norm $\|\cdot\|_\infty$ and the operations: if $f, g \in \mathcal{K}AA(\mathbb{R};\mathbb{X})$, then*
$$(f+g)(t) := f(t) +_{\mathbb{X}} g(t), \quad (f\cdot g)(t) := f(t) \times_{\mathbb{X}} g(t), \quad t \in \mathbb{R}.$$
(2) *Let $f \in \mathcal{K}AA(\mathbb{R};\mathbb{X})$. Then:*
   (a) *f is bounded, and if $\tilde{f}$ is the limit function in Definition 2.2, then*
   $$\|f\|_\infty = \|\tilde{f}\|_\infty.$$
   (b) *If $F : \mathbb{X} \to \mathbb{Y}$ is a continuous function, then $F \circ f : \mathbb{R} \to \mathbb{Y}$ is compact almost automorphic.*
   (c) *The range of f is a relatively compact subset of $\mathbb{X}$*
   (d) *If $a \in \mathcal{K}AA(\mathbb{R};\mathbb{R})$, then $F(t) := f(t - a(t))$ is a compact almost automorphic function from $\mathbb{R}$ to $\mathbb{X}$.*

As mentioned in the introduction, the concepts of bi-almost automorphy and bi-compact almost automorphy play a crucial role in this work. In the following definitions, we introduce these two important concepts.

Definition 2.5. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ be a measurable function. We say that f is *bi-almost automorphic* if, for every sequence $\{s_n'\}_{n\in\mathbb{N}} \subset \mathbb{R}$, there exist a subsequence $\{s_n\} \subseteq \{s_n'\}$ and a function $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ such that, the following pointwise limits are satisfied:
$$\lim_{n\to+\infty} f(t+s_n, s+s_n) = \tilde{f}(t,s),$$
$$\lim_{n\to+\infty} \tilde{f}(t-s_n, s-s_n) = f(t,s).$$

Definition 2.6. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ be a measurable function. We say that f is *compact bi-almost automorphic* if, for every sequence $\{s_n'\}_{n\in\mathbb{N}} \subset \mathbb{R}$, there exist a subsequence $\{s_n\} \subseteq \{s_n'\}$ and a function $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ such that, for any compact set $\mathbb{K} \subset \mathbb{R} \times \mathbb{R}$, the following limits hold:
$$\lim_{n\to+\infty} \sup_{(t,s)\in\mathbb{K}} \|f(t+s_n, s+s_n) - \tilde{f}(t,s)\|_{\mathbb{X}} = 0,$$
$$\lim_{n\to+\infty} \sup_{(t,s)\in\mathbb{K}} \|\tilde{f}(t-s_n, s-s_n) - f(t,s)\|_{\mathbb{X}} = 0.$$

The notion of exponential dichotomy for homogeneous non-autonomous systems is granted in the following definition.

Definition 2.7 ([15])**.** Let $\Phi(\cdot)$ be a fundamental matrix of system (1.2). We say that (1.2) admits an exponential dichotomy with parameters (α, K, P) if there exist positive constants α, K and a projection matrix P ($P^2 = P$) such that $\|G(t,s)\| \le Ke^{-\alpha|t-s|}$ for all $t, s \in \mathbb{R}$, where $G(\cdot,\cdot)$ is the Green function defined by:
$$G(t,s) := \begin{cases} \Phi(t)P\Phi^{-1}(s), & \text{if } t \ge s, \\ -\Phi(t)(I-P)\Phi^{-1}(s), & \text{if } t < s. \end{cases}$$
In this case, we say that system (1.2) has an (α, K, P)-exponential dichotomy.

Now we present the definition of a λ-bounded function, a notion that was fundamental in [14] for analyzing almost automorphic-type solutions to nonlinear abstract integral equations involving both advanced and delayed terms.

Definition 2.8 ([14]). Let $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a positive function. We say that a measurable function $C : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is λ-*bounded* if, for every $\tau \in \mathbb{R}$ and each $t, s \in \mathbb{R}$, the following inequality holds

$$\|C(t+\tau, s+\tau)\|_{\mathbb{X}} \leq \lambda(t,s). \tag{2.3}$$

Example 2.9. If $G(\cdot,\cdot)$ is the Green function defined in Definition 2.7, then $G(\cdot,\cdot)$ is measurable and λ-bounded with $\lambda(t,s) = Ke^{-\alpha|t-s|}$.

[14, Lemma 2.9] establishes that any λ-bounded, bi-almost automorphic function $C : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ satisfies inequality (2.3) (Definition 2.8). We complement this result in the next lemma, demonstrating that the limit function $\tilde{C}$ (see Definition 2.8) is likewise λ-bounded.

Lemma 2.10. *Suppose that the bi-almost automorphic function $C : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is λ-bounded. Then, its limit function $\tilde{C} : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ (see Definition 2.8) satisfies*

$$\|\tilde{C}(t+\tau, s+\tau)\|_{\mathbb{X}} \leq \lambda(t,s), \quad \forall \tau \in \mathbb{R}.$$

Proof. Let $\{s'_n\} \subset \mathbb{R}$ be arbitrary. Since C is bi-almost automorphic, there exist a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a function $\tilde{C}$ such that the following pointwise limits hold:

$$\tilde{C}(t,s) := \lim_{n\to+\infty} C(t+s_n, s+s_n), \quad C(t,s) = \lim_{n\to+\infty} \tilde{C}(t-s_n, s-s_n).$$

On the other hand, we have:

$$\|\tilde{C}(t+\tau, s+\tau)\|_{\mathbb{X}} \leq \|\tilde{C}(t+\tau, s+\tau) - C(t+\tau+s_n, s+\tau+s_n)\|_{\mathbb{X}} + \lambda(t,s).$$

Taking the limit as $n \to +\infty$ in the last inequality yields the desired result. □

This result will be invoked in Section 4. For further details on almost automorphic functions, we refer to [17, 28], and for exponential dichotomy, see the classical books [15, 23].

3. Compact almost automorphic dynamics of homogeneous non-autonomous differential equations

In this section we study the compact almost automorphic dynamics of the system (1.2). Our main result establishes that if the matrix A is compact almost automorphic and system (1.2) is exponentially dichotomous, then the system

$$y'(t) = B(t)y(t), \quad B \in \mathcal{H}(A), \tag{3.1}$$

is likewise exponentially dichotomous. Furthermore, the associated Green's function $G(\cdot,\cdot)$ is compact bi-almost automorphic. These results are rigorously formulated in the following theorem.

Theorem 3.1. *Suppose that system* (1.2) *admits an (α, K, P)-exponential dichotomy with fundamental matrix $\Phi(t)$, where $\Phi(0) = I$, and $A(\cdot)$ is compact almost automorphic. That is, for any arbitrary sequence $\{s'_n\} \subset \mathbb{R}$, there exists a subsequence $\{s_n\} \subset \{s'_n\}$ such that the following limits hold uniformly on compact subsets of the real line:*

$$\lim_{n\to+\infty} A(t+s_n) = B(t), \quad \lim_{n\to+\infty} B(t-s_n) = A(t)\,; \tag{3.2}$$

then

(1) *There exists a projection matrix P_0 such that the system*

$$y'(t) = B(t)y(t) \tag{3.3}$$

admits an (α, K, P_0)-exponential dichotomy.

(2) *The associated Green function of system* (1.2) *is compact bi-almost automorphic.*

As previously mentioned, the first part of this result parallels the findings for almost periodic systems, as demonstrated in [21, Theorem 7.6]. Prior to proving Theorem 3.1, we first establish the following lemma.

Lemma 3.2. *Under the hypotheses of Theorem 3.1, there exists a subsequence $\{\zeta_n\} \subset \{s_n\}$ such that the following limits hold:*

$$\lim_{n\to+\infty} \sup_{t\in\mathcal{C}} \|\Phi_n(t) - \Psi(t)\| = 0, \quad \lim_{n\to+\infty} \sup_{t\in\mathcal{C}} \|\Phi_n^{-1}(t) - \Psi^{-1}(t)\| = 0\,;$$

where, $\mathcal{C} \subset \mathbb{R}$ is compact, $\Phi_n(t) := \Phi(t+\zeta_n)\Phi^{-1}(\zeta_n)$, and Ψ is a fundamental matrix of (3.3)*.*

Proof. It suffices to consider the compact interval $\mathcal{C} = [-a, a]$, where $a > 0$. From the hypothesis, there exists a subsequence $\{s_n\} \subset \{s'_n\}$ such that the limits in (3.2) hold. For this subsequence, consider the sequence $\Phi_n^*(t) = \Phi(t+s_n)\Phi^{-1}(s_n)$, $n \in \mathbb{N}$. Note that for each $n \in \mathbb{N}$, $\Phi_n^*(0) = I$, and $\Phi_n^*(\cdot)$ is a fundamental matrix of the system

$$x'(t) = A(t+s_n)x(t).$$

By direct integration, it follows that

$$\Phi_n^*(t) = I + \int_0^t A(u+s_n)\Phi_n^*(u)\,du.$$

Let $M > 0$ be the supremum of $\|A(\cdot)\|$. Then, by the Gronwall-Bellman inequality, we have

$$\|\Phi_n^*(t)\| \le \|I\|e^{Mt} \le \|I\|e^{Ma}.$$

Thus, Φ_n^* is uniformly bounded on $\mathcal{C}$. Moreover, since

$$(\Phi_n^*(t))' = A(t+s_n)\Phi_n^*(t), \quad \text{for } n \in \mathbb{N} \text{ and } t \in \mathcal{C}, \tag{3.4}$$

the sequence of derivatives $(\Phi_n^*(\cdot))'$ is also uniformly bounded over $\mathcal{C}$.

By the Arzelà-Ascoli theorem, the sequence $\{\Phi_n^*\}$ has a subsequence $\{\Phi_n\}$ (i.e., there exists an associated subsequence $\{\zeta_n\} \subseteq \{s_n\}$) such that $\Phi_n(t) = \Phi(t+\zeta_n)\Phi^{-1}(\zeta_n)$ converges uniformly on $\mathcal{C}$. That is, there exists Ψ such that

$$\lim_{n\to+\infty} \Phi_n = \Psi, \quad \text{uniformly on } \mathcal{C}. \tag{3.5}$$

On the other hand, since the subsequence $\{\Phi_n\}$ satisfies (3.4), we have

$$\Phi'_n(t) = A(t+s_n)\Phi_n(t), \quad n \in \mathbb{N},\ t \in \mathcal{C}.$$

Thus, Φ'_n converges uniformly on $\mathcal{C}$ to $B(t)\Psi(t)$. Consequently, $\Psi(\cdot)$ is differentiable, and

$$\lim_{n\to+\infty} \Phi'_n(t) = \Psi'(t), \quad \text{on } \mathcal{C}.$$

Since $\Psi'(t) = B(t)\Psi(t)$ and $\Psi(0) = I$, it follows that Ψ is a fundamental matrix of (3.3). Therefore, $\Psi(\cdot)$ is nonsingular, and hence Ψ^{-1} exists on $\mathcal{C}$.

Now we prove that the following limit holds uniformly on $\mathcal{C}$,

$$\lim_{n\to+\infty} \Phi_n^{-1}(t) = \Psi^{-1}(t).$$

First, note that Φ_n^{-1} satisfies the equation

$$x'(t) = -x(t)A(t+\zeta_n),$$

and, by direct integration, we have

$$\Phi_n^{-1}(t) = I - \int_0^t \Phi_n^{-1}(u)A(u+\zeta_n)\,du.$$

Similarly,

$$\Psi^{-1}(t) = I - \int_0^t \Psi^{-1}(u)B(u)\,du.$$

Then,

$$\begin{aligned}
\|\Phi_n^{-1}(t) - \Psi^{-1}(t)\| &\le \int_0^t \|\Psi^{-1}(u)B(u) - \Phi_n^{-1}(u)A(u+\zeta_n)\|\,du \\
&\le \int_0^t \|\Psi^{-1}(u) - \Phi_n^{-1}(u)\|\,du\,\|A\|_\infty
\end{aligned}$$

$$+\int_0^t \|A(u+\zeta_n)-B(u)\|\,du\,\|\Psi^{-1}\|_\infty.$$

Since the limits in (3.2) are uniform on $\mathcal{C}$, for any $\epsilon>0$, there exists $N_0\in\mathbb{N}$ such that for all $n\geq N_0$,

$$\sup_{t\in\mathcal{C}}\|A(t+\zeta_n)-B(t)\|<\epsilon.$$

Therefore, from the inequality above, for $n\geq N_0$,

$$\|\Phi_n^{-1}(t)-\Psi^{-1}(t)\|\leq \epsilon a\|\Psi^{-1}\|_\infty+\int_0^t\|\Psi^{-1}(u)-\Phi_n^{-1}(u)\|\,du\,\|A\|_\infty.$$

By the Gronwall-Bellman inequality, we conclude that

$$\|\Phi_n^{-1}(t)-\Psi^{-1}(t)\|\leq \epsilon a\|\Psi^{-1}\|_\infty e^{\|A\|_\infty a}.$$

This completes the proof. □

We now prove the main result of this section.

Proof of Theorem 3.1. Let Φ be a fundamental matrix of (1.2), and let

$$G(t,s):=\begin{cases}\Phi(t)P\Phi^{-1}(s), & \text{if } t\geq s,\\ -\Phi(t)(I-P)\Phi^{-1}(s), & \text{if } t<s,\end{cases} \tag{3.6}$$

be its associated Green function satisfying $\|G(t,s)\|\leq Ke^{-\alpha|t-s|}$ for all $t,s\in\mathbb{R}$. To enhance clarity, we divide the proof into three steps. In step one, we prove the first part of this theorem, while steps two and three are dedicated to proving the second part.

Step 1: System (3.3) admits an (α,K,P_0)-exponential dichotomy. By Lemma 3.2, there exists a subsequence $\{\zeta_n\}\subset\{s_n\}$ such that the following limits hold uniformly on compact subsets of $\mathbb{R}$:

$$\lim_{n\to+\infty}\Phi_n(t)=\Psi(t)\quad\text{and}\quad\lim_{n\to+\infty}\Phi_n^{-1}(t)=\Psi^{-1}(t), \tag{3.7}$$

where $\Phi_n(t)=\Phi(t+\zeta_n)\Phi^{-1}(\zeta_n)$ and Ψ is a fundamental matrix of (3.3).

Since for all $n\in\mathbb{N}$, $\|\Phi(\zeta_n)P\Phi^{-1}(\zeta_n)\|\leq K$ and $\|\Phi(\zeta_n)Q\Phi^{-1}(\zeta_n)\|\leq K$, where $Q=I-P$, there exists a subsequence $\{\eta_n\}\subset\{\zeta_n\}$ such that:

$$\lim_{n\to+\infty}\Phi(\eta_n)P\Phi^{-1}(\eta_n)=:P_0,$$
$$\lim_{n\to+\infty}\Phi(\eta_n)Q\Phi^{-1}(\eta_n)=:Q_0.$$

Observe that $P_0^2=P_0$ and $P_0+Q_0=I$. Taking $n\to+\infty$ in the inequalities

$$\big\|\big(\Phi(t+\eta_n)\Phi^{-1}(\eta_n)\big)\Phi(\eta_n)P\Phi^{-1}(\eta_n)\big(\Phi(s+\eta_n)\Phi^{-1}(\eta_n)\big)^{-1}\big\|\leq Ke^{-\alpha(t-s)},\quad t\geq s,$$
$$\big\|\big(\Phi(t+\eta_n)\Phi^{-1}(\eta_n)\big)\Phi(\eta_n)Q\Phi^{-1}(\eta_n)\big(\Phi(s+\eta_n)\Phi^{-1}(\eta_n)\big)^{-1}\big\|\leq Ke^{-\alpha(s-t)},\quad t<s,$$

we conclude that

$$\|\Psi(t)P_0\Psi^{-1}(s)\|\leq Ke^{-\alpha(t-s)},\quad t\geq s,$$
$$\|\Psi(t)Q_0\Psi^{-1}(s)\|\leq Ke^{-\alpha(s-t)},\quad t<s.$$

Thus, there exists a projection matrix P_0 such that system (3.3) admits an (α,K,P_0)-exponential dichotomy with Green function

$$\tilde{G}(t,s):=\begin{cases}\Psi(t)P_0\Psi^{-1}(s), & \text{if } t\geq s,\\ -\Psi(t)Q_0\Psi^{-1}(s), & \text{if } t<s.\end{cases} \tag{3.8}$$

Step 2: The Green function $G(\cdot,\cdot)$ is bi-almost automorphic. Let $\{s'_n\}$ be an arbitrary sequence. By hypothesis and Step 1, there exists a subsequence $\{\eta_n\} \subset \{s'_n\}$ such that the following pointwise limit holds,

$$\lim_{n\to+\infty} G(t+\eta_n, s+\eta_n) = \tilde{G}(t,s),$$

where

$$G(t+\eta_n, s+\eta_n) := \begin{cases} \Phi_n(t)P_n\Phi_n^{-1}(s), & \text{if } t \geq s, \\ -\Phi_n(t)Q_n\Phi_n^{-1}(s), & \text{if } t < s, \end{cases} \tag{3.9}$$

with $\Phi_n(t) = \Phi(t+\eta_n)\Phi^{-1}(\eta_n)$, $P_n := \Phi(\eta_n)P\Phi^{-1}(\eta_n)$, and $Q_n := \Phi(\eta_n)Q\Phi^{-1}(\eta_n)$, and $\tilde{G}(\cdot,\cdot)$ is defined in (3.8).

We claim that there exists a subsequence $\{\xi_n\} \subseteq \{\eta_n\}$ such that

$$\lim_{n\to+\infty} \tilde{G}(t-\xi_n, s-\xi_n) = G(t,s).$$

Indeed, since $\Psi(\cdot)$, with $\Psi(0) = I$, is a fundamental matrix of (3.3), for each $n \in \mathbb{N}$, $\Psi_n(t) := \Psi(t-\eta_n)\Psi^{-1}(-\eta_n)$ is a fundamental matrix of the system

$$z'(t) = B(t-\eta_n)z.$$

Arguing as in Step 1, we can find a subsequence $\{\xi_n\} \subset \{\eta_n\}$ such that

$$\lim_{n\to+\infty} \Psi(-\xi_n)P_0\Psi^{-1}(-\xi_n) =: \tilde{P}_1,$$
$$\lim_{n\to+\infty} \Psi(-\xi_n)Q_0\Psi^{-1}(-\xi_n) =: \tilde{Q}_1.$$

Observe that $\tilde{P}_1^2 = \tilde{P}_1$ and $\tilde{P}_1 + \tilde{Q}_1 = I$. Moreover, as in the proof of Lemma 3.2, the limits $\lim_{n\to+\infty} B(t-\xi_n) = A(t)$ and $\lim_{n\to+\infty} \Psi_n(t) = \Upsilon(t)$ are uniform on compact subsets of $\mathbb{R}$, where $\Upsilon(t)$ is a fundamental matrix of system (1.2). Note that $\Upsilon(0) = I$. Thus, there exists a nonsingular matrix C such that $\Upsilon(t) = \Phi(t)C$. Since $\Upsilon(0) = \Phi(0) = I$, we have $C = I$, so $\Upsilon(t) = \Phi(t)$. Furthermore, since the projection for an exponential dichotomy is unique, $\tilde{P}_1 = P$ and $\tilde{Q}_1 = Q$. This reasoning implies that

$$\lim_{n\to+\infty} \tilde{G}(t-\xi_n, s-\xi_n) = G(t,s),$$

for each point $(t,s) \in \mathbb{R}^2$, as claimed.

Step 3: The Green function $G(\cdot,\cdot)$ is compact bi-almost automorphic. From Lemma 3.2, there exists a subsequence $\{\eta_n\} \subset \{s_n\}$ such that if $\mathcal{C} = [a,b]$ is a compact subset of $\mathbb{R}$, then the following limits hold:

$$\lim_{n\to+\infty} \sup_{t\in\mathcal{C}} \|\Phi_n(t) - \Psi(t)\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \sup_{t\in\mathcal{C}} \|\Phi_n^{-1}(t) - \Psi^{-1}(t)\| = 0, \tag{3.10}$$

where $\Phi_n(t) = \Phi(t+\eta_n)\Phi^{-1}(\eta_n)$ and Ψ is a fundamental matrix of (3.3). Additionally, there exist positive constants C_1, C_2, C_3 such that for every $n \in \mathbb{N}$ and $t \in \mathcal{C}$, we have:

$$\|\Phi_n(t)\| \leq C_1, \quad \|\Phi_n^{-1}(t)\| \leq C_2, \quad \|\Psi(t)\| \leq C_3. \tag{3.11}$$

Moreover,

$$P_n = \Phi(\eta_n)P\Phi^{-1}(\eta_n) \to P_0, \quad \text{as } n \to +\infty, \tag{3.12}$$
$$Q_n = \Phi(\eta_n)Q\Phi^{-1}(\eta_n) \to Q_0, \quad \text{as } n \to +\infty. \tag{3.13}$$

We claim that if $\mathbb{K} = [a,b] \times [a,b] \subset \mathbb{R}^2$, then

$$\lim_{n\to+\infty} \sup_{(t,s)\in\mathbb{K}} \|G(t+\eta_n, s+\eta_n) - \tilde{G}(t,s)\| = 0, \tag{3.14}$$

where $G(t+\eta_n, s+\eta_n)$ is defined in (3.9). Indeed:

Case 1: $t \geq s$.

$$\begin{aligned}
\|G(t+s_n, s+s_n) - \tilde{G}(t,s)\| &= \|\Phi_n(t)P_n\Phi_n^{-1}(s) - \Psi(t)P_0\Psi^{-1}(s)\| \\
&= \|\Phi_n(t)(P_n - P_0 + P_0)\Phi_n^{-1}(s) - \Psi(t)P_0\Psi^{-1}(s)\| \\
&= \|\Phi_n(t)(P_n - P_0)\Phi_n^{-1}(s) + \Phi_n(t)P_0\Phi_n^{-1}(s) - \Psi(t)P_0\Psi^{-1}(s)\| \\
&= \|\Phi_n(t)(P_n - P_0)\Phi_n^{-1}(s) + (\Phi_n(t) - \Psi(t))P_0\Phi_n^{-1}(s) \\
&\quad + \Psi(t)P_0(\Phi_n^{-1}(s) - \Psi^{-1}(s))\| \\
&\leq \|\Phi_n(t)\|\|P_n - P_0\|\|\Phi_n^{-1}(s)\| + \|\Phi_n(t) - \Psi(t)\|\|P_0\|\|\Phi_n^{-1}(s)\| \\
&\quad + \|\Psi(t)\|\|P_0\|\|\Phi_n^{-1}(s) - \Psi^{-1}(s)\| \\
&< A'\|P_n - P_0\| + B'\|\Phi_n(t) - \Psi(t)\| + C'\|\Phi_n^{-1}(s) - \Psi^{-1}(s)\|,
\end{aligned}$$

for some positive constants A', B', C'. Thus,

$$\begin{aligned}
\|G(t+s_n, s+s_n) - \tilde{G}(t,s)\| &< A'\|P_n - P_0\| + B'\sup_{t\in[a,b]}\|\Phi_n(t) - \Psi(t)\| \\
&\quad + C'\sup_{s\in[a,b]}\|\Phi_n^{-1}(s) - \Psi^{-1}(s)\|.
\end{aligned}$$

Case 2: $t < s$. Similarly to the case 1, we obtain

$$\begin{aligned}
\|G(t+s_n, s+s_n) - \tilde{G}(t,s)\| &= \|\Phi_n(t)Q_n\Phi_n^{-1}(s) - \Psi(t)Q_0\Psi^{-1}(s)\| \\
&< A'\|Q_n - Q_0\| + B'\sup_{t\in[a,b]}\|\Phi_n(t) - \Psi(t)\| \\
&\quad + C'\sup_{s\in[a,b]}\|\Phi_n^{-1}(s) - \Psi^{-1}(s)\|.
\end{aligned}$$

Therefore, from Cases 1 and 2, along with (3.10), (3.12), and (3.13), we conclude (3.14).

Analogously, a subsequence $\{\xi_n\} \subset \{\eta_n\}$ can be obtained such that the following limit holds on compact sets $\mathbb{K} \subset \mathbb{R}^2$:

$$\lim_{n\to+\infty}\sup_{(t,s)\in\mathbb{K}}\|\tilde{G}(t-\xi_n, s-\xi_n) - G(t,s)\| = 0. \qquad \square$$

Remark 3.3. Although $G(\cdot,\cdot)$ is measurable, discontinuous, and compact bi-almost automorphic, Theorem 3.1 does not contradict the result by Veech [31], in which he established that *on locally compact groups, Haar measurable almost automorphic functions are continuous.* Rather, this accentuate a fundamental distinction between:

- Two dimensional Bochner (compact) almost automorphic functions [13], where sequences are considered in the full Euclidean space $\mathbb{R}^2$ (as in this case), and
- Measurable (compact) bi-almost automorphic functions [12], where sequences are restricted to the proper subset $\Delta_2 \subset \mathbb{R}^2$ - the principal diagonal of $\mathbb{R}^2$.

The following lemma plays a pivotal role in establishing the invariance of the compact almost automorphic function space under convolution products with the Green function $G(\cdot,\cdot)$ as the kernel (Theorem 4.1).

Lemma 3.4 (Δ_2-like uniform continuity of the Green function $G(\cdot,\cdot)$)**.** *Let $\{t_n\}$ and $\{s_n\}$ be real sequences such that $|t_n - s_n| \to 0$ as $n \to +\infty$. Then, for each $(t,s) \in \mathbb{R}^2$, we have*

$$\|G(t+t_n, s+t_n) - G(t+s_n, s+s_n)\| \to 0 \quad \text{as } n \to +\infty.$$

Proof. Let us denote by $\mathcal{U} := \{(t,s) \in \mathbb{R}^2 : t < s\}$ and by $\mathcal{V} := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$. Define $G^2(t,s) := -\Phi(t)Q\Phi^{-1}(s)$ and $G^1(t,s) := \Phi(t)P\Phi^{-1}(s)$. Since $\mathcal{U}$ and $\mathcal{V}$ form a partition of $\mathbb{R}^2$, any point (t,s) belongs either to $\mathcal{U}$ or to $\mathcal{V}$.

Suppose $(t,s) \in \mathcal{U}$ and consider the sequence

$$\alpha_n(t,s) := \|G^2(t+t_n, s+t_n) - G^2(t+s_n, s+s_n)\|. \tag{3.15}$$

We claim that $\alpha_n(t,s) \to 0$ as $n \to +\infty$. To prove this, let $\{\alpha'_n(t,s)\} \subset \{\alpha_n(t,s)\}$ be a convergent subsequence converging to $\psi_0(t,s)$, where

$$\alpha'_n(t,s) := \|G^2(t+t'_n, s+t'_n) - G^2(t+s'_n, s+s'_n)\|,$$

with $\{t'_n\} \subset \{t_n\}$ and $\{s'_n\} \subset \{s_n\}$. We must show that $\psi_0(t,s) = 0$.

Since $G^2(\cdot,\cdot)$ is compact bi-almost automorphic on $\mathcal{U}$, there exists a subsequence $\{s''_n\} \subset \{s'_n\}$ and a function $\tilde{G}^2$ such that the following limits hold:

$$\lim_{n\to+\infty} \sup_{(t,s)\in\mathbb{K}} \|G^2(t+s''_n, s+s''_n) - \tilde{G}^2(t,s)\| = 0, \tag{3.16}$$

$$\lim_{n\to+\infty} \sup_{(t,s)\in\mathbb{K}} \|\tilde{G}^2(t-s''_n, s-s''_n) - G^2(t,s)\| = 0, \tag{3.17}$$

where $\mathbb{K}$ is a compact subset of $\mathcal{U}$. As a consequence, the limit function $\tilde{G}^2$ is continuous on $\mathcal{U}$.

Since $\{|t_n - s_n|\}$ is bounded, there exists $R > 0$ such that $t_n - s_n \in [-R, R]$ (a compact interval in $\mathbb{R}$). Therefore, the points $(t_n - s_n, t_n - s_n) \in L_R$, where $L_R := \{(z,z) \in \mathbb{R}^2 : |z| \le \sqrt{2}R\}$ is a compact subset of (the diagonal of) $\mathbb{R}^2$. This implies that $L_R(t,s) := (t,s) + L_R$ is a compact subset of $\mathcal{U}$.

Now, from the inequalities

$$\begin{aligned}
\alpha''_n(t,s) &= \|G^2(t+t''_n, s+t''_n) - G^2(t+s''_n, s+s''_n)\| \\
&\le \|G^2(t+t''_n - s''_n + s''_n, s+t''_n - s''_n + s''_n) - \tilde{G}^2(t+t''_n - s''_n, s+t''_n - s''_n)\| \\
&\quad + \|\tilde{G}^2(t+t''_n - s''_n, s+t''_n - s''_n) - \tilde{G}^2(t,s)\| \\
&\quad + \|\tilde{G}^2(t,s) - G^2(t+s''_n, s+s''_n)\| \\
&\le \sup_{(z_1,z_2)\in L_R(t,s)} \|G^2(z_1+s''_n, z_2+s''_n) - \tilde{G}^2(z_1,z_2)\| \\
&\quad + \|\tilde{G}^2(t+t''_n - s''_n, s+t''_n - s''_n) - \tilde{G}^2(t,s)\| \\
&\quad + \|\tilde{G}^2(t,s) - G^2(t+s''_n, s+s''_n)\|,
\end{aligned}$$

and using (3.16) and the continuity of $\tilde{G}^2$, we conclude that $\alpha''_n(t,s) \to 0$ as $n \to +\infty$. This implies that $\psi_0(t,s) = 0$, and hence $\alpha_n(t,s) \to 0$ as $n \to +\infty$, as claimed.

Analogously, it can be shown that if $(t,s) \in \mathcal{V}$, then

$$\lim_{n\to+\infty} \|G^1(t+t_n, s+t_n) - G^1(t+s_n, s+s_n)\| = 0. \qquad \square$$

4. Compact almost automorphic solutions of non-autonomous linear differential equations

In this section, we prove that if system (1.2) is exponentially dichotomous with a compact almost automorphic matrix A, then for a compact almost automorphic function f, the unique solution x of system (1.1), given by (1.3), is also compact almost automorphic. Before that, we establish the invariance of the space $\mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$ under convolution products with the Green's function $G(\cdot,\cdot)$ as the kernel.

4.1. Invariance of $\mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$ under convolution products with kernel $G(\cdot,\cdot)$.

Theorem 4.1. *Let $A(\cdot)$ be a compact almost automorphic matrix, and suppose that system* (1.2) *has an (α, K, P)-exponential dichotomy with Green function $G(\cdot,\cdot)$. Then the operator $\mathcal{G}_1$, defined by*

$$\mathcal{G}_1 u(t) := \int_{\mathbb{R}} G(t,s)u(s)\,ds, \tag{4.1}$$

leaves invariant the space $\mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$.

Proof. Let $u \in \mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$. By Proposition 2.3, it suffices to prove that $\mathcal{G}_1 u$ is almost automorphic and uniformly continuous. We will accomplish this in two steps.

Step 1: $\mathcal{G}_1 u$ is almost automorphic. Let $\{s'_n\}$ be any real sequence. Since $G(\cdot,\cdot)$ is compact bi-almost automorphic and u is compact almost automorphic, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$ such that the following uniform limits hold on compact subsets of $\mathbb{R}\times\mathbb{R}$:

$$\lim_{n\to+\infty} G(t+s_n, s+s_n) = \tilde{G}(t,s), \quad \lim_{n\to+\infty} \tilde{G}(t-s_n, s-s_n) = G(t,s),$$

and on compact subsets of $\mathbb{R}$, the following uniform limits also hold:

$$\lim_{n\to+\infty} u(t+s_n) = \tilde{u}(t), \quad \lim_{n\to+\infty} \tilde{u}(t-s_n) = u(t).$$

In particular, these limits are pointwise.

Let $v := \mathcal{G}_1 u$. Since $G(\cdot,\cdot)$ is λ-bounded for $\lambda(t,s) = Ke^{-\alpha|t-s|}$, $t,s\in\mathbb{R}$, by Lemma 2.10 and using the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to+\infty} v(t+s_n) = \tilde{v}(t), \quad \lim_{n\to+\infty} \tilde{v}(t-s_n) = v(t), \tag{4.2}$$

where

$$\tilde{v}(t) := \int_{-\infty}^{+\infty} \tilde{G}(t,s)\tilde{u}(s)\,ds.$$

Thus, v is almost automorphic.

Step 2: $\mathcal{G}_1 u$ is uniformly continuous. Let $\{t_n\}$ and $\{s_n\}$ be two sequences in $\mathbb{R}$ such that $|t_n - s_n| \to 0$ as $n\to+\infty$. Then

$$\begin{aligned}
|\mathcal{G}_1 u(t_n) - \mathcal{G}_1 u(s_n)| &\le \int_{\mathbb{R}} \|G(t_n, s+t_n) - G(s_n, s+s_n)\| |u(s+t_n)|\,ds \\
&\quad + \int_{\mathbb{R}} \|G(s_n, s+s_n)\| |u(s+t_n) - u(s+s_n)|\,ds \\
&=: I_1(n) + I_2(n)\,,
\end{aligned}$$

where:

$$\begin{aligned}
I_1(n) &:= \int_{\mathbb{R}} \|G(t_n, s+t_n) - G(s_n, s+s_n)\| |u(s+t_n)|\,ds\,, \\
I_2(n) &:= \int_{\mathbb{R}} \|G(s_n, s+s_n)\| |u(s+t_n) - u(s+s_n)|\,ds\,.
\end{aligned}$$

By Lemma 2.10, the Δ_2-like uniform continuity of $G(\cdot,\cdot)$ (Lemma 3.4), the uniform continuity of u, and the Lebesgue Dominated Convergence Theorem, we conclude that

$$\lim_{n\to+\infty} I_1(n) = \lim_{n\to+\infty} I_2(n) = 0. \qquad \square$$

The proof of the following theorem is analogous to that of Theorem 4.1.

Theorem 4.2. *Let $A(\cdot)$ be a compact almost automorphic matrix, and suppose that system* (1.2) *has an (α, K, P)-exponential dichotomy with Green function $G(\cdot,\cdot)$. Then the operator $\mathcal{G}_2$, defined by*

$$\mathcal{G}_2 u(t) := \int_{-\infty}^{t} G(t,s)u(s)\,ds, \tag{4.3}$$

leaves invariant the space $\mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$.

Remark 4.3. Here we present a counterexample to [1, Corollary 1]. Let us consider $f \in AA(\mathbb{R};\mathbb{R}) \setminus \mathcal{K}AA(\mathbb{R};\mathbb{R})$ and let $g(t,s) = f(t)\exp(-\alpha|t-s|)$ where $\alpha > 0$, then we have that $g(t,s)$ is bi-almost automorphic, $|g(t,s)| \le ||f||_\infty e^{-\alpha|t-s|}$. Let $u = 1$ be the constant function, which is compact almost automorphic. Then, the new function

$$F(t) = \int_{-\infty}^{t} g(t,s)u(s)ds\,,$$

is almost automorphic, but is not compact almost automorphic. In fact, $F(t) = f(t)/\alpha$.

Remark 4.4. From the proof of Theorem 4.1, we observe that ensuring the invariance of the space $\mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$ under the convolution products defined in (4.1) and (4.3) requires the Green's function $G(\cdot,\cdot)$ to be Δ_2-like uniformly continuous (as established in Lemma 3.4). Here, we complement the previous remark by showing that neither the Δ_2-like uniform continuity nor the compact bi-almost automorphy of $G(\cdot,\cdot)$ can be deduced solely from its λ-boundedness (e.g., if $\lambda(t,s) = Ke^{-\alpha|t-s|}$) and its bi-almost automorphic properties. Indeed, let $f \in \mathcal{K}AA(\mathbb{R};\mathbb{R})$ with $\inf_{t\in\mathbb{R}} |f(t)| \geq \delta > 0$, and take $g \in AA(\mathbb{R};\mathbb{R}) \setminus \mathcal{K}AA(\mathbb{R};\mathbb{R})$. Then, the function $G(t,s) := f(t)g(s)e^{-\alpha|t-s|}$ is λ-bounded, with $\lambda(t,s) = ||f||_\infty ||g||_\infty e^{-\alpha|t-s|}$ and is bi-almost automorphic but fails to be compact bi-almost automorphic or Δ_2-like uniformly continuous. To see the last assertion, suppose that $G(\cdot,\cdot)$ is Δ_2-like uniformly continuous. Pick two real sequences $\{t_n\}$ and $\{s_n\}$ such that $|t_n - s_n| \to 0$ as $n \to +\infty$, then from the Δ_2-like uniform continuity of G, the uniform continuity of f, and the following inequality:

$$\delta|g(t_n) - g(s_n)| \leq ||g||_\infty |f(t_n) - f(s_n)| + |G(t_n,t_n) - G(s_n,s_n)|\,,$$

we conclude that g is uniformly continuous, a contradiction.

Proposition 4.5. *If $a(\cdot)$ is compact almost automorphic, then*

$$G_0(t,s) := \int_s^t a(\xi)\,d\xi \quad \text{and} \quad G_1(t,s) := \exp\left(\int_s^t a(\xi)\,d\xi\right)$$

are compact bi-almost automorphic.

Of course, if in Proposition 4.5, $a(\cdot)$ is only almost automorphic, then $G_0(\cdot,\cdot)$ and $G_1(\cdot,\cdot)$ are bi-almost automorphic. Note that, $G_0(\cdot,\cdot)$ and $G_1(\cdot,\cdot)$ are not necessarily bounded functions, to see this consider $a(\cdot)$ as a positive almost automorphic function with $\inf\limits_{t\in\mathbb{R}} a(t) > \delta_0 > 0$.

4.2. **Compact almost automorphy of the solution.** The next theorem states the main result of this section.

Theorem 4.6. *Let $A(\cdot)$ be a compact almost automorphic matrix and $f \in \mathcal{K}AA(\mathbb{R};\mathbb{R}^p)$. If the system*

$$x'(t) = A(t)x(t) \tag{4.4}$$

has an (α,K,P)-exponential dichotomy with Green function $G(\cdot,\cdot)$, then the linear system

$$x'(t) = A(t)x(t) + f(t)\,,$$

has a unique compact almost automorphic solution x, given by

$$x(t) = \int_{\mathbb{R}} G(t,s)f(s)\,ds.$$

Moreover, if $\Phi(\cdot)$ is the fundamental matrix solution of system (4.4) *and P is the projection, then the solution x can also be expressed as*

$$x(t) = \int_{-\infty}^t \Phi(t)P\Phi^{-1}(s)f(s)\,ds + \int_t^\infty \Phi(t)(I-P)\Phi^{-1}(s)f(s)\,ds. \tag{4.5}$$

Note that if, in the previous theorem, f is assumed to be almost automorphic (rather than compact almost automorphic), then the solution x is likewise almost automorphic.

5. Existence and exponential stability of a positive compact almost automorphic solution of some delayed biological models

In this section, we study the existence and global exponential stability of a unique positive compact almost automorphic solution for the following biological model with nonlinear harvesting terms and mixed delays,

$$\dot{x}(t) = -\alpha(t)x(t) + \sum_{i=1}^n \beta_i(t)f_i\big(\lambda_i(t)x(t-\tau_i(t))\big) + b(t)H\big(x(t-\sigma(t))\big). \tag{5.1}$$

For the results in this section, we follow closely the work of Abbas et al. [1]. We begin by introducing some standard preliminaries and notation.

Let f be a real and bounded continuous function defined on its domain $D_f \subset \mathbb{R}$, then $\overline{f}$, $\underline{f}$ are defined as follows:

$$\overline{f} := \sup_{t\in D_f} f(t), \quad \underline{f} := \inf_{t\in D_f} f(t).$$

We also define the positive constants τ and $\overline{\lambda}$ as follows:

$$\tau := \max\big\{ \max_{1\le i\le n} \overline{\tau_i}, \overline{\sigma}\big\}, \quad \overline{\lambda} := \max_{1\le i\le n} \overline{\lambda_i}\,, \quad \underline{\lambda} := \min_{1\le i\le n} \underline{\lambda_i}\,.$$

Let $C([-\tau,0],\mathbb{R})$ denote the Banach space of continuous functions from $[-\tau,0]$ to $\mathbb{R}$ under the norm $\|\phi\|_\tau = \sup_{\theta\in[-\tau,0]}|\phi(\theta)|$. Let $\mathcal{C}^+$ be the cone of non-negative functions in $C([-\tau,0],\mathbb{R})$, i.e.,

$$\mathcal{C}^+ = \{\phi \in C([-\tau,0],\mathbb{R}) : \phi(t) \ge 0\},$$

and define the set

$$\mathcal{C}_0^+ = \{\phi \in \mathcal{C}^+ : \phi(0) > 0\}.$$

Because of biological interpretations, the set of admissible initial conditions for equation (5.1) is considered in $\mathcal{C}_0^+$.

If $x(\cdot)$ is defined on the interval $[t_0-\tau,\sigma]$ with $t_0,\sigma\in\mathbb{R}$, then the function $x_t \in C([-\tau,0],\mathbb{R})$ is defined by $x_t(\theta) := x(t+\theta)$ for all $\theta\in[-\tau,0]$ and $t_0 \le t \le \sigma$. Therefore, the initial condition for equation (5.1) is a function, i.e.

$$x_{t_0} = \phi, \quad \phi \in \mathcal{C}_0^+. \tag{5.2}$$

A solution of the IVP (5.1)-(5.2) is denoted by $x(t;t_0,\phi)$, but for convenience, we will denote it by $x(t)$.

5.1. **Existence and uniqueness.** As a first hypothesis, we assumed the following:

- (A1) The functions $\alpha(t)$, $\sigma(t)$, $\beta_i(t)$, $\lambda_i(t)$, and $\tau_i(t)$ for $i = 1,2,\dots,n$ are positive compact almost automorphic functions on $\mathbb{R}$ and $b(t)$ is compact almost automorphic on $\mathbb{R}$.
- (A2) $\underline{\alpha} > 0$ and, for every $i\in\{1,2,\cdots,n\}$, $\underline{\beta_i} > 0$ and $\underline{\lambda_i} > 0$.
- (A3) The function $H : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is bounded and Lipschitz continuous, i.e., there exists a positive constant L_H such that:

$$|H(a)-H(b)| \le L_H|a-b| \quad \text{for } a,b\in\mathbb{R}_0^+.$$

Proposition 5.1. *If conditions* (A1)–(A3) *hold, then any solution x of equation* (5.1) *with initial condition $\phi\in\mathcal{C}_0^+$ satisfies*

$$\frac{\sum_{i=1}^n \underline{\beta_i}\,\underline{f_i} + \underline{b}\,\underline{H}}{\overline{\alpha}} \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le \frac{\sum_{i=1}^n \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H}}{\underline{\alpha}}.$$

Proof. From (5.1), we have

$$-\overline{\alpha}\,x(t) + \sum_{i=1}^n \underline{\beta}_i\,\underline{f_i} + \underline{b}\,\underline{H} \le x'(t) \le -\underline{\alpha}\,x(t) + \sum_{i=1}^n \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H}\,.$$

Then, from the comparison principle [2, Lemma 1.1], we have

$$x(t) \le x(0)e^{-\underline{\alpha}t} + \frac{\sum_{i=1}^n \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H}}{\underline{\alpha}}\left(1-e^{-\underline{\alpha}\,t}\right)\,. \tag{5.3}$$

Also, from the inequality

$$-x'(t) \le -\overline{\alpha}(-x(t)) - \Big(\sum_{i=1}^n \underline{\beta}_i\,\underline{f_i} + \underline{b}\,\underline{H}\Big)$$

and, using [2, Lemma 1.1] again, we have:

$$-x(t) \le -x(0)e^{-\overline{\alpha}t} - \frac{\sum_{i=1}^n \underline{\beta}_i\,\underline{f_i} + \underline{b}\,\underline{H}}{\overline{\alpha}}\big(1-e^{-\overline{\alpha}t}\big)\,;$$

that is,

$$x(t) \geq x(0)e^{-\overline{\alpha}t} + \frac{\sum_{i=1}^{n} \underline{\beta}_i \, \underline{f_i} + \underline{b}\, \underline{H}}{\overline{\alpha}} \left(1 - e^{-\overline{\alpha}t}\right) . \tag{5.4}$$

Now, the conclusion follows from inequalities (5.3) and (5.4). □

Remark 5.2. We remark that

(a) If in Proposition 5.1 the inequality

$$0 \leq \frac{\sum_{i=1}^{n} \underline{\beta_i} \, \underline{f_i} + \underline{b}\, \underline{H}}{\overline{\alpha}}$$

holds, then it implies that all solutions of equation (5.1) are biologically meaningful.

(b) Unlike in [1, 34], our Assumption (A3) does not require $H(0) = 0$. This leads to different lower and upper bounds for the solution x in Proposition 5.1.

(c) In our Assumption (A1), the function b is not required to be a non-negative compact almost automorphic function. This contrasts with the work of Abbas et al. [1], where the non-negativity of b is essential to establish the analogous result to Proposition 5.1 (see [1, Theorem 2]).

Let us state the following additional assumptions:

(A4) For all $1 \leq i \leq n$, the functions $f_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are Lipschitz continuous and attain their maximum value over $\mathbb{R}_0^+$, i.e., $\overline{f_i} = f_i(m_i^*)$ with $m_i^* \in \mathbb{R}_0^+$, and f_i is non-increasing for $x > m_i^*$.

(A5) There exist two positive constants γ_1 and γ_2 such that:

$$0 \leq \frac{\overline{m^*}}{\underline{\lambda}} < \gamma_1 < \frac{1}{\overline{\alpha}}\Big(\sum_{i=1}^{n} \underline{\beta_i}\, f_i(\overline{\lambda}\gamma_2) + \underline{b}\,\underline{H}\Big),$$
$$\frac{1}{\underline{\alpha}}\Big(\sum_{i=1}^{n} \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H}\Big) < \gamma_2,$$

where $\overline{m^*} = \max_{1 \leq i \leq n} m_i^*$.

(A6) For all $1 \leq i \leq n$, there exist positive numbers ℓ_{f_i}, which are the Lipschitz constants of f_i on $[\overline{m^*}, +\infty)$.

Lemma 5.3. *Let* $\Omega_0 := \{\phi \in \mathcal{C}^+ : \gamma_1 < \phi(t) < \gamma_2,\ t \in [-\tau, 0]\}$. *If conditions* (A1)–(A5) *are satisfied, then for every initial data* $\phi \in \Omega_0$, *the solution* $x(t)$ *of equation* (5.1) *satisfies*

$$\gamma_1 < x(t) < \gamma_2, \quad t \in [t_0, \zeta(\phi)),$$

and its existence interval can be extended to $[t_0, +\infty)$.

Proof. First, we prove that $x(t) < \gamma_2$ for $t \in [t_0, \zeta(\phi))$. Suppose this is not true. Then there exists $t_1 \in [t_0, \zeta(\phi))$ such that

$$x(t_1) = \gamma_2, \quad x(t) < \gamma_2 \quad \forall t \in [t_0 - \tau, t_1).$$

Using (A5), we arrive at the contradiction

$$\begin{aligned} 0 &\leq x'(t_1) \\ &= -\alpha(t_1)x(t_1) + \sum_{i=1}^{n} \beta_i(t_1) f_i\big(\lambda_i(t_1)x(t_1 - \tau_i(t_1))\big) + b(t_1)H\big(x(t_1 - \sigma(t_1))\big) \\ &< -\underline{\alpha}\gamma_2 + \sum_{i=1}^{n} \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H} < 0. \end{aligned}$$

Similarly, suppose that the inequality: $\gamma_1 < x(t)$ for $t \in [t_0, \zeta(\phi))$, does not hold. Then there exists $t_2 \in [t_0, \zeta(\phi))$ such that

$$x(t_2) = \gamma_1, \quad x(t) > \gamma_1 \quad \forall t \in [t_0 - \tau, t_2).$$

Now, since $\tau_i(t_2) \le \tau$ implies $t_0 - \tau < t_2 - \tau \le t_2 - \tau_i(t_2) \le t_2$, it follows that $x(t_2 - \tau_i(t_2)) > \gamma_1$. But, the inequality $x(t_2 - \tau_i(t_2)) < \gamma_2$ also holds, then using (A1), (A2), and (A5) we have

$$\bar{\lambda}\gamma_2 > \overline{\lambda_i}\gamma_2 > \lambda_i(t_2)\,x(t_2 - \tau_i(t_2)) > \lambda_i(t_2)\,\gamma_1 > \underline{\lambda_i}\gamma_1 > \underline{\lambda}\gamma_1 > \overline{m^*}\,,$$

which implies, using (A4), that

$$f_i\left(\lambda_i(t_2)\,x(t_2 - \tau_i(t_2))\right) > f_i\left(\bar{\lambda}\gamma_2\right)\,.$$

With the previous calculations, and using (A3), we arrive at the contradiction

$$\begin{aligned} 0 &\ge x'(t_2) \\ &= -\alpha(t_2)x(t_2) + \sum_{i=1}^{n} \beta_i(t_2) f_i\big(\lambda_i(t_2)x(t_2 - \tau_i(t_2))\big) + b(t_2)H\big(x(t_2 - \sigma(t_2))\big) \\ &> -\overline{\alpha}\gamma_1 + \sum_{i=1}^{n} \underline{\beta_i}\, f_i(\overline{\lambda}\gamma_2) + \underline{b}\,\underline{H} > 0. \end{aligned}$$

From [30, Theorem 3.2], it follows that $\zeta(\phi) = +\infty$. □

The next theorem establishes the existence of a unique positive compact almost automorphic solution of the model (5.1).

Theorem 5.4. *Assume* (A1)–(A6) *hold. If*

$$\bar{b}\,L_H + \sum_{i=1}^{n} \overline{\beta_i}\,\overline{\lambda_i}\,\ell_{f_i} < \underline{\alpha} \tag{5.5}$$

holds, then equation (5.1) *has a unique compact almost automorphic solution in*

$$\Omega = \{x \in \mathcal{K}AA(\mathbb{R}, \mathbb{R}^+) : \gamma_1 \le x(t) \le \gamma_2,\ \forall t \in \mathbb{R}\}\,.$$

Proof. We define the operator:

$$(\mathfrak{M}x)(t) := \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}(Nx)(s)\,ds, \tag{5.6}$$

where $N : C(\mathbb{R}) \to C(\mathbb{R})$ is defined as

$$N(x)(s) := \sum_{i=1}^{n} \beta_i(s) f_i\big(\lambda_i(s)x(s - \tau_i(s))\big) + b(s)H\big(x(s - \sigma(s))\big). \tag{5.7}$$

We now prove that Ω is closed, $\mathfrak{M}(\Omega) \subset \Omega$, and $\mathfrak{M}$ is a contraction. The conclusion will then follow from the Banach's contraction principle.

- Ω is closed. Let $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to x$ uniformly on $\mathbb{R}$. From Proposition 2.4 it follows that $x \in \mathcal{K}AA(\mathbb{R}, \mathbb{R})$. Now, given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that
$$\|x_n - x\|_\infty < \epsilon, \quad \forall n \ge N_0.$$
Thus, for all $t \in \mathbb{R}$ and $\forall n \ge N_0$, we have
$$\begin{gathered} 0 \le x(t) = |x(t)| \le \|x_n - x\|_\infty + |x_n(t)| < \epsilon + \gamma_2\,, \\ \gamma_1 - \epsilon < x_n(t) - \epsilon < x(t)\,. \end{gathered}$$
Since $\epsilon > 0$ is arbitrary, from the last inequalities we obtain $\gamma_1 \le x(t) \le \gamma_2$ for all $t \in \mathbb{R}$. Therefore, $x \in \Omega$.
- $\mathfrak{M}(\Omega) \subset \Omega$. If u is compact almost automorphic, then from Proposition 2.4, Nu is compact almost automorphic, and from Proposition 4.5 and Theorem 4.2, $\mathfrak{M}(u)$ is also compact almost automorphic. Moreover, since $u \in \Omega$, we also have:
$$\sum_{i=1}^{n} \underline{\beta_i}\, f_i(\overline{\lambda}\gamma_2) + \underline{b}\,\underline{H} \le (Nu)(s) \le \sum_{i=1}^{n} \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H}\,, \quad \forall s \in \mathbb{R}\,.$$

Therefore,

$$0 < \gamma_1 < \frac{1}{\overline{\alpha}}\Big(\sum_{i=1}^{n} \underline{\beta_i}\, f_i(\overline{\lambda}\gamma_2) + \underline{b}\,\underline{H}\Big) \le (\mathfrak{M}u)(t) \le \frac{1}{\underline{\alpha}}\Big(\sum_{i=1}^{n} \overline{\beta_i}\,\overline{f_i} + \overline{b}\,\overline{H}\Big) < \gamma_2\,,$$

for all $t \in \mathbb{R}$.

- $\mathfrak{M}$ is a contraction. This follows from the inequality:

$$\|\mathfrak{M}(u) - \mathfrak{M}(v)\|_\infty \le \kappa\|u - v\|_\infty, \quad \forall u, v \in \Omega,$$

where

$$\kappa := \frac{1}{\underline{\alpha}}\Big(\overline{b}L_H + \sum_{i=1}^{n} \overline{\beta_i}\,\overline{\lambda_i}\,\ell_{f_i}\Big) < 1.$$

□

5.2. **Global exponential stability.** In the present section, we prove the global exponential stability of the compact almost automorphic solution to equation (5.1).

Definition 5.5 ([5, 34])**.** Let $x^*(t)$, $x(t)$ be solutions of equation (5.1) with initial conditions $x^*_{t_0}(s) = \varphi^*(s)$ and $x_{t_0}(s) = \varphi(s)$, $s \in [-\tau, 0]$. Then x^* is said to be globally exponential stable if, there exist constants $\lambda > 0$ and $M_\varphi > 1$ such that

$$|x(t) - x^*(t)| \le M_\varphi\|\varphi - \varphi^*\|_\tau e^{-\lambda t}, \quad t \ge 0\,.$$

Next, we state the Halanay's inequality as is given in [1, 34].

Lemma 5.6. *Let t_0 be a real number and $\bar{\tau}$ be a non-negative number. If $v : [t_0 - \bar{\tau}, \infty) \to \mathbb{R}^+$ satisfies*

$$\frac{d}{dt}v(t) \le -\alpha v(t) + \beta\big[\sup_{s\in[t-\bar{\tau},t]} v(s)\big], \quad t \ge t_0,$$

where α and β are constants with $\alpha > \beta > 0$, then

$$v(t) \le \|v_{t_0}\|_{\bar{\tau}} e^{-\eta(t-t_0)} \quad \text{for } t \ge t_0,$$

where η is the unique positive solution of

$$\eta + \beta e^{\eta\bar{\tau}} - \alpha = 0\,.$$

The proof of the following theorem closely follows that of [1, Theorem 4]. We present it here for the sake of completeness.

Theorem 5.7. *If assumptions* (A1)–(A6) *are satisfied and* (5.5) *holds, then the unique compact almost automorphic solution of the system* (5.1) *in the region Ω is globally exponentially stable.*

Proof. Let $x^*(t)$ be the unique compact almost automorphic solution of the delayed differential equation (5.1) (which is given by Theorem 5.4), and let$x(t)$ be another solution such that $\gamma_1 \le x(t) \le \gamma_2$ for $t \ge 0$. Let us define the new function

$$v(t) := x(t) - x^*(t)\,.$$

We have

$$\dot{v}(t) = -\alpha(t)v(t) + (Nx)(t) - (Nx^*)(t),$$

where N is defined in (5.7). Therefore,

$$v(t) = v(0)e^{-\int_0^t \alpha(\xi)d\xi} + \int_0^t e^{-\int_s^t \alpha(\xi)d\xi}\,[(Nx)(s) - (Nx^*)(s)]\,ds\,, \quad t \ge 0\,.$$

It follows that

$$|v(t)| \le |v(0)e^{-\int_0^t \alpha(z)dz}| + \int_0^t |e^{-\int_s^t \alpha(z)dz}||(Nx)(s) - (Nx^*)(s)|\,,$$

since $x^*(t) \in \Omega$, and $x(t)$ satisfies $\gamma_1 \le x(t) \le \gamma_2$ for $t \ge 0$; then, from (A5) and (A6), we have

$$|v(t)| \le \|v\|_\tau e^{-\underline{\alpha}t} + \int_0^t e^{-\underline{\alpha}(t-s)}\Big(\sum_{i=1}^{n} \overline{\beta_i}\,\overline{\lambda_i}\ell_{f_i} + \overline{b}L_H\Big) \sup_{\sigma\in[s-\tau,s]} |v(\sigma)|ds\,.$$

From inequality (5.5) it follows that $\underline{\alpha} > \sum_{i=1}^{n} \overline{\beta_i}\,\overline{\lambda_i}\ell_{f_i} + \bar{b}L_H > 0$, then by Halanay's inequality - Lemma 5.6 - it follows that there exist positive constants η and M such that

$$|v(t)| \le M\|v\|_\tau e^{-\eta t}\,, \quad t \ge 0\,,$$

where η is the real solution of the characteristic equation

$$\eta = -\underline{\alpha} + \Big(\sum_{i=1}^{n} \overline{\beta_i}\,\overline{\lambda_i}\ell_{f_i} + \bar{b}\,L_H\Big)e^{\eta\tau}\,.$$

Therefore, $|x(t) - x^*(t)| \le M\|x_0 - x_0^*\|_\tau e^{-\eta t}$ for $t \ge 0$. □

Remark 5.8. In [34, Theorem 4.4], the authors established the global exponential stability of the unique pseudo compact almost automorphic solution for their delayed equation. However, their proof considers an arbitrary solution $u(t)$ without ensuring the required bounds $\gamma_1 \le u(t) \le \gamma_2$ for $t \ge 0$, which are necessary, under their assumptions (A5) and (A6), to properly apply the Lipschitz conditions on the functions f_i.

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