*Electronic Journal of Qualitative Theory of Differential Equations*
2012, No. **33**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

# Forced oscillation of hyperbolic equations with mixed nonlinearities

Yutaka Shoukaku[a1] and Norio Yoshida[b]

[a]Faculty of Engineering, Kanazawa University, Kanazawa 920-1192, Japan
[b]Department of Mathematics, University of Toyama, Toyama 930-8555, Japan

**ABSTRACT.** In this paper, we consider the mixed nonlinear hyperbolic equations with forcing term via Riccati inequality. Some sufficient conditions for the oscillation are derived by using Young's inequality and integral averaging method.

**2000 Mathematics Subject Classification:** 34K11, 35B05, 35R10.
**Keywords:** Forced oscillation; hyperbolic equations; Riccati inequality; interval criteria

## 1. Introduction

We shall provide oscillation results for the hyperbolic equation

$$\begin{aligned}\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}u(x,t)\right)+p(t)\frac{\partial}{\partial t}u(x,t)-a(t)\Delta u(x,t)&\\ +c(x,t,u)=f(x,t),\ (x,t)\in\Omega\equiv G\times(0,\infty),&\end{aligned}\tag{E}$$

with

$$\begin{aligned}c(x,t,u) &= \sum_{i=0}^{2} q_i(x,t)\varphi_i(u(x,t))\\ &= q_0(x,t)u(x,t)+q_1(x,t)|u(x,t)|^{\beta-1}u(x,t)\\ &\qquad +q_2(x,t)|u(x,t)|^{\gamma-1}u(x,t),\end{aligned}$$

where $\beta > 1$, $0 < \gamma < 1$, $\Delta$ is the Laplacian in $\mathbb{R}^n$ and $G$ is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary $\partial G$.

We consider the following boundary conditions

$$u=\psi \qquad \text{on} \qquad \partial G\times(0,\infty), \tag{B1}$$

$$\frac{\partial u}{\partial \nu}+\mu u=\tilde{\psi} \qquad \text{on} \qquad \partial G\times(0,\infty), \tag{B2}$$

where $\nu$ denotes the unit exterior normal vector to $\partial G$ and $\psi,\tilde{\psi}\in C(\partial G\times(0,\infty);\mathbb{R})$, $\mu\in C(\partial G\times(0,\infty);[0,\infty))$.

We assume throughout this paper that:

[1]Corresponding author. E-mail address: shoukaku@t.kanazawa-u.ac.jp

(H1) $r(t) \in C^1([0,\infty);(0,\infty))$, $p(t) \in C([0,\infty);\mathbb{R})$,
$a(t) \in C([0,\infty);[0,\infty))$, $q_i(x,t) \in C(\overline{\Omega};[0,\infty))$ $(i = 0, 1, 2)$,
$f(x,t) \in C(\overline{\Omega};\mathbb{R})$.

**Definition 1.** A solution $u(x,t) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of Eq. (E) is said to be *oscillatory* in $\Omega$ if $u$ has a zero in $G \times (t,\infty)$ for any $t > 0$.

**Definition 2.** We say that functions $(H_1, H_2)$ belong to a function class $\mathbb{H}$, denoted by $(H_1, H_2) \in \mathbb{H}$, if $H_1, H_2 \in C(D;[0,\infty))$ satisfy

$$H_i(t,t) = 0,\ H_i(t,s) > 0\ (i = 1,2) \quad \text{for } t > s,$$

where $D = \{(t,s) : 0 < s \le t < \infty\}$. Moreover, the partial derivatives $\partial H_1/\partial t$ and $\partial H_2/\partial s$ exist on $D$ and satisfy

$$\frac{\partial H_1}{\partial t}(s,t) = h_1(s,t)H_1(s,t) \quad \text{and} \quad \frac{\partial H_2}{\partial s}(t,s) = -h_2(t,s)H_2(t,s),$$

where $h_1,\ h_2 \in C(D;\mathbb{R})$.

There is an increasing interest in oscillation problems for hyperbolic equations with forcing terms. There are many papers dealing with nonlinear hyperbolic equations (see, e.g., [2, 6, 7, 9–16]) under the following restrict conditions

$$\text{(C)} \qquad \varphi_i'(u) \ge K_0 \quad \text{or} \quad \varphi_i(u)/u \ge K_0$$

for some constant $K_0 > 0$. In recent years there are a number of papers [1,3,8,16,17] which obtained oscillation criteria for the second order forced ordinary differential equation with mixed nonlinearities. In these papers, they utilized the arithmetic-geometric mean inequality to establish the interval oscillation criteria. However, those metods in [1, 3, 8, 16, 17] cannot apply to the Eq. (E). Therefore, the aim of this paper is to obtain oscillation theorems without restrictions (C) for Eq. (E) by using the different inequality in [1, 3, 8, 16, 17].

This paper is organized as follows. In the next section we introduce the main ideas by using Young's inequality (see, [4]), which plays an important role in establishing oscillation theorems for superlinear and sublinear hyperbolic equations. In Section 3 we use the well-known Riccati type transformation. The latest section contains the main results and an example which illustrates our results.

## 2. Reduction to one-dimensional problems

In this section we reduce the multi-dimensional oscillation problems for (E) to one-dimensional oscillation problems. It is known that the first eigenvalue $\lambda_1$ of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in} \quad G, \\ w &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in $G$. The notations in this paper are as follows:

$$\begin{aligned} &U(t) = K_\Phi \int_G u(x,t)\Phi(x)dx, \quad \tilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)dx, \\ &F(t) = K_\Phi \int_G f(x,t)\Phi(x)dx, \quad \tilde{F}(t) = \frac{1}{|G|}\int_G f(x,t)dx, \\ &\Psi(t) = K_\Phi \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS, \quad \tilde{\Psi}(t) = \frac{1}{|G|}\int_{\partial G} \tilde{\psi} dS, \\ &q_i(t) = \min_{x\in\overline{G}} q_i(x,t) \ (i = 0,1,2), \end{aligned}$$

where $K_\Phi = (\int_G \Phi(x)dx)^{-1}$ and $|G| = \int_G dx$.

The following lemma is of basic importance for our later considerations.
**Lemma 1**(Young's Inequality). *If $p > 1$ and $q > 1$ are conjugate numbers, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then for any $u, v \in \mathbb{R}$*

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \geq |uv|$$

*and equality holds iff $u = |v|^{q-2}v$.*

**Theorem 1.** *If the functional differential inequalities*

$$(r(t)y'(t))' + p(t)y'(t) + q(t)y(t) \leq \pm F_\Psi(t) \tag{1}$$

*have no eventually positive solutions, then every solution $u(x,t)$ of the problem* (E), (B1) *is oscillatory in $\Omega$, where*

$$\begin{aligned} q(t) &= q_0(t) + \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{\frac{1-\beta}{\beta-\gamma}} q_1(t)^{\frac{1-\gamma}{\beta-\gamma}} q_2(t)^{\frac{\beta-1}{\beta-\gamma}}, \\ F_\Psi(t) &= F(t) - a(t)\Psi(t). \end{aligned}$$

**Proof.** Suppose to the contrary that there is a nonoscillatory solution $u$ of the problem (E), (B1). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$ because the case $u(x,t) < 0$ is handled similarily. It follows from Lemma 1 that

$$\begin{aligned}
& q_0(x,t)u(x,t) + q_1(x,t)|u(x,t)|^{\beta-1}u(x,t) + q_2(x,t)|u(x,t)|^{\gamma-1}u(x,t) \\
\geq\ & q_0(t)u(x,t) + q_1(t)|u(x,t)|^{\beta-1}u(x,t) + q_2(t)|u(x,t)|^{\gamma-1}u(x,t) \\
=\ & q_0(t)u(x,t) + u(x,t)\left(q_1(t)|u(x,t)|^{\beta-1} + q_2(t)|u(x,t)|^{\gamma-1}\right) \\
\geq\ & q_0(t)u(x,t) + u(x,t)\left(\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{\frac{1-\beta}{\beta-\gamma}} q_1(t)^{\frac{1-\gamma}{\beta-\gamma}} q_2(t)^{\frac{\beta-1}{\beta-\gamma}}\right) \\
=\ & q(t)u(x,t),\ (x,t) \in G \times (t_0, \infty),
\end{aligned}$$

and hence, Eq. (E) can be written in the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}u(x,t)\right) + p(t)\frac{\partial}{\partial t}u(x,t) - a(t)\Delta u(x,t) \\
&\qquad + q(t)u(x,t) \leq f(x,t),\ (x,t) \in G \times (t_0, \infty).
\end{aligned} \tag{2}$$

Multiplying (2) by $K_\Phi \Phi(x)$ and integrating over $G$, we obtain

$$\begin{aligned}
&(r(t)U'(t))' + p(t)U'(t) - a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx \\
&\qquad + q(t)U(t) \leq F(t),\ t \geq t_0.
\end{aligned}$$

Using Green's formula, we see that

$$\begin{aligned}
& K_\Phi \int_G \Delta u(x,t)\Phi(x)dx \\
=\ & -K_\Phi \int_{\partial G}\left(\psi \frac{\partial \Phi}{\partial \nu}(x)\right) dS - \lambda_1 K_\Phi \int_G u(x,t)\Phi(x)dx \leq -\Psi(t)
\end{aligned} \tag{3}$$

for $t \geq t_0$. Combining (2) with (3) yields

$$(r(t)U'(t))' + p(t)U'(t) + q(t)U(t) \leq F_\Psi(t),\ t \geq t_0.$$

Therefore $U(t)$ is an eventually positive solution of (1) with $+F_\Psi(t)$. This contradicts the hypothesis and completes the proof.

**Theorem 2.** *If the functional differential inequalities*

$$(r(t)y'(t))' + p(t)y'(t) + q(t)y(t) \leq \pm \tilde{F}_\Psi(t) \tag{4}$$

*have no eventually positive solutions, then every solution $u(x,t)$ of the problem* (E), (B2) *is oscillatory in* $\Omega$, *where*

$$\tilde{F}_{\Psi}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t).$$

**Proof.** Suppose to the contrary that there is a nonoscillatory solution $u$ of problem (E), (B2). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Dividing (2) by $|G|$ and integrating over $G$, we have

$$\begin{aligned}\left(r(t)\tilde{U}'(t)\right)' + p(t)\tilde{U}'(t) - \frac{a(t)}{|G|}\int_G \Delta u(x,t)dx \\ +q(t)\tilde{U}(t) \le \tilde{F}(t),\ t \ge t_0. \end{aligned} \tag{5}$$

It follows from Green's formula that

$$\begin{aligned} &\frac{1}{|G|}\int_G \Delta u(x,t)dx \\ = &\frac{1}{|G|}\int_{\partial G}\left(-\mu u(x,t) + \tilde{\psi}\right) dS \le \tilde{\Psi}(t),\ t \ge t_0. \end{aligned} \tag{6}$$

Combining (5) with (6) yields

$$\left(r(t)\tilde{U}'(t)\right)' + p(t)\tilde{U}'(t) + q(t)\tilde{U}(t) \le \tilde{F}_{\Psi}(t),\ t \ge t_0.$$

Hence $\tilde{U}(t)$ is an eventually positive solution of (4) with $+\tilde{F}_{\Psi}(t)$. This contradicts the hypothesis and completes the proof.

## 3. Second order functional differential inequalities

We consider sufficient conditions for every solution $y(t)$ of the functional differential inequality

$$(r(t)y'(t))' + p(t)y'(t) + q(t)y(t) \le f(t) \tag{7}$$

to have no eventually positive solution, where $f(t) \in C([0,\infty);\mathbb{R})$.
**Theorem 3.** *Assume that*

(H2) *for any $T > 0$ there exists an interval $[t_k, t_{k+1}] \subset [T, \infty)$ such that*

$$f(t) \le 0,\ t \in [t_k, t_{k+1}].$$

*If the Riccati inequality*

$$z'(t) + \frac{1}{2}\frac{1}{P(t)}z^2(t) \le -Q(t) \tag{8}$$

*has no solution on* $[t_k, t_{k+1}]$, *then* (7) *has no eventually positive solution, where* $\phi(t) \in C^1((0,\infty);(0,\infty))$ *and*

$$\begin{aligned} P(t) &= \phi(t)r(t), \\ Q(t) &= \phi(t)q(t) - \frac{1}{2}\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)^2 \frac{r(t)}{\phi(t)}. \end{aligned}$$

**Proof.** Suppose that $y(t)$ is a positive solution of (7) on $[t_0,\infty)$ for some $t_0 > 0$. From the hypothesis (H2) there exists an interval $I = [t_k, t_{k+1}] \subset [t_0,\infty)$ such that $f(t) \le 0$ in $I$, and so

$$(r(t)y'(t))' + p(t)y'(t) + q(t)y(t) \le 0,\ t \in I. \tag{9}$$

If we set

$$w(t) = \frac{r(t)y'(t)}{y(t)},$$

then we obtain

$$\begin{aligned} w'(t) &= \frac{(r(t)y'(t))'}{y(t)} - r(t)y'(t)\frac{y'(t)}{y^2(t)} \\ &\le -\frac{p(t)}{r(t)}w(t) - q(t) - \frac{1}{r(t)}w^2(t),\ t \in I. \end{aligned} \tag{10}$$

Multiplying (10) by $\phi(t)$, we obtain

$$\phi(t)w'(t) + \frac{\phi(t)p(t)}{r(t)}w(t) + \frac{\phi(t)}{r(t)}w^2(t) \le -\phi(t)q(t), \tag{11}$$

and hence

$$\begin{aligned} (\phi(t)w(t))' + &\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)w(t) \\ &+\frac{\phi(t)}{r(t)}w^2(t) \le -\phi(t)q(t),\ t \in I. \end{aligned} \tag{12}$$

By Young's inequality (cf. Lemma 1), we have

$$\left|\left(\frac{\phi(t)p(t)}{r(t)} - \phi'(t)\right)w(t)\right|$$

$$
\begin{aligned}
&= \left| \left( \frac{\phi(t)p(t)}{r(t)} - \phi'(t) \right) \left( \frac{\phi(t)}{r(t)} \right)^{-\frac{1}{2}} \left( \frac{\phi(t)}{r(t)} \right)^{\frac{1}{2}} w(t) \right| \\
&\leq \frac{1}{2} \left( \left( \frac{\phi(t)p(t)}{r(t)} - \phi'(t) \right)^2 \left( \frac{\phi(t)}{r(t)} \right)^{-1} + \left( \frac{\phi(t)}{r(t)} \right) w^2(t) \right),
\end{aligned}
$$

which means that

$$
\begin{aligned}
&\left( \frac{\phi(t)p(t)}{r(t)} - \phi'(t) \right) w(t) \\
\geq\ & -\frac{1}{2} \left( \frac{\phi(t)}{r(t)} \right) w^2(t) - \frac{1}{2} \left( \frac{\phi(t)p(t)}{r(t)} - \phi'(t) \right)^2 \left( \frac{r(t)}{\phi(t)} \right), \ t \in I. \quad (13)
\end{aligned}
$$

Combining (12) with (13), we have

$$
\begin{aligned}
&(\phi(t)w(t))' + \frac{1}{2} \left( \frac{\phi(t)}{r(t)} \right) w^2(t) \\
\leq\ & -\phi(t)q(t) + \frac{1}{2} \left( \frac{\phi(t)p(t)}{r(t)} - \phi'(t) \right)^2 \left( \frac{r(t)}{\phi(t)} \right), \ t \in I. \quad (14)
\end{aligned}
$$

We define

$$
z(t) = \phi(t)w(t),
$$

then the above inequality reduces to

$$
\begin{aligned}
&z'(t) + \frac{1}{2} \left( \frac{1}{\phi(t)r(t)} \right) z^2(t) \\
\leq\ & -\phi(t)q(t) + \frac{1}{2} \left( \frac{\phi(t)p(t)}{r(t)} - \phi'(t) \right)^2 \left( \frac{r(t)}{\phi(t)} \right), \ t \in I.
\end{aligned}
$$

Therefore $z(t)$ is a solution of (8) on $I$. This contradicts the hypothesis and completes the proof.

**Theorem 4.** *Assume that* (H2) *holds, and let* $\phi(t) \in C^1((T_0, \infty); (0, \infty))$ *for some* $T_0 > 0$. *If for each* $T \geq T_0$ *there exist* $(H_1, H_2) \in \mathbb{H}$ *and* $a = t_k, b = t_{k+1}, c \in \mathbb{R}$ *such that* $T \leq a < c < b$ *and*

$$
\begin{aligned}
&\frac{1}{H_1(c,a)} \int_a^c H_1(s,a) \left\{ q(s) - \frac{1}{4} r(s) \lambda_1^2(s,a) \right\} \phi(s) ds \\
&\quad + \frac{1}{H_2(b,c)} \int_c^b H_2(b,s) \left\{ q(s) - \frac{1}{4} r(s) \lambda_2^2(b,s) \right\} \phi(s) ds > 0, \quad (15)
\end{aligned}
$$

*then* (7) *has no eventually positive solution, where*

$$\begin{aligned}\lambda_1(s,t) &= \frac{\phi'(s)}{\phi(s)} - \frac{p(s)}{r(s)} + h_1(s,t),\\ \lambda_2(t,s) &= \frac{\phi'(s)}{\phi(s)} - \frac{p(s)}{r(s)} - h_2(t,s).\end{aligned}$$

**Proof.** Suppose that $y(t)$ is a positive solution of (7) on $[t_0,\infty)$ for some $t_0 \geq T_0 > 0$. Then there exists an interval $[a,b]$ such that $t_0 \leq a < b$, and hence $y(t) > 0$ in $(a,b)$. Proceeding as in the proof of Theorem 3, there exists a positive solution $w(s)$ of (11) on $[a.b]$. Multiplying (11) by $H_2(t,s)$ and integrating over $[c,t]$ for $t \in [c,b)$, we have

$$\begin{aligned}&\int_c^t H_2(t,s)q(s)\phi(s)ds\\ \leq\ & -\int_c^t H_2(t,s)w'(s)\phi(s)ds - \int_c^t H_2(t,s)\frac{p(s)}{r(s)}w(s)\phi(s)ds\\ &\quad -\int_c^t H_2(t,s)\frac{1}{r(s)}w^2(s)\phi(s)ds\\ \leq\ & H_2(t,c)w(c)\phi(c) + \frac{1}{4}\int_c^t H_2(t,s)\lambda_2^2(t,s)r(s)\phi(s)ds\\ & -\int_c^t H_2(t,s)\left\{\sqrt{\frac{1}{r(s)}}w(s) - \frac{1}{2}\lambda_2(t,s)\sqrt{r(s)}\right\}^2\phi(s)ds,\end{aligned}$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)\left\{q(s) - \frac{1}{4}r(s)\lambda_2^2(t,s)\right\}\phi(s)ds \leq w(c)\phi(c).$$

Letting $t \to b-0$ in the above, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q(s) - \frac{1}{4}r(s)\lambda_2^2(b,s)\right\}\phi(s)ds \leq w(c)\phi(c). \tag{16}$$

On the other hand, multiplying (11) by $H_1(s,t)$ and integrating over $[t,c]$ for $t \in (a,c]$, we obtain

$$\begin{aligned}&\int_t^c H_1(s,t)q(s)\phi(s)ds\\ \leq\ & -\int_t^c H_1(s,t)w'(s)\phi(s)ds - \int_t^c H_1(s,t)\frac{p(s)}{r(s)}w(s)\phi(s)ds\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_t^c H_1(s,t)\frac{1}{r(s)}w^2(s)\phi(s)ds \\
&\le \; -H_1(c,t)w(c)\phi(c) + \frac{1}{4}\int_t^c H_1(s,t)\lambda_1^2(s,t)r(s)\phi(s)ds \\
&\quad - \int_t^c H_1(s,t)\left\{\sqrt{\frac{1}{r(s)}}w(s) - \frac{1}{2}\lambda_1(s,t)\sqrt{r(s)}\right\}^2\phi(s)ds,
\end{aligned}
$$

and so

$$
\frac{1}{H_1(c,t)}\int_t^c H_1(s,t)\left\{q(s) - \frac{1}{4}r(s)\lambda_1^2(s,t)\right\}\phi(s)ds \le -w(c)\phi(c).
$$

Letting $t \to a+0$ in the above, we obtain

$$
\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q(s) - \frac{1}{4}r(s)\lambda_1^2(s,a)\right\}\phi(s)ds \le -w(c)\phi(c). \tag{17}
$$

Adding (16) and (17), we easily obtain the following

$$
\begin{aligned}
&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q(s) - \frac{1}{4}r(s)\lambda_1^2(s,a)\right\}\phi(s)ds \\
&\quad + \frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q(s) - \frac{1}{4}r(s)\lambda_2^2(b,s)\right\}\phi(s)ds \le 0,
\end{aligned}
$$

which contradicts the condition (15). This contradiction proves that Theorem 4 holds.

**Theorem 5.** *Assume that* (H2) *holds. If*

$$
\limsup_{t\to\infty}\int_T^t H_1(s,T)\left\{q(s) - \frac{1}{4}r(s)\lambda_1^2(s,T)\right\}\phi(s)ds > 0 \tag{18}
$$

*and*

$$
\limsup_{t\to\infty}\int_T^t H_2(t,s)\left\{q(s) - \frac{1}{4}r(s)\lambda_2^2(t,s)\right\}\phi(s)ds > 0 \tag{19}
$$

*for some functions* $(H_1, H_2) \in \mathbb{H}$ *and each* $T \ge 0$*, then* (7) *has no eventually positive solution, where* $\phi(t) \in C^1((T_0,\infty);(0,\infty))$ *for some* $T_0 > 0$.

**Proof.** For any $T \ge t_0$, let $a = T$. In (18) we choose $T = a$. Then there exists $c > a$ such that

$$
\int_a^c H_1(s,a)\left\{q(s) - \frac{1}{4}r(s)\lambda_1^2(s,a)\right\}\phi(s)ds > 0 \tag{20}
$$

(cf. [14, Theorem 8.8.5]). In (19) we choose $T = c$. Then there exists $b > c$ such that

$$\int_c^b H_2(b,s)\left\{q(s) - \frac{1}{4}r(s)\lambda_2^2(b,s)\right\}\phi(s)ds > 0. \tag{21}$$

Combining (20) and (21), we obtain (15). The conclusion follows from Theorem 4, and the proof is completed.

## 4. Oscillation results

Using the Riccati inequality, we derive sufficient conditions for every solution of the hyperbolic equation (E) to be oscillatory. We are going to use the following lemma which is due to Usami [5].

**Lemma 2.** *If there exists a function $\phi(t) \in C^1([T_0,\infty);(0,\infty))$ such that*

$$\int_{T_1}^{\infty}\left(\frac{\bar{p}(t)|\phi'(t)|^{\beta}}{\phi(t)}\right)^{\frac{1}{\beta-1}}dt < \infty, \ \int_{T_1}^{\infty}\frac{1}{\bar{p}(t)(\phi(t))^{\beta-1}}dt = \infty,$$
$$\int_{T_1}^{\infty}\phi(t)\bar{q}(t)dt = \infty$$

*for some $T_1 \geq T_0$, then the Riccati inequality*

$$x'(t) + \frac{1}{\beta}\frac{1}{\bar{p}(t)}|x(t)|^{\beta} \leq -\bar{q}(t)$$

*has no solution on $[T,\infty)$ for all large $T$, where $\beta > 1$, $\bar{p}(t) \in C([T_0,\infty);(0,\infty))$ and $\bar{q}(t) \in C([T_0,\infty);\mathbb{R})$.*

Combining Theorems 1–5 and Lemma, we obtain the following theorems.
**Theorem 6.** *Assume that the hypothesis* (H1) *holds, and that:*

(H3) *for any $t > 0$ there exists $t_k, t_{k+1}, t_{k+2}$ such that $T \leq t_k < t_{k+1} < t_{k+2}$ and*

$$F_{\Psi}(t)\ [or\ \tilde{F}_{\Psi}(t)] \ = \ \begin{cases} \leq 0, & t \in [t_k, t_{k+1}], \\ \geq 0, & t \in [t_{k+1}, t_{k+2}]. \end{cases}$$

*If*

$$\int_{T_1}^{\infty}\left(\frac{P(t)\phi'(t)^2}{\phi(t)}\right)dt < \infty, \quad \int_{T_1}^{\infty}\frac{1}{P(t)\phi(t)}dt = \infty,$$
$$\int_{T_1}^{\infty}\phi(t)Q(t)dt = \infty,$$

*then every solution* $u(x,t)$ *of* (E), (B1) (*or* (E), (B2)) *is oscillatory in* $\Omega$.

**Theorem 7.** *Assume that the hypotheses* (H1) *and* (H3) *hold. If for each* $T > 0$, *there exist functions* $(H_1, H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a = t_k, b = t_{k+1}, d = t_{k+2}, c, \tilde{c} \in \mathbb{R}$ *such that* $T \le a < c < b < \tilde{c} < d$, (15) *holds and*

$$\frac{1}{H_1(\tilde{c},b)} \int_b^{\tilde{c}} H_1(s,b) \left\{ q(s) - \frac{1}{4} r(s) \lambda_1^2(s,b) \right\} \phi(s) ds$$
$$+ \frac{1}{H_2(d,\tilde{c})} \int_{\tilde{c}}^d H_2(d,s) \left\{ q(s) - \frac{1}{4} r(s) \lambda_2^2(d,s) \right\} \phi(s) ds > 0,$$

*then every solution* $u(x,t)$ *of* (E), (B1) (*or* (E), (B2)) *is oscillatory in* $\Omega$.

**Theorem 8.** *Assume that the hypotheses* (H1) *and* (H3) *hold. If* (18) *and* (19) *hold for some functions* $(H_1, H_2) \in \mathbb{H}$ *and each* $T \ge 0$, *then every solution* $u(x,t)$ *of* (E), (B1) (*or* (E), (B2)) *is oscillatory in* $\Omega$.

**Example.** We consider the hyperbolic equation

$$\begin{aligned} &\frac{\partial}{\partial t}\left( e^t \frac{\partial}{\partial t} u(x,t) \right) - 2e^t \frac{\partial}{\partial t} u(x,t) - a(t) u_{xx}(x,t) \\ &\qquad + \frac{3}{4} e^{2t} u(x,t) + \frac{1}{2} e^{\frac{t}{2}} |u(x,t)|^2 u(x,t) + 3 |u(x,t)|^{-\frac{2}{3}} u(x,t) \\ &= \sin x \sin t, \ (x,t) \in (0,\pi) \times (0,\infty), \qquad (22) \\ &u(0,t) = u(\pi,t) = 0, \ t > 0. \qquad (23) \end{aligned}$$

Here $r(t) = e^t$, $p(t) = -2e^t$, $q_0(t) = (3/4)e^{2t}$, $q_1(t) = (1/2)e^{t/2}$, $\beta = 3$, $q_2(t) = 3$, $\gamma = 1/3$ and $f(x,t) = \sin x \sin t$. By direct calculation we have $F(t) = F_\Psi(t) = \frac{\pi}{4} \sin t$ and

$$q(t) = \frac{3}{4} e^{2t} + 4 \times 3^{-\frac{3}{4}} \times \left( \frac{1}{2} e^{\frac{t}{2}} \right)^{\frac{1}{4}} \times 3^{\frac{3}{4}} = e^{2t}.$$

Choosing $\phi(t) = e^{-t}$, we obtain

$$Q(t) = e^{-t} \times q(t) - \frac{1}{2} \left( \frac{e^{-t} \times (-e^t)}{e^t} + e^{-t} \right)^2 \frac{e^t}{e^{-t}} = e^t,$$

and so

$$\begin{aligned} &\int^\infty \left( \frac{P(t)\phi'(t)^2}{\phi(t)} \right) dt = \int^\infty e^{-t} dt < \infty, \\ &\int^\infty \frac{1}{P(t)\phi(t)} dt = \int^\infty e^t dt = \infty, \\ &\int^\infty \phi(t) Q(t) dt = \int^\infty (1) dt = \infty. \end{aligned}$$

Hence all conditions of Theorem 6 are satisfied. Moreover, the hypotheses (18) and (19) of Theorem 8 hold, since

$$\limsup_{t\to\infty} \int_T^t (e^T - e^s)^2 \left\{ e^{2s} - \frac{1}{4} e^s \left( \frac{2e^T}{e^T - e^s} \right)^2 \right\} e^{-2s} ds > 0$$

and

$$\limsup_{t\to\infty} \int_T^t (e^t - e^s)^2 \left\{ e^{2s} - \frac{1}{4} e^s \left( \frac{2e^s}{e^t - e^s} \right)^2 \right\} e^{-2s} ds > 0,$$

where

$$\phi(t) = e^{-2t} \quad \text{and} \quad H_1(s,t) = H_2(t,s) = (e^t - e^s)^2.$$

It follows from Theorems 6 or 8 that for every solution $u$ of the problem (22), (23) is oscillatory in $(0,\pi) \times (0,\infty)$.

# References


[1] Bai Y., Liu L., New oscillation criteria for second-order delay differential equations with mixed nonlinearities, Discrete Dynamics in Nature and Society, 2010 , **2010**: 1–9.

[2] Cui S., Xu Z., Interval oscillation theorems for second order nonlinear partial delay differential equations, Differ Equ Appl, 2009, **1**: 379–391.

[3] Hassan T. S., Erbe L., Peterson A., Forced oscillation of second order differential equations with mixed nonlinearities, Acta Math Sci, 2011, **2**: 613–626.

[4] Hardy G., Littlewood J. E., Polya G., Inequalities, Second Edition, Cambridge Mathematical Library, Cambridge, 1988.

[5] Usami H., Some oscillation theorem for a class of quasilinear elliptic equations, Ann Mat Pura Appl, 1998 **175**: 277–283.

[6] Shoukaku Y., Forced oscillations of nonlinear hyperbolic equations with functional arguments via Riccati method, Appl Appl Math, 2010, Special Issue No.1: 122–153.

[7] Shoukaku Y., Yoshida N., Oscillations of nonlinear hyperbolic equations with functional arguments via Riccati method, Appl Math Comput, 2010, **217**: 143–151.

[8] Li C., Chen S., Oscillation of second-order functional differential equations with mixed nonlinearities and oscillatory potentials, Appl Math Comput, 2009, **210**: 504–507.

[9] Li W. N., Oscillation for solutions of partial differential equations with delays, Demonstratio Math, 2000, **33**: 319–332.

[10] Li W. T., Interval oscillation criteria for second order nonlinear differential equations with damping, Taiwanese J Math, 2003, **7**: 461–475.

[11] Li W. T., Huo H. F., Interval oscillation criteria for nonlinear second-order differential equations, Indian J Pure Appl Math, 2001, **32**: 1003–1014.

[12] Wang J., Meng F., Liu S., Integral average method for oscillation of second order partial differential equations with delays, Appl Math Comput, 2007, **187**: 815–823.

[13] Wang J., Meng F., Liu S., Interval oscillation criteria for second order partial differential equations with delays, J Comput Appl Math, 2008, **212**: 397–405.

[14] Yoshida N., Oscillation Theory of Partial Differential Equations, World Scientific Publishing Co. Pte. Ltd., 2008.

[15] Zhong Y. H., Yuan Y. H., Oscillation criteria of hyperbolic equations with continuous deviating arguments, Kyungpook Math J, 2007, **47**: 347–356.

[16] Sun Y. G., Wong J. S. W., Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J Math Anal Appl, 2007, **334**: 549–560.

[17] Sun Y., Meng F. W., Interval criteria for oscillation of second-order differential equations with mixed nonlinearities, Appl Math Comput, 2008, **198**: 375–381.

(Received January 13, 2012)