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About Nondecreasing Solutions for First Order Neutral Functional Differential Equations*

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Abstract

Conditions that solutions of the first order neutral functional differential equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0, \omega],$$

are nondecreasing are obtained. Here $A : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$, $B : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ and $S : L^{\infty}{}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ are linear continuous operators, A and B are positive operators, $C_{[0,\omega]}$ is the space of continuous functions and $L^{\infty}_{[0,\omega]}$ is the space of essentially bounded functions defined on $[0, \omega]$. New tests on positivity of the Cauchy function and its derivative are proposed. Results on existence and uniqueness of solutions for various boundary value problems are obtained on the basis of the maximum principles.

1 Preliminary

Our paper is devoted to the maximum principles for first order neutral functional differential equation.

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0, \omega], \tag{1.1}$$

where $A : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$, $B : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$, $S : L^{\infty}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ are linear continuous Volterra operators, A and B are positive operators, the spectral radius $\rho(S)$ of the operator S is less than one, here $C_{[0,\omega]}$ is the space of continuous functions, $L^{\infty}_{[0,\omega]}$ is the space of essentially bounded functions defined

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on $[0, \omega]$. We consider this equation in the space of absolutely continuous functions $D_{[0,\omega]}$. By solutions of equation (1.1) we mean functions $x : [0, \omega] \to R^1$ from the space $D_{[0,\omega]}$ which satisfy it almost everywhere in $[0, \omega]$ and such that $x' \in L^\infty_{[0,\omega]}$.

We mean the Volterra operators according to the classical Tikhonov's definition.

Definition 1.1. *An operator T is called Volterra if any two functions x_1 and x_2 coinciding on an interval $[0, a]$ have the equal images on $[0, a]$, i.e. $(Tx_1)(t) = (Tx_2)(t)$ for $t \in [0, a]$ and for every $0 < a \le \omega$.*

Neutral functional differential equations have their own history (see, for example, [11, 13, 14, 17] and also the bibliography therein). Various results on existence and uniqueness of boundary value problems for neutral equations and their stability were obtained in [1], where also the basic results about the representation of solutions were presented. Note also in this connection the papers [3, 4, 8, 12], where results on nonoscillation and positivity of the Cauchy and Green's functions for neutral functional differential equations were obtained. All results about positivity of solutions of neutral equations were obtained under assumption that the operator $S : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ is positive. The first results about maximum principles and nondecreasing solutions in the case of negative operator S were obtained in [7]. In this paper we study a general case of the operator S and do not assume nor positivity and nor negativity of this operator. Results about nondecreasing solutions of homogeneous and nonhomogeneous equations are obtained.

Let us note here that the operator $S : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ in equation (1.1) can be, for example, of the following forms

$$(Sy)(t) = \sum_{j=1}^{m} q_j(t) y(\tau_j(t)), \quad \text{where } \tau_j(t) \le t, \ t \in [0, \omega], \tag{1.2}$$

$$y(\tau_j(t)) = 0 \text{ if } \tau_j(t) < 0,$$

or

$$(Sy)(t) = \sum_{i=1}^{n} \int_0^t k_i(t, s) y(s) ds, \quad t \in [0, \omega], \tag{1.3}$$

where $q_j(t)$ are essentially bounded measurable functions, $\tau_j(t)$ are measurable functions for $j = 1, ..., m$, and $k_i(t, s)$ are summable with respect to s and measurable essentially bounded with respect to t for $i = 1, ..., n$. All linear combinations of operators (1.2) and (1.3) and their superpositions are also allowed.

Properties of operator (1.2) were studied, for example, in [9, 10]. To achieve the action of operator (1.2) in the space of essentially bounded functions $L^\infty_{[0,\omega]}$, we have for each j to assume that $mes\,\{t : \tau_j(t) = c\} = 0$ for every constant c. We suppose everywhere below that this condition is fulfilled. It is known that the spectral radius of the integral operator (1.3), considered on every finite interval $t \in [0, \omega]$, is equal to zero (see, for example, [1]). Let us note the sufficient conditions of the fact that the spectral radius $\rho(S)$ of the operator S, defined by formula (1.2), is less than one. Define the set $\kappa_\varepsilon^j = \{t \in [0,\omega] : \ t - \tau_j(t) \le \varepsilon\}$ and $\kappa_\varepsilon = \bigcup_{j=1}^m \kappa_\varepsilon^j$. If there exists such ε that $mes\ (\kappa_\varepsilon) = 0$, then on every finite interval $t \in [0,\omega]$ the spectral radius of the operator S, defined by formula (1.2) for $t \in [0,\omega]$, is zero. In the case $mes\ (\kappa_\varepsilon) > 0$, the spectral radius of the operator S defined by (1.2) on the finite interval $t \in [0,\omega]$ is less than one if $ess\sup_{t\in\kappa_\varepsilon} \sum_{j=1}^m |q_j(t)| < 1$. The inequality $ess\sup_{t\in[0,\infty)} \sum_{j=1}^m |q_j(t)| < 1$ implies that the spectral radius $\rho(S)$ of the operator S considered on the semiaxis $t \in [0,+\infty)$ (i.e. in the case $\omega = \infty$) and defined by (1.2), satisfies the inequality $\rho(S) < 1$. We also assume that τ_j are nondecreasing functions for $j = 1,...,m$.

In the case, when $\rho(S) < 1$, it is known [1] that the general solution of equation (1.1) has the representation

$$x(t) = \int_0^t C(t,s)f(s)ds + X(t)x(0), \tag{1.4}$$

where the kernel $C(t,s)$ is called the Cauchy function, and $X(t)$ is the solution of the homogeneous equation $(Mx)(t) = 0$, $t \in [0,\omega]$, satisfying the condition $X(0) = 1$. On the basis of representation (1.4), the results about differential inequalities (under corresponding conditions, solutions of inequalities are greater or less than solution of the equation) can be formulated in the form of positivity of the Cauchy function $C(t,s)$ and the solution $X(t)$. Results about comparison of solutions for delay differential equations solved with respect to the derivative (i.e. in the case when S is the zero operator) were obtained in [6, 12, 15], where assertions on existence and uniqueness of solutions of various boundary value problems for first order functional differential equations were obtained.

Our assertions are based on the assumption that the operator A is a dominant among two operators A and B.

In the case, when the spectral radius of the operator $S : {L^\infty}_{[0,\omega]} \to L^\infty_{[0,\omega]}$ is less than one, we can rewrite equation (1.1) in the equivalent form

$$(Nx)(t) \equiv x'(t) - (I-S)^{-1}(A-B)x(t) = (I-S)^{-1}f(t), \quad t \in [0,\omega], \tag{1.5}$$

and its general solution can be written in the form

$$x(t) = \int_0^t C_0(t,s)(I-S)^{-1}f(s)ds + X(t)x(0), \tag{1.6}$$

where $C_0(t,s)$ is the Cauchy function of equation (1.5) [1]. Note that this approach in the study of neutral equations was first used in the paper [8]. Below in the paper we use the fact that the Cauchy function $C_0(t,s)$ coincides with the fundamental function of equation (1.5). It is also clear that

$$\int_0^t C(t,s)f(s)ds = \int_0^t C_0(t,s)(I-S)^{-1}f(s)ds. \tag{1.7}$$

Positivity of $C_0(t,s)$ can be obtained for equation (1.5) using results of [6, 12, 15]. In the case of positive operator S, we get that the operator $(I-S)^{-1} = I + S +$

$S^2+...$ is positive, and positivity of $C(t,s)$ follows from equality (1.7). Without assumption about positivity of the operator S we cannot make conclusion about positivity of solutions. In this paper we demonstrate that for corresponding classes of the right hand sides f, solutions are nondecreasing.

2 Several Denotations and Remarks

In this paragraph we consider the equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0,\omega], \tag{2.1}$$

where $A : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ and $B : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ are positive linear continuous Volterra operators, $S : L^{\infty}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ and the spectral radius $\rho(S)$ of the operator S is less than one.

These operators A and B are $u-$bounded operators and according to [16], they can be written in the form of the Stieltjes integrals

$$(Ax)(t) = \int_0^t x(\xi)d_\xi a(t,\xi) \text{ and } (Bx)(t) = \int_0^t x(\xi)d_\xi b(t,\xi), \quad t \in [0,\omega], \tag{2.2}$$

respectively, where the functions $a(\cdot,\xi)$ and $b(\cdot,\xi) : [0,\omega] \to R^1$ are measurable for $\xi \in [0,\omega]$, $a(t,\cdot)$ and $b(t,\cdot) : [0,\omega] \to R^1$ has the bounded variation for almost all $t \in [0,\omega]$ and $\bigvee_{\xi=0}^{t} a(t,\xi)$, $\bigvee_{\xi=0}^{t} b(t,\xi)$ are essentially bounded.

Consider for convenience equation (2.1) in the following form

$$(Mx)(t) \equiv x'(t)-(Sx')(t)-\int_0^t x(\xi)d_\xi a(t,\xi)+\int_0^t x(\xi)d_\xi b(t,\xi) = f(t), \quad t \in [0,\omega]. \tag{2.3}$$

Consider also the homogeneous equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0,\omega], \tag{2.5}$$

and the following auxiliary equations (which are analogs of the so called s-trancated equations defined first in [2])

$$(M_s x)(t) \equiv x'(t)-(S_s x')(t)-(A_s x)(t)+(B_s x)(t) = 0, \quad t \in [s,\omega], \ s \geq 0, \tag{2.6}$$

where the operators $A_s : C_{[s,\omega]} \to L^{\infty}_{[s,\omega]}$ and $B_s : C_{[s,\omega]} \to L^{\infty}_{[s,\omega]}$ are defined by the formulas

$$(A_s x)(t) = \int_s^t x(\xi)d_\xi a(t,\xi) \text{ and } (B_s x)(t) = \int_s^t x(\xi)d_\xi b(t,\xi), \quad t \in [s,\omega], \tag{2.7}$$

and the operator $S_s : L^{\infty}_{[s,\omega]} \to L^{\infty}_{[s,\omega]}$ is defined by the equality

$$(S_s y_s)(t) = (Sy)(t), \text{ where } y_s(t) = y(t) \text{ for } t \geq s \text{ and } y(t) = 0 \text{ for } t < s. \tag{2.8}$$

For operators defined by equalities (1.2) and (1.3) we have

$$(S_s y)(t) = \sum_{j=1}^{m} q_j(t) y(\tau_j(t)), \quad \text{where } \tau_j(t) \le t, \quad y(\tau_j(t)) = 0 \text{ if } \tau_j(t) < s, \tag{2.9}$$

$$t \in [s, \omega],$$

and

$$(S_s y)(t) = \sum_{i=1}^{n} \int_s^t k_i(t,s) y(s) ds, \quad t \in [s, \omega], \tag{2.10}$$

respectively. It is clear that $\rho(S_s) < 1$ for every $s \in [0, +\infty)$ if $\rho(S) < 1$.

Functions from the space $D_{[s,\omega]}$ of absolutely continuous functions $x : [s, \omega] \to R^1$, $x' \in L^\infty_{[s,\omega]}$, satisfying equation (2.6) almost everywhere in $[s, \omega]$, we call solutions of this equation.

3 About Nondecreasing Solutions of Neutral Equations

Let us consider the equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0, \omega]. \tag{3.1}$$

where $A : C_{[0,\omega]} \to L^\infty_{[0,\omega]}$ and $B : C_{[0,\omega]} \to L^\infty_{[0,\omega]}$ are positive linear continuous Volterra operators, the operator $S : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ admits the representation $S = S^+ - S^-$, where $S^+, S^- : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ are positive operators, and its spectral radius $\rho(S)$ is less than one. If the operator S is not positive, then the operator $(I - S)^{-1} = I + S + S^2 + S^3 + \ldots$ is not, generally speaking, a positive operator. This is the main difficulty in the study of positivity of the solution x and its derivative x'. All previous results about the positivity of solutions for this equation assumed positivity of the operator S (see, for example, [3, 4, 12]) or its negativity [7].

Let us define the operator $|S| : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ by the formula $(|S|\, y)(t) = ((S^+ + S^-)y)(t),\ t \in [0, \omega]$.

Theorem 3.1. *Assume that the spectral radius $\rho(|S|)$ of the operator $|S| : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ is less than one, A, B are positive operators,*

$$\varphi_s(t) \equiv ((A_s - B_s)1)(t) \ge 0 \text{ for } t \in [s, \omega], \tag{3.2}$$

for every $s \in [0, \omega)$, and there exists an essentially bounded function ψ_s satisfying the inequalities

$$\psi_s(t) - (|S_s|\, \psi_s)(t) \ge \varphi_s(t),\ 2\varphi_s(t) \ge \psi_s(t), \quad t \in [s, \omega]. \tag{3.3}$$

Then the solution x of the homogeneous equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0, \omega], \tag{3.4}$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0$, $x'(t) \geq 0$ for $t \in [0, +\infty)$, and for every nonnegative nondecreasing function $f \in L^{\infty}_{[0,\omega]}$ satisfying inequality $2f(t) \geq \phi(t)$, where essentially bounded ϕ satisfies the inequality $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, the solution x of equation (3.1) is nonnegative and nondecreasing for $t \in [0, \omega]$.

Remark 3.1. If essinf $_{0 \leq s \leq t \leq \omega} \varphi_s(t) > 0$, and the first of inequalities (3.3) is fulfilled, then the spectral radius of the operator $|S| : L^{\infty}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ is less than one[18].

Remark 3.2. The condition

$$(A1)(t) \geq (B1)(t) \text{ for } \ t \in [0, \omega],$$

cannot be set instead of condition (3.2) as Example 2.1 in the paper [7], obtained in the case of zero operator S , demonstrates.

In Theorem 3.1 we have to verify conditions (3.2) and (3.3) for all positive s. In the case of corresponding inequalities between deviating arguments we can do with these inequalities for $s = 0$ only.

Below we consider the case of the operator S defined by the equality

$$(Sy)(t) = q(t)y(r(t)), \quad \text{where } r(t) \leq t, \quad y(r(t)) = 0 \text{ if } \ r(t) < 0, \ \ t \in [0, \omega], \tag{3.5}$$

for this operator the operator $|S|$ is of the form

$$(|S|\,y)(t) = |q(t)|\,y(r(t)), \quad \text{where } r(t) \leq t, \quad y(r(t)) = 0 \text{ if } \ r(t) < 0, \ \ t \in [0, \omega]. \tag{3.6}$$

Note that we do not assume for operator S nor $q(t) \geq 0$ for $t \in [0, +\infty)$ nor $q(t) \leq 0$ for $t \in [0, \omega]$. Define the function

$$\chi(t, s) = \begin{cases} 0, & t < s, \\ 1, & t \geq s. \end{cases}$$

Let us start with the equation

$$x'(t) - q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) = f(t), \quad t \in [0, \omega], \tag{3.7}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \ \xi < 0. \tag{3.8}$$

Theorem 3.2. *Assume that the spectral radius $\rho(|S|)$ of the operator $|S| : L^{\infty}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ defined by equality (3.6) is less than one, $r(t)$, $h(t)$ and $g(t)$ are* *nondecreasing functions, the coefficients satisfy the inequalities* $a(t) \geq 0,\ b(t) \geq 0,\ g(t) \geq h(t)$ *and*

$$\varphi(t) \equiv a(t)\chi(g(t),0) - b(t)\chi(h(t),0) \geq 0 \text{ for } t \in [0,\omega], \tag{3.9}$$

and there exists an essentially bounded function ψ *satisfying the inequalities*

$$\psi(t) - |q(t)|\,\psi(r(t))\chi(r(t),0) \geq \varphi(t),\ 2\varphi(t) \geq \psi(t),\quad t \in [0,\omega]. \tag{3.10}$$

Then the solution x *of the equation*

$$x'(t) - q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) = 0, \quad t \in [0,\omega], \tag{3.11}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \ \xi < 0, \tag{3.12}$$

such that $x(0) > 0$, *satisfies the inequalities* $x(t) \geq 0,\ x'(t) \geq 0$ *for* $t \in [0,\omega]$ *and in the case of nondecreasing nonnegative function* $f \in L^{\infty}_{[0,\omega]}$, *satisfying the inequality* $2f(t) \geq \phi(t)$, *with essentially bounded function* ϕ *satisfying the inequality* $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, *the solution* x of *equation (3.7) is nonnegative and nondecreasing for* $t \in [0,\omega]$.

Consider the equation

$$\begin{gathered} x'(t) - q(t)x'(r(t)) + \\ + \sum_{i=1}^{m} \left\{ - \int_{g_{1i}(t)}^{g_{2i}(t)} x(\xi)d_{\xi}a_i(t,\xi) + \int_{h_{1i}(t)}^{h_{2i}(t)} x(\xi)d_{\xi}b_i(t,\xi) \right\} = f(t), \qquad (3.13) \\ t \in [0,\omega], \\ x(\xi) = x'(\xi) = 0 \text{ for } \ \xi < 0. \end{gathered}$$

Theorem 3.3. *Let the spectral radius* $\rho(|S|)$ *of the operator* $|S| : L^{\infty}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ *defined by equality (3.6) be less than one,* $r(t)$ *be a nondecreasing function and the functions* $a_i(t,\xi)$ *and* $b_i(t,\xi)$ *be nondecreasing functions with respect to* $\xi,\ \ 0 \leq h_{1i}(t) \leq h_{2i}(t) \leq g_{1i}(t) \leq g_{2i}(t) \leq t$, *and the following inequalities be fulfilled*

$$\varphi(t) \equiv \bigvee_{\xi=g_{1i}(t)}^{g_{2i}(t)} a_i(t,\xi) - \bigvee_{\xi=h_{1i}(t)}^{h_{2i}(t)} b_i(t,\xi) \geq 0, \tag{3.14}$$

for $t \in [0,\omega],\ i = 1,...,m$, and there exists an essentially bounded function ψ satisfying condition (3.10), *then the solution* x *of the equation*

$$x'(t) - q(t)x'(r(t)) + \sum_{i=1}^{m} \left\{ - \int_{g_{1i}(t)}^{g_{2i}(t)} x(\xi)d_{\xi}a_i(t,\xi) + \int_{h_{1i}(t)}^{h_{2i}(t)} x(\xi)d_{\xi}b_i(t,\xi) \right\} = 0, \tag{3.15}$$

$$t \in [0,\omega],$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0, x'(t) \geq 0$ for $t \in [0,\omega]$ and in the case of nondecreasing nonnegative function $f \in L^\infty_{[0,\omega]}$, satisfying the inequality $2f(t) \geq \phi(t)$, with essentially bounded function ϕ satisfying the inequality $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, the solution x of equation (3.13) is nonnegative and nondecreasing for $t \in [0,\omega]$.

Consider the integro-differential equation

$$x'(t) - q(t)x'(r(t)) - \sum_{i=1}^{m} \int_{g_{1i}(t)}^{g_{2i}(t)} m_i(t,\xi)x(\xi)d\xi +$$

$$+\sum_{i=1}^{m} \int_{h_{1i}(t)}^{h_{2i}(t)} k_i(t,\xi)x(\xi)d\xi = f(t), \quad t \in [0,\omega], \tag{3.16}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \xi < 0,$$

Denote $h^0(t) = \max\{h(t), 0\}$.

Theorem 3.4. *Let the spectral radius $\rho(|S|)$ of the operator $|S| : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ defined by equality (3.6) be less than one, $r(t)$ be a nondecreasing function and $h_{1i}(t) \leq h_{2i}(t) \leq g_{1i}(t) \leq g_{2i}(t) \leq t$, $k_i(t,\xi) \geq 0$, $m_i(t,\xi) \geq 0$ for $t,\xi \in [0,\omega]$, the following inequalities be fulfilled*

$$\varphi(t) \equiv \int\limits_{g^0_{1i}(t)}^{g^0_{2i}(t)} m_i(t,\xi)d\xi - \int\limits_{h^0_{1i}(t)}^{h^0_{2i}(t)} k_i(t,\xi)d\xi, \tag{3.17}$$

$t \in [0,+\infty)$, $i = 1,...,m$, and there exists an essentially bounded function ψ satisfying condition (3.10), then the solution x of the equation

$$x'(t) - q(t)x'(r(t)) - \sum_{i=1}^{m} \int_{g_{1i}(t)}^{g_{2i}(t)} m_i(t,\xi)x(\xi)d\xi + \sum_{i=1}^{m} \int_{h_{1i}(t)}^{h_{2i}(t)} k_i(t,\xi)x(\xi)d\xi = 0, \tag{3.18}$$

$$t \in [0,\omega],$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0, x'(t) \geq 0$ for $t \in [0,+\infty)$ and in the case of nondecreasing nonnegative function $f \in L^\infty_{[0,\omega]}$, satisfying the inequality $2f(t) \geq \phi(t)$, with essentially bounded function ϕ satisfying the inequality $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, the solution x of equation (3.16) is nonnegative and nondecreasing for $t \in [0,\omega]$.

Consider the equation

$$x'(t) - q(t)x'(r(t)) - \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + b(t)x(h(t)) = f(t), \quad t \in [0,\omega], \tag{3.19}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \xi < 0.$$

In the following assertion the integral term is dominant.

Theorem 3.5. *Let the spectral radius $\rho(|S|)$ of the operator $|S| : L^{\infty}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ defined by equality (3.6) be less than one, $r(t)$ be a nondecreasing function, $q(t) \geq 0$, $b(t) \geq 0$, $m(t,\xi) \geq 0$, $h(t) \leq g_1(t) \leq g_2(t) \leq t$ for $t, \xi \in [0,\omega]$, and the following inequality be fulfilled*

$$\varphi(t) \equiv \int_{g_1^0(t)}^{g_2^0(t)} m(t,\xi)d\xi - b(t)\chi(h(t),0), \quad t \in [0,\omega], \tag{3.20}$$

and there exists an essentially bounded function ψ satisfying condition (3.10). Then the solution x of the equation

$$x'(t) - q(t)x'(r(t)) - \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + b(t)x(h(t)) = 0, \quad t \in [0,\omega], \tag{3.21}$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0, x'(t) \geq 0$ for $t \in [0,\omega]$ and in the case of nondecreasing nonnegative function $f \in L^{\infty}_{[0,\omega]}$, satisfying the inequality $2f(t) \geq \phi(t)$, with essentially bounded function ϕ satisfying the inequality $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, the solution x of *equation (3.19) is nonnegative and nondecreasing for $t \in [0,\omega]$.*

Consider the equation

$$x'(t)-q(t)x'(r(t))-a(t)x(g(t))+\int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi = f(t), \quad t \in [0,\omega], \tag{3.22}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \xi < 0.$$

In the following assertion the term $a(t)x(g(t))$ is dominant.

Theorem 3.6. *Let the spectral radius $\rho(|S|)$ of the operator $|S| : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ defined by equality (3.6) be less than one, $r(t)$ be a nondecreasing function, $h_1(t) \leq h_2(t) \leq g(t) \leq t$, $q(t) \geq 0$, $k(t,\xi) \geq 0$, $a(t) \geq 0$ for $t, \xi \in [0,\omega]$, and the following inequality*

$$\varphi(t) \equiv a(t) - \int_{h_1^0(t)}^{h_2^0(t)} k(t,\xi)d\xi \geq 0, \quad t \in [0,\omega], \tag{3.23}$$

be fulfilled, and there exists an essentially bounded function ψ satisfying condition (3.10). *Then the solution x of the equation*

$$x'(t) - q(t)x'(r(t)) - a(t)x(g(t)) + \int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi = 0, \quad t \in [0,\omega], \tag{3.24}$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0, x'(t) \geq 0$ for $t \in [0,\infty)$ and in the case of nondecreasing nonnegative function $f \in L^{\infty}_{[0,\omega]}$, satisfying the *inequality* $2f(t) \geq \phi(t)$, *with essentially bounded function* ϕ *satisfying the inequality* $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, *the solution* x *of equation (3.22) is nonnegative and nondecreasing for* $t \in [0, \omega]$.

Consider now the equation

$$x'(t) - q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) -$$

$$- \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + \int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi = f(t), \quad t \in [0, \omega], \tag{3.25}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \xi < 0.$$

In the following assertion we do not assume inequalities $k(t,\xi) \leq m(t,\xi)$ or $b(t) \leq a(t)$. Here the sum $a(t)x(g(t)) + \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi$ is a dominant term.

Theorem 3.7. *Let the spectral radius* $\rho(|S|)$ *of the operator* $|S| : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ *defined by equality (3.6) be less than one,* $r(t)$ *be a nondecreasing function,* $k(t,\xi) \geq 0$, $m(t,\xi) \geq 0$, $a(t) \geq 0$, $b(t) \geq 0$, $q(t) \geq 0$, $h(t) \leq g_1(t) \leq g_2(t) \leq t$, $h_1(t) \leq h_2(t) \leq g(t) \leq t$ *for* $t, \xi \in [0, \omega]$, *the following inequalities be fulfilled*

$$\int_{g_1^0(t)}^{g_2^0(t)} m(t,\xi)d\xi - b(t)\chi(h(t), 0) \geq 0, \; t \in [0, \omega], \tag{3.26}$$

$$a(t)\chi(g(t), 0) - \int_{h_1^0(t)}^{h_2^0(t)} k(t,\xi)d\xi \geq 0, \; t \in [0, \omega], \tag{3.27}$$

and there exists an essentially bounded function ψ *satisfying condition (3.10) with*

$$\varphi(t) \equiv \int_{g_1^0(t)}^{g_2^0(t)} m(t,\xi)d\xi + a(t)\chi(g(t), 0) - b(t)\chi(h(t), 0) - \int_{h_1^0(t)}^{h_2^0(t)} k(t,\xi)d\xi \geq 0,$$

$$t \in [0, \omega],$$

. *Then the solution* x *of the equation*

$$x'(t) - q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) -$$

$$- \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + \int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi = 0, \tag{3.28}$$

$$t \in [0, \omega],$$

such that $x(0) > 0$, *satisfies the inequalities* $x(t) \geq 0, x'(t) \geq 0$ *for* $t \in [0, \omega]$ *and in the case of nondecreasing nonnegative function* $f \in L^\infty_{[0,\omega]}$, *satisfying the inequality* $2f(t) \geq \phi(t)$, *with essentially bounded function* ϕ *satisfying the inequality* $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, *the solution* x *of equation (3.25) is nonnegative and nondecreasing for* $t \in [0, \omega]$.

Remark 3.3. The inequalities $\phi - |S|\,\phi \geq f,\ 2f \geq \phi$ for the operator $|S|$, defined by (3.6), nondecreasing function f and $\phi(t) = (I - |S|)^{-1} f$ are fulfilled if $|q| \leq \frac{1}{2}$. Actually in this case we get $(I - |S|)^{-1} f(t) = (I + |S| + |S|^2 + ...)f(t) \leq (I + |S| + |S|^2 + ...) \operatorname{esssup}_{t\in[0,t]} f(s) = \frac{1}{1-|S|} \operatorname{esssup}_{t\in[0,t]} f(s) = \frac{1}{1-|S|} f(t) \leq 2f(t)$.

Consider the equation

$$x'(t) - q(t)x'(r(t)) - ax(g(t)) + bx(h(t)) = f(t), \quad t \in [0,\omega], \tag{3.29}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \ \xi < 0.$$

with constant coefficients a and b.

Corollary 3.1. *Assume that $r(t)$ is increasing and $h(t)$ and $g(t)$ are nondecreasing functions, the coefficients satisfy the inequalities $a > b \geq 0,\ |q(t)| \leq \frac{1}{2},\ t - \varepsilon \geq g(t) \geq h(t) \geq 0$, where ε is a positive constant, and there exists an essentially bounded function ψ satisfying the inequalities*

$$\psi(t) - |q(t)|\,\psi(r(t))\chi(r(t),0) \geq a - b \geq \frac{1}{2}\psi(t), \ \ t \in [0,\omega]. \tag{3.30}$$

Then the solution x of the homogeneous equation

$$x'(t) - q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) = 0, \quad t \in [0,\omega], \tag{3.31}$$

$$x(\xi) = x'(\xi) = 0 \text{ for } \ \xi < 0,$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0,\ x'(t) \geq 0$ for $t \in [0,\omega]$ and in the case of nondecreasing nonnegative function $f \in L^\infty_{[0,\omega]}$, the solution x of equation (3.29) is nonnegative and nondecreasing for $t \in [0,\omega]$.

4. Proofs.

Let us write equation (3.1) in the form

$$(I - S)x'(t) = Ax(t) - Bx(t) + f(t). \tag{4.1}$$

The spectral radius $\rho(|S|)$ of the operator $|S| : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ is less than one, then the spectral radius $\rho(S)$ of the operator $S : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ is also less than one. In this case there exists a bounded operator $(I - S)^{-1} : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$. We can write equation (4.1) in the form

$$(Nx)(t) \equiv x'(t) - \sum_{n=0}^{\infty}(S^n(A - B)x)(t) = \sum_{n=0}^{\infty}(S^n f)(t). \tag{4.2}$$

Denote by $C_0(t,s)$ the Cauchy function of the equation $Nx = 0$, which is also the fundamental function of equation (3.1).

Proofs of Theorems 3.1 - 3.7 are based on the following auxiliary assertions.

Lemma 4.1 [6]. *Let S be the zero operator. Then the following two assertions are equivalent:*

1) for every positive s there exists a positive function $v_s \in D_{[s,\infty)}$ such that $(M_s v_s)(t) \leq 0$ for $t \in [s, \omega]$,

2) the Cauchy function $C(t,s)$ of equation (3.1) is positive for $0 \leq s \leq t \leq \omega$.

Lemma 4.2. *Let $A : C_{[0,\infty)} \to L^\infty_{[0,\infty)}$, $B : C_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be positive Volterra operators, the spectral radius $\rho(|S|)$ of the operator $|S| : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be less than one and*

$$(A_s 1)(t) \geq (B_s 1)(t), \; ((A_s - B_s)1)(t) \geq \sum_{j=1}^{\infty} (|S_s|^j (A_s - B_s)1)(t), \quad t \in [s, \omega], \tag{4.3}$$

for every nonnegative s. Then $C_0(t,s) > 0$ and $\frac{\partial}{\partial t} C_0(t,s) \geq 0$ for $0 \leq s \leq t < \omega$.

Proof. Lemma 4.1 is true for equation (4.2). Let us set $v_s(t) \equiv 1, \; t \in [s, \omega]$ in the assertion 1 of Lemma 4.1. Condition (3.24) implies, according to Lemma 4.1, that $C_0(t,s) > 0$ for $0 \leq s \leq t \leq \omega$. It is clear that $C_0(t,s) \geq 0$ for $0 \leq s \leq t \leq \omega$. Let us prove that $\frac{\partial}{\partial t} C_0(t,s) \geq 0$ *for* $0 \leq s \leq t \leq \omega$. We use the fact that the function $C_0(t,s)$, as a function of argument t for each fixed positive s, satisfies the equation

$$(N_s x)(t) \equiv x'(t) - \sum_{n=0}^{\infty} (S_s^n (A_s - B_s) x)(t) = 0, \; t \in [s, \omega], \tag{4.4}$$

and the condition $x(s) = 1$.

The following integral equation

$$x(t) = \int_s^t \sum_{n=0}^{\infty} (S_s^n (A_s - B_s) x)(\xi) d\xi + 1 \tag{4.5}$$

is equivalent to equation (4.4) with the condition $x(s) = 1$.

The spectral radius of the operator $T : C_{[s,\omega]} \to C_{[s,\omega]}$, defined by the equality

$$(Tx)(t) = \int_s^t \sum_{n=0}^{\infty} (S_s^n (A_s - B_s) x)(\xi) d\xi, \quad t \in [s, \omega], \tag{4.6}$$

is zero for every positive number ω [1]. Let us build the sequence

$$x_{m+1}(t) = \int_s^t \sum_{n=0}^{\infty} (S_s^n (A_s - B_s) x_m)(\xi) \, d\xi + 1, \tag{4.7}$$

where the iterations start with the constant $x_0(t) \equiv 1$ for $t \in [s, \omega]$.

The sequence of functions $x_m(t)$ converges in the space $C_{[s,\omega]}$ to the unique solution $x(t)$ of equation (4.5) on the interval $[s, \omega]$. It is clear that this solution is absolutely continuous. It follows from the fact that all operators are Volterra ones, that the solution $y(t)$ of equation (4.4) with the initial condition $y(s) = 1$ and the solution $x(t)$ of equation (4.5) coincide for $t \in [s, \omega]$.

Condition (4.3) and the inequality $\rho(|S|) < 1$ imply nonnegativity of the derivatives

$$x'_{m+1}(t) = \sum_{n=0}^{\infty} (S_s^n (A_s - B_s) x_m)(t), \; t \in [s, \omega]. \tag{4.8}$$

Repeating the argumentation used in the proof of Lemma 2.4[7], we obtain that this sequence of nondecreasing functions x_m converges to the nondecreasing solution x, i.e $\frac{\partial}{\partial t} C_0(t, s) \geq 0$ for $0 \leq s \leq t \leq \omega$.

Concerning nonhomogeneous equation (3.1) we propose the following assertion.

Lemma 4.3. *Let* $A : C_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ and $B : C_{[0,\infty)} \to L^{\infty}_{[0,\omega]}$ *be positive Volterra operators, the spectral radius* $\rho(|S|)$ *of the operator* $|S|$ *be less than one and condition (4.3) be fulfilled for every nonnegative* s*. Then the solution* x *of the homogeneous equation*

$$(Mx)(t) \equiv x'(t) + (Sx')(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0, \omega], \tag{4.9}$$

such that $x(0) \geq 0$*, satisfies inequalities* $x(t) \geq 0$*,* $x'(t) \geq 0$ *for* $t \in [0, \omega]$*. If in addition the nonnegative nondecreasing function* $f \in L^{\infty}_{[0,\infty)}$ *satisfies the inequality*

$$f(t) \geq \sum_{j=1}^{\infty} (|S|^j f)(t), \quad t \in [0, \omega], \tag{4.10}$$

then the solution x of *equation (3.1) is nonnegative and nondecreasing for every positive nondecreasing* f*.*

Proof of Lemma 4.3. Assertions about nonnegativity of solution x of the homogeneous equation $Mx = 0$ and its derivative follows from the equalities $x(t) = C_0(t, 0)$ and $x'(t) = \frac{\partial}{\partial t} C_0(t, 0)$ and Lemma 4.2. From the representation of solutions of equation (4.2) we can write

$$x(t) = \int_0^t C_0(t, s) \sum_{n=0}^{+\infty} (S^n f)(s) ds + x(0) C_0(t, 0), \tag{4.11}$$

and

$$x'(t) = \sum_{n=0}^{+\infty} (S^n f)(t) + \int_0^t \frac{\partial}{\partial t} C_0(t, s) \sum_{n=0}^{+\infty} (S^n f)(s) ds + x(0) \frac{\partial}{\partial t} C_0(t, 0). \tag{4.12}$$

It is clear now that the inequality (4.10) and nonnegativity of $\frac{\partial}{\partial t} C_0(t, s)$ for $0 \leq s \leq t < \omega$ imply the inequalities $x(t) \geq 0$, $x'(t) \geq 0$for $t \in [0, \omega]$.

To prove Theorem 3.1 let us note the following. Conditions (3.2) and (3.3) imply condition (4.3). Nonnegative nondecreasing function $f \in L^{\infty}_{[0,\omega]}$ satisfying inequality $2f(t) \geq \phi(t)$, where essentially bounded ϕ satisfies the inequality $\phi(t) - (|S|\,\phi)(t) \geq f(t)$, satisfies also condition (4.10).

Concerning proofs of Theorems 3.2-3.7 let us note the following. The fact that the spectral radius $\rho(|S|) < 1$ allows us to write equation (4.1) in form (4.2). Conditions of each of Theorems 3.2-3.7 imply that for $x(0) > 0$ we get

$$x'(t) = \sum_{n=0}^{\infty} (S^n(A-B)x)(t) + \sum_{n=0}^{\infty} (S^n f)(t) \geq 0. \tag{4.13}$$

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