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Asymptotic behavior of positive solutions of odd order Emden-Fowler type differential equations in the framework of regular variation

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Abstract. The asymptotic behavior of solutions of the odd-order differential equation of Emden-Fowler type

$$x^{(2n+1)}(t) + q(t)|x(t)|^{\gamma}\text{sgn}\, x(t) = 0,$$

is studied in the framework of regular variation, under the assumptions that $0 < \gamma < 1$ and $q(t) : [a, \infty) \to (0, \infty)$ is regularly varying function. It is shown that complete and accurate information can be acquired about the existence of all possible positive solutions and their asymptotic behavior at infinity.

Keywords: odd-order differential equation, intermediate solution, regularly varying function, slowly varying function, asymptotic behavior of solutions

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1 Introduction

The objective of this paper is to make a detailed study of the existence and the asymptotic behavior of positive solutions of the nonlinear odd-order differential equation

$$\text{(A)} \qquad x^{(2n+1)}(t) + q(t)|x(t)|^{\gamma}\text{sgn}\, x(t) = 0,$$

where γ is a constant such that $0 < \gamma < 1$ and $q : [a, \infty) \to (0, \infty)$ is a continuous function. Equation (A) is often referred to as *sublinear differential equation* in this case, while equation (A) for which $\gamma > 1$ is called *superlinear differential equation.*

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A solution $x(t)$ of (A) existing in an infinite interval of the form $[T_x, \infty)$ is said to be *proper* if

$$\sup\{|x(t)| : t \geq T\} > 0 \quad \text{for any} \;\; T \geq T_x.$$

A proper solution is called *oscillatory* if it has an infinite sequence of zeros clustering at infinity and *nonoscillatory* otherwise. Thus, a nonoscillatory solution is eventually positive or eventually negative.

Sublinear equation (A) may have both oscillatory and nonoscillatory solutions on $[t_0, \infty)$ for some $t_0 > a$.

Theorem A. *Any proper solution $x(t)$ of sublinear equation* (A) *is either oscillatory or satisfies*

$$|x^{(i)}(t)| \downarrow 0 \quad \textit{as} \;\; t \to \infty, \quad i = 0, 1, \ldots, 2n, \tag{1.1}$$

if and only if

$$\int_a^\infty t^{(2n-1)\gamma} q(t) dt = \infty\,. \tag{1.2}$$

Our main interest is in nonoscillatory solutions of equation (A). If $x(t)$ satisfies (A), then so does $-x(t)$, and so in studying nonoscillatory solutions of (A) it suffices to restrict our attention to its (eventually) positive solutions. Let $\mathcal{P}$ denote the set of eventually positive solutions of equation (A), while $\mathcal{P}_k$, $0 \leq k \leq 2n+1$ denote the set of all $x \in \mathcal{P}$ satisfying

$$\begin{cases} x^{(i)}(t) > 0, & t \geq T_x, \quad 0 \leq i \leq k; \\ (-1)^{i-k} x^{(i)}(t) > 0, & t \geq T_x, \quad k \leq i \leq 2n+1. \end{cases} \tag{1.3}$$

By the well-known Kiguradze's lemma (see [5]) every positive solution $x(t) \in \mathcal{P}$ falls into one and only one class $\mathcal{P}_k$ with k such that $k \in \{0, 2, \ldots, 2n\}$. In other words, the set $\mathcal{P}$ has the decomposition

$$\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_{2n}\,. \tag{1.4}$$

Since $x^{(i)}(t)$, $i \in \{0, 1, \cdots, 2n\}$, are eventually monotone, they tend to finite or infinite limits as $t \to \infty$, i.e.

$$\lim_{t\to\infty} x^{(i)}(t) = \omega_j \in [0, \infty] \iff \lim_{t\to\infty} \frac{x(t)}{t^i} = \text{const} \in [0, \infty], \qquad i \in \{0,\; 1, \cdots, 2n\}.$$

If $x \in \mathcal{P}_k$, then the set of its asymptotic values $\{\omega_i : i = 0, 1, 2, \ldots, 2n\}$ falls into one of the following three cases:

$$\begin{cases} \omega_0 = \omega_1 = \cdots = \omega_{k-1} = \infty, \;\; \omega_k \in (0, \infty), \;\; \omega_{k+1} = \omega_{k+2} = \cdots = \omega_{2n} = 0; \\ \omega_0 = \omega_1 = \cdots = \omega_{k-1} = \infty, \;\; \omega_k = \omega_{k+1} = \cdots = \omega_{2n} = 0; \\ \omega_0 = \omega_1 = \cdots = \omega_{k-2} = \infty, \;\; \omega_{k-1} \in (0, \infty), \;\; \omega_k = \omega_{k+1} = \cdots = \omega_{2n} = 0, \end{cases} \tag{1.5}$$

for $k \in \{2,\; 4, \ldots, 2n\}$, or into one of the following two cases:

$$\begin{cases} \omega_0 = \omega_1 = \cdots = \omega_{2n} = 0; \\ \omega_0 \in (0, \infty), \;\; \omega_1 = \omega_2 = \cdots = \omega_{2n} = 0\,, \end{cases} \tag{1.6}$$

in case $k = 0$.

For simplicity of notation we introduce the symbols $\sim$ and $\prec$ to denote the asymptotic equivalence and the asymptotic dominance of two positive functions $f(t)$ and $g(t)$:

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1,$$

$$f(t) \prec g(t), \quad t \to \infty \quad \Longleftrightarrow \quad g(t) \succ f(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty\,,$$

and the classes of positive solutions:

$$\begin{aligned}
\mathcal{P}(\mathrm{I}_j) &= \left\{ x \in \mathcal{P} \,:\, \lim_{t\to\infty} \frac{x(t)}{t^j} = \text{const} > 0 \right\} \\
&= \left\{ x \in \mathcal{P} \,:\, x(t) \sim c_j t^j, \;\; t \to \infty, \;\; c_j > 0 \right\}, \quad j \in \{0,\ 1, \cdots, 2n\} \\
\mathcal{P}(\mathrm{II}_k) &= \left\{ x \in \mathcal{P} \,:\, \lim_{t\to\infty} \frac{x(t)}{t^{k-1}} = \infty \text{ and } \lim_{t\to\infty} \frac{x(t)}{t^k} = 0 \right\} \\
&= \left\{ x \in \mathcal{P} \,:\, t^{k-1} \prec x(t) \prec t^k, \;\; t \to \infty \right\}, \quad k \in \{2,\ 4, \cdots, 2n\}\,, \\
\mathcal{P}(\mathrm{II}_0) &= \left\{ x \in \mathcal{P} \,:\, \lim_{t\to\infty} x(t) = 0 \right\} = \left\{ x \in \mathcal{P} \,:\, x(t) \prec 1, \;\; t \to \infty \right\},
\end{aligned}$$

All solutions of types $\mathcal{P}(\mathrm{I}_j)$, $j \in \{0,\ 1, \cdots, 2n\}$ are collectively called *primitive solutions*, solutions of types $\mathcal{P}(\mathrm{II}_k)$, $j \in \{1,\ 2, \cdots, 2n\}$ will be referred to as *intermediate solutions* of (A), while solutions of type $\mathcal{P}(\mathrm{II}_0)$ are called *decaying solutions*. Thus,

$$\begin{aligned}
\mathcal{P}_0 &= \mathcal{P}(\mathrm{I}_0) \cup \mathcal{P}(\mathrm{II}_0), \\
\mathcal{P}_k &= \mathcal{P}(\mathrm{I}_{k-1}) \cup \mathcal{P}(\mathrm{II}_k) \cup \mathcal{P}(\mathrm{I}_k) \quad \text{for every} \quad k \in \{2,\ 4, \cdots, 2n\},
\end{aligned}$$

which due to (1.4) means that we have the following classification of positive solutions of equation (A) according to their asymptotic behavior at infinity:

$$\mathcal{P} = \mathcal{P}(\mathrm{I}) \cup \mathcal{P}(\mathrm{II}), \quad \text{where} \;\; \mathcal{P}(\mathrm{I}) = \bigcup_{j=0}^{2n} \mathcal{P}(\mathrm{I}_j), \quad \mathcal{P}(\mathrm{II}) = \bigcup_{i=0}^{n} \mathcal{P}(\mathrm{II}_{2i})\,.$$

Sharp criteria for the existence of solutions belonging to $\mathcal{P}(\mathrm{I}_j)$, $j \in \{0,\ 1, \cdots, 2n\}$ and $\mathcal{P}(\mathrm{II}_k)$, $k \in \{2,\ 4, \ldots, 2n\}$ can be given explicitly (for the proof see Kiguradze, Chanturia [6, Theorem 16.9], Kusano, Naito [13] and Tanaka [14]).

Theorem 1.1 *Equation* (A) *has a positive solution* $x(t) \in \mathcal{P}(\mathrm{I}_j)$, $j \in \{0,\ 1, \cdots, 2n\}$ *if and only if*

$$Q_j = \int_a^\infty t^{2n-j(1-\gamma)} q(t)\,dt < \infty. \tag{1.7}$$

Theorem 1.2 *Sublinear equation* (A) *has a positive solution* $x(t) \in \mathcal{P}(\mathrm{II}_k)$, $k \in \{2,\ 4, \ldots, 2n\}$ *if and only if*

$$Q_k = \int_a^\infty t^{2n-k(1-\gamma)} q(t)\,dt < \infty \tag{1.8}$$

and

$$Q_{k-1} = \int_a^\infty t^{2n-(k-1)(1-\gamma)} q(t)dt = \infty. \tag{1.9}$$

Therefore, the following questions naturally arise:

(i) If (1.2) holds, does (A) really possess decaying positive solutions? If so, what can be said about the exact asymptotic decay of such solutions?

(ii) Is it possible to determine the accurate asymptotic behavior at infinity of intermediate solutions of equation (A)?

The recent development of the study of second order differential equations by means of regular variation (in the sense of Karamata) as demonstrated in the papers [3], [4], [7], [8], [11], [12] seem to suggest the possibility of investigating the higher-order problems in the framework of regularly varying functions, more specifically, by limiting ourselves to equation (A) with regularly varying coefficient $q(t)$. The objective of this paper is to show that theory of regular variation can provide us with full information about the existence and asymptotic behavior of positive solutions of the odd order differential equation (A) with regularly varying coefficient $q(t)$.

Recently, Evtukhov and Samoilenko in [2] studied the differential equation

$$x^{(m)} = \alpha_0 q(t)\varphi(x), \tag{1.10}$$

where $\alpha_0 \in \{-1, 1\}$, $q : [a, \omega) \to (0, +\infty)$ is a continuous function, $-\infty < a < \omega \leq +\infty$ and $\varphi : \Delta_{Y_0} \to (0, +\infty)$ is a continuous regularly varying function of index $\gamma \neq 1$ as $y \to Y_0$, $Y_0 \in \{-\infty, 0, +\infty\}$ and Δ_{Y_0} is a one sided neoghborhood of Y_0. They gave sharp conditions for the existence of $P_\omega(Y_0, \lambda_0)-$solutions possessing certain asymptotic behavior, where $-\infty \leq \lambda_0 \leq +\infty$. Such solutions are defined on an interval $[t_0, \omega) \subset (a, \omega)$ and satisfy conditions

$$\begin{aligned} &\lim_{t\uparrow\omega} y(t) = Y_0, \qquad \lim_{t\uparrow\omega} y^{(k)}(t) \in \{-\infty, 0, +\infty\}, \ (k = 1, 2, \ldots, n-1), \\ &\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0\,. \end{aligned} \tag{1.11}$$

The condition imposed on the function $q(t)$ in main results of [2] means actually that it is either of regular or rapid variation. However, this fact is neither used nor mentioned by Evtukhov and Samoilenko, which makes their method of proofs different from ours and the statements on solutions somewhat weaker than ours. Some comments along this line is given in the last section of our paper.

2 Basic properties of regularly varying functions

The class of regularly varying functions was introduced in 1930 by J. Karamata by the following:

Definition 2.1 A measurable function $f : [0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0. \tag{2.1}$$

We denote by RV(ρ) the set of all regularly varying functions of index ρ. If in particular $\rho = 0$, we often use SV instead of RV(0) and refer to members of SV as *slowly varying* functions. It is clear that an RV(ρ)-function $f(t)$ is expressed as $f(t) = t^\rho L(t)$ with $L(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation.

Definition 2.2 A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^\rho L(t)$ with $L(t) \in \mathrm{SV}$ satisfying

$$\lim_{t\to\infty} L(t) = \text{const} > 0.$$

Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) denotes the set of all trivial RV(ρ)-functions (or the set of all nontrivial RV(ρ)-functions).

Typical examples of slowly varying functions are all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\right\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm.

The following result concerns operations which preserve slow variation.

Proposition 2.1 *Let $L(t), L_1(t), L_2(t)$ be slowly varying. Then, $(L(t))^\alpha$ for any $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$ as $t \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition 2.2 *If $L(t) \in \mathrm{SV}$, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^\varepsilon L(t) = \infty, \qquad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

The following result, termed *Karamata's integration theorem*, will play a central role in establishing our main results in Sections 3.

Proposition 2.3 *Let $L(t) \in \mathrm{SV}$. Then,*

(i) *if $\alpha > -1$,*

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if $\alpha < -1$,*

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0\,.$$

The reader is referred to Bingham, Goldie and Teugels [1] for the most complete exposition of theory of regular variation and its applications and to Marić [16] for the comprehensive survey of results up to 2000 on the asymptotic analysis of second order linear and nonlinear ordinary differential equations in the framework of regular variation.

3 Intermediate regularly varying solutions of (A)

We first study intermediate regularly varying solutions of equation (A) with regularly varying coefficient $q(t)$. Thus, in what follows the function $q(t)$ is assumed to be regularly varying of index σ expressed as

$$q(t) = t^{\sigma} l(t), \quad l(t) \in \mathrm{SV}\,, \tag{3.1}$$

and k is assumed to be even integer such that $2 \le k \le 2n$. Let $x(t) \in \mathcal{P}(\mathrm{II}_k)$ be a regularly varying solution of (A). In view of (1.3) there exists positive constants c_1, c_2 and $T > a$ such that

$$c_1 t^{k-1} \le x(t) \le c_2 t^k, \qquad t \ge T\,.$$

Thus, if the regularity index of $x(t)$ is ρ, i.e. $x(t) = t^{\rho}\xi(t)$, $\xi(t) \in \mathrm{SV}$, then clearly $\rho \in [k-1, k]$ and $\xi(t) \to \infty$ or $\xi(t) \to 0$ as $t \to \infty$ according as $\rho = k-1$ or $\rho = k$. Therefore, the class of regularly varying solutions of type $\mathcal{P}(\mathrm{II}_k)$, if non-empty, is divided into three types of subclasses composed of regularly varying solutions belonging respectively to

$$\text{ntr-RV}(k-1), \quad \mathrm{RV}(\rho) \ \text{ with } \ \rho \in (k-1, k), \quad \text{ntr-RV}(k)\,. \tag{3.2}$$

It will be shown that the class of regularly varying solutions of type $\mathcal{P}(\mathrm{II}_k)$ of equation (A) coincides with only one of the three subsets in (3.2), depending on the regularity index of $q(t)$, and that all members belonging to that subset has one and the same asymptotic behavior at infinity.

Moreover, since $x(t) \in \mathcal{P}(\mathrm{II}_k) \subset \mathcal{P}_k$, in view of (1.5), we may integrate (A) $(2n+1-k)-$times from t to ∞ and then $k-$times from t_0 to t, to get

$$x(t) = \sum_{j=0}^{k-1} x^{(j)}(t_0)\frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{2n-k}}{(2n-k)!}\, q(r)x(r)^{\gamma} dr ds, \quad t \ge t_0. \tag{3.3}$$

Using that $x(t)/t^k \to 0$, $t \to \infty$, from (3.3) we obtain the integral asymptotic relation

$$\int_a^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{2n-k}}{(2n-k)!}\, q(r)x(r)^{\gamma} dr ds \sim x(t), \quad t \to \infty, \tag{3.4}$$

which can be concerned as an "approximation" of (3.3) at infinity. Common way of determining the desired intermediate solution of (A) would be solving the integral equation (3.3) with the help of fixed point technique. A closed convex subset $\mathcal{X}$ of $C[t_0, \infty)$, which should be chosen in a such way that appropriate integral operator $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send it into a relatively compact subset of $C[t_0, \infty)$, will be here found by means of regularly varying functions of index $\rho \in [k-1, k]$ which satisfy the integral asymptotic relation (3.4).

Let us interpret the conditions (1.8) and (1.9) in the language of regular variation. Since

$$\int_a^\infty t^{2n-k(1-\gamma)} q(t)dt = \int_a^\infty t^{2n-k(1-\gamma)+\sigma} l(t)dt$$

it is easy to see that

$$\begin{aligned} &\int_a^\infty t^{2n-k(1-\gamma)} q(t)dt < \infty \\ &\quad \Longleftrightarrow (i) \ \ \sigma < k(1-\gamma) - 2n - 1, \ \text{or} \\ &\qquad\quad (ii) \ \ \sigma = k(1-\gamma) - 2n - 1 \ \text{ and } \int_a^\infty \frac{l(t)}{t}\, dt = \int_a^\infty t^{2n-k(1-\gamma)} q(t)dt < \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_a^\infty t^{2n-(k-1)(1-\gamma)} q(t)dt = \infty \\ &\quad \Longleftrightarrow (i) \ \ \sigma > (k-1)(1-\gamma) - 2n - 1, \ \text{or} \\ &\qquad\quad (ii) \ \ \sigma = (k-1)(1-\gamma) - 2n - 1 \ \text{ and } \int_a^\infty \frac{l(t)}{t}\, dt = \int_a^\infty t^{2n-(k-1)(1-\gamma)} q(t)dt = \infty. \end{aligned}$$

This observation, with the statement of Theorem 1.2, suggests us to carry out the study of intermediate solutions belonging to the class $\mathcal{P}(\mathrm{II}_k)$ by distinguishing the cases:

$$\sigma = (k-1)(1-\gamma) - 2n - 1 \ \text{ and } \ Q_{k-1} = \infty, \tag{3.5}$$

$$\sigma \in \big((k-1)(1-\gamma) - 2n - 1, k(1-\gamma) - 2n - 1\big), \tag{3.6}$$

$$\sigma = k(1-\gamma) - 2n - 1 \ \text{ and } \ Q_k < \infty. \tag{3.7}$$

Actually, we verify that the above conditions, respectively, are necessary and sufficient for the existence of three types of regularly varying solutions of (A) listed in (3.2) with precise asymptotic behavior at infinity and that the regularity index ρ of such solution is uniquely determined by γ, n and the regularity index σ of $q(t)$.

For the proof of our main results we make use of the following lemma - general L'Hospital's rule (see [15]):

Lemma 3.1 *Let $f, g \in C^1[T, \infty)$ and*

$$\lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \quad \textit{for all large } t$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \textit{and} \quad g'(t) < 0 \quad \textit{for all large } t\,.$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}\,.$$

We first show two preparatory results.

Lemma 3.2 *If $q(t) \in \mathrm{RV}(\sigma)$ and $x(t) = t^{\rho}\xi(t)$, $l(t) \in \mathrm{SV}$ is a solution of* (A) *such that*

$$t^{k-1} \prec x(t) \prec t^{k}, \quad t \to \infty$$

for some $k \in \{2,\ 4, \dots, 2n\}$, then one of the following three statements holds:
(i) $\sigma + \rho\gamma = k - 2n - 1$, $x(t) \in \mathrm{RV}(k)$ *and*

$$x(t) \sim \frac{t^{k}}{(2n-k)! \cdot k!} \int_{t}^{\infty} s^{-1} l(s)\xi(s)^{\gamma} ds, \quad t \to \infty. \tag{3.8}$$

(ii) $k - 2n - 2 < \sigma + \rho\gamma < k - 2n - 1$, $x(t) \in \mathrm{RV}\big(\sigma + \rho\gamma + 2n + 1\big)$ *and*

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2n+1} l(t)\xi(t)^{\gamma}}{P_k Q_k}, \quad t \to \infty \tag{3.9}$$

where

$$P_k = \prod_{i=1}^{2n+1-k} [-(\sigma + \rho\gamma + i)], \quad Q_k = \prod_{i=1}^{k} (\sigma + \rho\gamma + 2n + 1 - k + i)\,.$$

(iii) $\sigma + \rho\gamma = k - 2n - 2$, $x(t) \in \mathrm{RV}(k-1)$ *and*

$$x(t) \sim \frac{t^{k-1}}{(2n+1-k)! \cdot (k-1)!} \int_{t_0}^{t} s^{-1} l(s)\xi(s)^{\gamma} ds, \quad t \to \infty. \tag{3.10}$$

Proof. Since $x(t) \in \mathcal{P}(\mathrm{II}_k) \subset \mathcal{P}_k$, due to (1.5), $w_i = x^{(i)}(\infty) = 0$ for $i \in \{k, k+1, \dots, 2n\}$, so that function $q(t)x(t)^{\gamma} = t^{\sigma+\rho\gamma} l(t)\xi(t)^{\gamma}$ is $2n+1-k$ times integrable on $[t_0, \infty)$, implying that $\sigma + \rho\gamma \le k - 2n - 1$. Note that integration of (A) $2n+1-k$ times on $[t, \infty)$ gives

$$x^{(k)}(t) = \int_{t}^{\infty} \frac{(s-t)^{2n-k}}{(2n-k)!} s^{\sigma+\rho\gamma} l(s)\xi(s)^{\gamma} ds, \qquad t \ge t_0\,. \tag{3.11}$$

We distinguish the following four cases:
(i) $\sigma + \rho\gamma = k - 2n - 1$,
(ii) $k - 2n - 2 < \sigma + \rho\gamma < k - 2n - 1$,
(iii) $\sigma + \rho\gamma = k - 2n - 2$ and
(iv) $\sigma + \rho\gamma < k - 2n - 2$.

(i) Let $\sigma + \rho\gamma = k - 2n - 1$. If $k = 2n$, then by (3.11) and Proposition 2.3 (iii)

$$x^{(2n)}(t) = \int_{t}^{\infty} s^{-1} l(s)\xi(s)^{\gamma} ds \in \mathrm{SV}, \qquad t \ge t_0. \tag{3.12}$$

Integrating (3.12) $2n$ times on $[t_0, t]$ and applying Karamata's integration theorem (Proposition 2.3 (i)) we obtain

$$x(t) \sim \frac{t^{2n}}{(2n)!} \int_{t}^{\infty} s^{-1} l(s)\xi(s)^{\gamma} ds \in \mathrm{RV}(2n), \quad t \to \infty. \tag{3.13}$$

If $k \in \{2, 4, \ldots, 2n-2\}$, integration of (A) $2n-k$ times on $[t, \infty)$ gives

$$-x^{(k+1)}(t) \sim \frac{t^{-1} l(t) \xi(t)^{\gamma}}{(2n-k)!}, \quad t \to \infty, \tag{3.14}$$

which integrating once more on $[t, \infty)$ with the application of Karamata's integration theorem (Proposition 2.3 (ii)) yields

$$x^{(k)}(t) \sim \frac{1}{(2n-k)!} \int_t^{\infty} s^{-1} l(s) \xi(s)^{\gamma} ds \in \mathrm{SV}\,. \tag{3.15}$$

Further integration of (3.15) k times on $[t_0, t]$ gives (3.8). Noting that (3.8) with $k = 2n$ is identical to (3.13), we prove that in this case for any $k \in \{2,\ 4, \ldots, 2n\}$ regularly varying solution $x(t)$ of (A) satisfies (3.8).

(ii) Let $k-2n-2 < \sigma+\rho\gamma < k-2n-1$. Applying Karamata's integration theorem (Proposition 2.3 (ii)) k times on the right-hand side of (3.11) we have

$$x^{(k)}(t) \sim \frac{t^{\sigma+\rho\gamma+2n+1-k} l(t) \xi(t)^{\gamma}}{P_k}, \quad t \to \infty. \tag{3.16}$$

Since $\sigma+\rho\gamma+2n+1-k > -1$ we may integrate (3.16) k times from t_0 to t. Repeated application of Proposition 2.3 (i) then shows that $x(t)$ satisfies (3.9). Thus, by (3.9) $x(t) \in \mathrm{RV}(\sigma+\rho\gamma+2n+1)$, where $\sigma+\rho\gamma+2n+1 \in (k-1, k)$.

(iii) Let $\sigma+\rho\gamma = k-2n-2$. Then, (3.16) becomes

$$x^{(k)}(t) \sim \frac{t^{-1} l(t) \xi(t)^{\gamma}}{(2n+1-k)!}, \quad t \to \infty\,,$$

and integrating this k times on $[t_0, t]$ we see via Karamata's integration theorem that $x(t)$ satisfies (3.10).

(iv) Let $\sigma+\rho\gamma < k-2n-2$. This case is impossible because since $\sigma+\rho\gamma+2n+1-k < -1$ the right-hand side of (3.16) is integrable on $[t_0, \infty)$, contradicting the fact that $x^{(k-1)}(\infty) = \infty$. ⊠

Lemma 3.3 *Suppose that* $q(t) \in \mathrm{RV}(\sigma)$. *Let* $k \in \{2,\ 4, \ldots, 2n\}$ *and let constant* ρ *be defined by*

$$\rho = \frac{\sigma+2n+1}{1-\gamma}\,. \tag{3.17}$$

(i) If (3.5) *holds, the function* $X_k(t) \in \mathrm{RV}(k-1)$ *defined on* $[a, \infty)$ *by*

$$X_k(t) = t^{k-1} \left[\frac{1-\gamma}{(2n+1-k)! \cdot (k-1)!} \int_a^t s^{2n-(k-1)(1-\gamma)} q(s) ds \right]^{\frac{1}{1-\gamma}}, \tag{3.18}$$

satisfies the asymptotic relation (3.4).
(ii) If (3.6) *holds, the function* $Y_k(t) \in \mathrm{RV}(\rho)$ *defined on* $[a, \infty)$ *by*

$$Y_k(t) = \left[\frac{t^{2n+1} q(t)}{L(\rho, k)} \right]^{\frac{1}{1-\gamma}} \tag{3.19}$$

where

$$L(\rho,k)=\prod_{i=0}^{k-1}(\rho-i)\cdot\prod_{i=k}^{2n}(i-\rho)\,. \tag{3.20}$$

satisfies the asymptotic relation (3.4).
(iii) If (3.7) *holds, the function* $Z_k(t)\in\mathrm{RV}(k)$ *defined on* $[a,\infty)$ *by*

$$Z_k(t)=t^k\left[\frac{1-\gamma}{(2n-k)!\cdot k!}\int_t^\infty s^{2n-k(1-\gamma)}q(s)ds\right]^{\frac{1}{1-\gamma}}, \tag{3.21}$$

satisfies the asymptotic relation (3.4).

Proof. (A) Let (3.5) holds. Then, using that $X_k(t)=t^{k-1}\xi_k(t)\in\mathrm{RV}(k-1)$ where

$$\xi_k(t)=\left[\frac{1-\gamma}{(2n+1-k)!\cdot(k-1)!}\int_a^t s^{2n-(k-1)(1-\gamma)}q(s)ds\right]^{\frac{1}{1-\gamma}}\in\mathrm{SV}$$

and $q(t)X_k(t)^\gamma=t^{k-2-2n}l(t)\xi_k(t)^\gamma\in\mathrm{RV}(k-2-2n)$, by Proposition 2.3 (ii), we have

$$\begin{aligned}\int_t^\infty\frac{(s-t)^{2n-k}}{(2n-k)!}\,q(s)X_k(s)^\gamma ds &= \int_t^\infty\frac{(s-t)^{2n-k}}{(2n-k)!}\,s^{k-2-2n}l(s)\xi_k(s)^\gamma ds\\ &\sim \frac{t^{-1}l(t)\xi_k(t)^\gamma}{(2n+1-k)!},\quad t\to\infty\,.\end{aligned} \tag{3.22}$$

Integration of (3.22) on $[a,t]$ then implies

$$\int_a^t\int_s^\infty\frac{(r-s)^{2n-k}}{(2n-k)!}\,q(r)X_k(r)^\gamma drds\sim\int_a^t\frac{s^{-1}l(s)\xi_k(s)^\gamma}{(2n+1-k)!}\,ds\in\mathrm{SV}\,, \tag{3.23}$$

so Karamata's integration theorem applied to (3.23) $k-1$ times gives

$$\begin{aligned}\int_a^t\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{2n-k}}{(2n-k)!}q(r)X_k(r)^\gamma drds &\sim \int_a^t\frac{(t-s)^{k-2}}{(k-2)!}\int_a^s\frac{r^{-1}l(r)\xi_k(r)^\gamma}{(2n+1-k)!}\,drds\\ &\sim \frac{t^{k-1}}{(k-1)!}\int_a^t\frac{s^{-1}l(s)\xi_k(s)^\gamma}{(2n+1-k)!}\,ds,\ t\to\infty\,.\end{aligned} \tag{3.24}$$

Since by straightforward computations we get

$$\int_a^t\frac{s^{-1}l(s)\xi_k(s)^\gamma}{(2n+1-k)!}ds=\frac{1}{(2n+1-k)!}\int_a^t s^{2n-(k-1)(1-\gamma)}q(s)\xi_k(s)^\gamma ds=(k-1)!\,\xi_k(t)\,, \tag{3.25}$$

asymptotic relation (3.4) for $X_k(t)$ follows instantly by combining (3.24) and (3.25).

(B) Let $\sigma\in\Big((k-1)(1-\gamma)-2n-1,k(1-\gamma)-2n-1\Big)$. Using that

$$Y_k(t)=\Big[L(\rho,k)^{-1}\,t^{2n+1}q(t)\Big]^{\frac{1}{1-\gamma}}=t^\rho\Big[L(\rho,k)^{-1}l(t)\Big]^{\frac{1}{1-\gamma}}$$

and

$$q(t)Y_k(t)^\gamma = t^{\sigma+\rho\gamma}\big(L(\rho,k)\big)^{-\frac{\gamma}{1-\gamma}} l(t)^{\frac{1}{1-\gamma}} = t^{\rho-2n-1}\big(L(\rho,k)\big)^{-\frac{\gamma}{1-\gamma}} l(t)^{\frac{1}{1-\gamma}},$$

we obtain for $t\to\infty$

$$\int_t^\infty \frac{(s-t)^{2n-k}}{(2n-k)!}\, q(s)Y_k(s)^\gamma ds \sim \left(\prod_{i=k}^{2n}(i-\rho)\right)^{-1} t^{\rho-k}\big(L(\rho,k)\big)^{-\frac{\gamma}{1-\gamma}} l(t)^{\frac{1}{1-\gamma}}$$

Nothing that $\rho - k > -1$ and integrating the above k times on $[a,t]$ then gives

$$\begin{aligned}
&\int_a^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty \frac{(r-s)^{2n-k}}{(2n-k)!}\, q(r)Y_k(r)^\gamma dr ds \\
&\quad\sim \left(\prod_{i=k}^{2n}(i-\rho)\right)^{-1} \big(L(\rho,k)\big)^{-\frac{\gamma}{1-\gamma}} \int_a^t \frac{(t-s)^{k-1}}{(k-1)!}\, s^{\rho-k} l(s)^{\frac{1}{1-\gamma}} ds \\
&\quad\sim \left(\prod_{i=k}^{2n}(i-\rho)\cdot\prod_{i=0}^{k-1}(\rho-i)\right)^{-1} \big(L(\rho,k)\big)^{-\frac{\gamma}{1-\gamma}} t^\rho l(t)^{\frac{1}{1-\gamma}} = Y_k(t), \quad t\to\infty\,.
\end{aligned}$$

(C) Let (3.7) hold. Then, $Z_k(t)\in \mathrm{RV}(k)$ is expressed as

$$Z_k(t) = t^k \eta_k(t), \quad \eta_k(t) = \left[\frac{1-\gamma}{(2n-k)!\cdot k!}\int_t^\infty s^{2n-k(1-\gamma)} q(s)ds\right]^{\frac{1}{1-\gamma}} \in \mathrm{SV}$$

and $q(t)Z_k(t)^\gamma = t^{k-2n-1} l(t)\eta_k(t)^\gamma \in \mathrm{RV}(k-2n-1)$. Applying Karamata's integration theorem we have

$$\begin{aligned}
\int_t^\infty \frac{(s-t)^{2n-k}}{(2n-k)!}\, q(s)Z_k(s)^\gamma ds &= \int_t^\infty \frac{(s-t)^{2n-k}}{(2n-k)!}\, s^{k-2n-1} l(s)\eta_k(s)^\gamma ds \\
&\sim \int_t^\infty \frac{s^{-1} l(s)\eta_k(s)^\gamma}{(2n-k)!}\, ds \in \mathrm{SV}\,, \quad t\to\infty\,.
\end{aligned} \tag{3.26}$$

Using that $t^{-1}l(t) = t^{2n-k(1-\gamma)}q(t)$ and the definition of $\eta_k(t)$ we calculate

$$\begin{aligned}
&\int_t^\infty \frac{s^{-1} l(s)\eta_k(s)^\gamma}{(2n-k)!}\, ds \\
&= \int_t^\infty \frac{s^{2n-k(1-\gamma)}\, q(s)}{(2n-k)!}\left[\frac{1-\gamma}{(2n-k)!\cdot k!}\int_t^\infty s^{2n-k(1-\gamma)} q(s)ds\right]^{\frac{\gamma}{1-\gamma}} ds \\
&= k!\left[\frac{1-\gamma}{(2n-k-1)!\cdot k!}\int_t^\infty s^{2n-k(1-\gamma)} q(s)ds\right]^{\frac{1}{1-\gamma}} = k!\eta_k(t)\,.
\end{aligned} \tag{3.27}$$

Integration of (3.26) k times on $[a,t]$, with the help of Proposition 2.3 (i) and (3.27), gives

$$\begin{aligned}
\int_a^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty \frac{(r-s)^{2n-k}}{(2n-k)!}\, q(r)Z_k(r)^\gamma dr ds &\sim \frac{t^k}{k!}\int_t^\infty \frac{s^{-1} l(s)\eta_k(s)^\gamma}{(2n-k)!}\, ds \\
&\sim t^k\,\eta_k(t) = Z_k(t), \quad t\to\infty\,. \ \square
\end{aligned}$$

Now, we state and prove the main results of this section establishing necessary and sufficient condition for the existence of regularly varying solution belonging to the classes listed in (3.2).

Theorem 3.1 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses nontrivial* $\mathrm{RV}(k-1)-$*solutions if and only if* (3.5) *holds, in which case the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$x(t) \sim t^{k-1}\left[\frac{1-\gamma}{(2n+1-k)! \cdot (k-1)!}\int_a^t s^{2n-(k-1)(1-\gamma)}q(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.28}$$

Theorem 3.2 *Let ρ and $L(\rho, k)$ be constants defined, respectively, by* (3.17) *and* (3.20) *and suppose that $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses regularly varying solutions of index $\rho \in (k-1, k)$ if and only if* (3.6) *holds, and the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$x(t) \sim \left[\frac{t^{2n+1}q(t)}{L(\rho,k)}\right]^{\frac{1}{1-\gamma}}, \qquad t \to \infty. \tag{3.29}$$

Theorem 3.3 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses nontrivial* $\mathrm{RV}(k)-$*solutions if and only if* (3.7) *holds, in which case the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$x(t) \sim t^{k}\left[\frac{1-\gamma}{(2n-k)! \cdot k!}\int_t^\infty s^{2n-k(1-\gamma)}q(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.30}$$

Proof of the "only if" part of Theorem 3.1: Suppose that $x(t) \in ntr - \mathrm{RV}(k-1)$. Then, $\rho = k-1$ and $x(t) = t^{k-1}\xi(t)$, $\xi(t) \in \mathrm{SV}$ such that $\xi(t) \to \infty$ as $t \to \infty$. It is clear that only case (iii) of Lemma 3.2 is admissible for $x(t)$. Therefore, $\sigma = (k-1)(1-\gamma) - 2n - 1$ and $x(t)$ satisfies (3.10) for all $k \in \{2,\ 4, \dots, 2n\}$, which is equivalent to

$$\xi(t) \sim \frac{1}{(2n+1-k)! \cdot (k-1)!}\int_{t_0}^t s^{-1}l(s)\xi(s)^{\gamma}ds, \quad t \to \infty. \tag{3.31}$$

The right-hand side of (3.31), denoted by $\eta(t)$, satisfies the differential asymptotic relation

$$\eta(t)^{-\gamma}\eta'(t) \sim \frac{t^{-1}l(t)}{(2n+1-k)! \cdot (k-1)!} = \frac{t^{2n-(k-1)(1-\gamma)}q(t)}{(2n+1-k)! \cdot (k-1)!}, \quad t \to \infty. \tag{3.32}$$

From (3.32), since $\eta(t) \to \infty$, $t \to \infty$, we conclude that $Q_{k-1} = \infty$ and

$$\begin{aligned}\eta(t) \ &\sim \ \left[\frac{1-\gamma}{(2n+1-k)! \cdot (k-1)!}\int_{t_0}^t s^{2n-(k-1)(1-\gamma)}q(s)ds\right]^{\frac{1}{1-\gamma}} \\ &\sim \ \left[\frac{1-\gamma}{(2n+1-k)! \cdot (k-1)!}\int_{a}^t s^{2n-(k-1)(1-\gamma)}q(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,\end{aligned}$$

showing the truth of asymptotic formula (3.28) for $x(t)$.

Proof of the "only if" part of Theorem 3.2: Suppose that $x(t) \in \mathrm{RV}(\rho)$ for some $\rho \in (k-1, k)$ is the solution of (A). Then, clearly only the statement (ii) of Lemma 3.2 could hold.

Thus, we must have $\rho = \sigma + \rho\gamma + 2n + 1$, which implies that ρ is given by (3.17). This combined with $\rho \in (k-1, k)$ determines the range of σ to be

$$\sigma \in ((k-1)(1-\gamma) - 2n - 1, k(1-\gamma) - 2n - 1).$$

Using (3.17) we have $P_k Q_k = L(\rho, k)$, and (3.9) can be rewritten as

$$x(t) \sim \frac{t^{2n+1} q(t) x(t)^{\gamma}}{L(\rho, k)}, \quad t \to \infty,$$

so that asymptotic formula for $x(t)$ is be given by (3.29).

Proof of the "only if" part of Theorem 3.3: Suppose that $x(t) \in ntr - \mathrm{RV}(k)$. Then, $\rho = k$ and $x(t) = t^k \xi(t)$, $\xi(t) \in \mathrm{SV}$ such that $\xi(t) \to 0$ as $t \to \infty$. For such $x(t)$ only case (i) in Lemma 3.2 is possible and $x(t)$ must satisfy the asymptotic relation (3.8) for each $k \in \{2,\ 4, \ldots, 2n\}$. This means that $\sigma = k(1-\gamma) - 2n - 1$ and from (3.8) we have

$$\xi(t) \sim \frac{1}{(2n-k)! \cdot k!} \int_t^{\infty} s^{-1} l(s) \xi(s)^{\gamma} ds, \quad t \to \infty. \tag{3.33}$$

Let $\eta(t)$ denote the right-hand side of (3.33). Then, (3.33) is transformed into the differential asymptotic relation

$$-\eta(t)^{-\gamma} \eta'(t) \sim \frac{t^{-1} l(t)}{(2n-k)! \cdot k!} = \frac{t^{2n-k(1-\gamma)} q(t)}{(2n-k)! \cdot k!}, \quad t \to \infty. \tag{3.34}$$

Noting that $\eta(t) \to 0$, $t \to \infty$, we see from (3.34) that $Q_k < \infty$ and integration of (3.34) on $[t, \infty)$ gives

$$\eta(t) \sim \left[\frac{1-\gamma}{(2n-k)! \cdot k!} \int_t^{\infty} s^{2n-k(1-\gamma)} q(s) ds \right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

which implies the validity of (3.30).

Proof of the "if" part of Theorems 3.1, 3.2 and 3.3: Suppose that either (3.5) or (3.6) or (3.7) holds for $q(t) \in \mathrm{RV}(\sigma)$ and let ρ, $L(\rho, k)$ be defined by (3.17), (3.20). We perform simultaneous proof of all three theorems, so to simplify notation we introduce the function $\Phi_k(t)$ on $[a, \infty)$ by

$$\Phi_k(t) = \begin{cases} t^{k-1} \left[\dfrac{1-\gamma}{(2n+1-k)! \cdot (k-1)!} \displaystyle\int_a^t s^{2n-(k-1)(1-\gamma)} q(s) ds \right]^{\frac{1}{1-\gamma}}, & \text{if (3.5) holds;} \\ \left[\dfrac{t^{2n+1} q(t)}{L(\rho, k)} \right]^{\frac{1}{1-\gamma}}, & \text{if (3.6) holds;} \\ t^k \left[\dfrac{1-\gamma}{(2n-k)! \cdot k!} \displaystyle\int_t^{\infty} s^{2n-k(1-\gamma)} q(s) ds \right]^{\frac{1}{1-\gamma}}, & \text{if (3.7) holds.} \end{cases}$$

By Lemma 3.3 the function $\Phi_k(t)$ satisfies (3.4). Thus, there exists $T_1 > a$ such that

$$\int_{T_1}^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r) \Phi_k(r)^{\gamma} dr ds \le 2\Phi_k(t), \quad t \ge T_1. \tag{3.35}$$

Let such a T_1 be fixed. From (3.4) we have

$$\int_{T_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)\Phi_k(r)^{\gamma} drds \sim \Phi_k(t), \quad t \to \infty,$$

so that there exists $T_2 > T_1$ such that

$$\int_{T_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)\Phi_k(r)^{\gamma} drds \geq \frac{\Phi_k(t)}{2}, \quad t \geq T_2. \tag{3.36}$$

Let $T_2 > T_1$ be a fixed constant such that (3.36) hold, $m \in (0,1)$ be a fixed positive constant such that

$$m^{1-\gamma} \leq \frac{1}{2}, \tag{3.37}$$

and choose a constant $M > 1$ such that

$$M^{1-\gamma} \geq 4 \quad \text{and} \quad M \geq 2m \frac{\Phi_k(T_2)}{\Phi_k(T_1)}. \tag{3.38}$$

We define the set $\mathcal{X}$ to be the set of continuous functions $x(t)$ on $[T_1, \infty)$ satisfying

$$\begin{cases} m\Phi_k(T_2) \leq x(t) \leq M\Phi_k(t), & \text{for } T_1 \leq t \leq T_2\,, \\ m\Phi_k(t) \leq x(t) \leq M\Phi_k(t), & \text{for } t \geq T_2\,. \end{cases} \tag{3.39}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_1, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_1, \infty)$. We now define the integral operator

$$\mathcal{F}x(t) = m\Phi_k(T_2) + \int_{T_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)x(r)^{\gamma} drds, \quad t \geq T_1 \tag{3.40}$$

and let it act on the set $\mathcal{X}$ defined above. It can be shown that $\mathcal{F}$ is a self-map on $\mathcal{X}$ and sends $\mathcal{X}$ continuously on a relatively compact subset of $C[T_1, \infty)$.

(a) $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$: Let $x(t) \in \mathcal{X}$. Using (3.38), (3.39) and (3.40) we get

$$\begin{aligned} \mathcal{F}x(t) &\leq m\Phi_k(T_2) + \frac{M^{\gamma}}{2} \int_{T_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)\Phi_k(r)^{\gamma} drds \\ &\leq \frac{M}{2}\,\Phi_k(T_1) + \frac{M}{2}\,\Phi_k(t) \leq \frac{M}{2}\,\Phi_k(t) + \frac{M}{2}\,\Phi_k(t) = M\,\Phi_k(t), \quad t \geq T_1\,. \end{aligned}$$

On the other hand, using (3.37), (3.39) and (3.40) we have

$$\mathcal{F}x(t) \geq m\,\Phi_k(T_2) \quad \text{for} \quad T_1 \leq t \leq T_2\,,$$

and

$$\begin{aligned} \mathcal{F}x(t) &\geq m^{\gamma} \int_{T_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)\Phi_k(r)^{\gamma} drds \\ &\geq m^{\gamma} \frac{\Phi_k(t)}{2} \geq m\,\Phi_k(t), \quad t \geq T_2\,. \end{aligned}$$

This shows that $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

(b) $\mathcal{F}(\mathcal{X})$ *is relatively compact:* The inclusion $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$ implies that $\mathcal{F}(\mathcal{X})$ is locally uniformly bounded on $[T_1, \infty)$. For all $x(t) \in \mathcal{X}$, we have from (3.40)

$$0 \le (\mathcal{F}x)'(t) \le M^\gamma \int_{T_1}^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)\Phi_k(r)^\gamma dr ds, \quad t \ge T_1.$$

Thus, it follows that $\mathcal{F}(\mathcal{X})$ is locally equicontinuous on $[T_1, \infty)$. The relative compactness of the set $\mathcal{X}$ then follows from the Arzela-Ascoli lemma.

(c) $\mathcal{F}$ *is a continuous map on* $\mathcal{X}$*:* Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging to $x(t) \in \mathcal{X}$ as $n \to \infty$ on any compact subinterval of $[T_1, \infty)$. We need to verify that $\mathcal{F}x_n(t) \to \mathcal{F}x(t)$ uniformly on compact subintervals of $[T_1, \infty)$. From the inequality

$$\begin{aligned} |\mathcal{F}x_n(t) - \mathcal{F}x(t)| &\le \int_{T_1}^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)|x_n(r)^\gamma - x(r)^\gamma| dr ds \\ &\le \frac{t^k}{k!} \int_{T_1}^\infty \frac{(s-T_1)^{2n-k-1}}{(2n-k-1)!} q(s)|x_n(s)^\gamma - x(s)^\gamma| dr ds, \quad t \ge T_1. \end{aligned}$$

using that $q(t)\big|x_n(t)^\gamma - x(t)^\gamma| \to 0$ as $n \to \infty$ at each point $t \in [T_1, \infty)$ and $q(t)\big|x_n(t)^\gamma - x(t)^\gamma| \le M^\gamma q(t)\Phi_k(t)^\gamma$ for $t \ge T_1$, while $q(t)\Phi_k(t)^\gamma$ is integrable on $[T_1, \infty)$, by the application of the Lebesgue dominated convergence theorem, we conclude that $\mathcal{F}x_n(t) \to \mathcal{F}x(t)$ uniformly on any compact subinterval of $[T_1, \infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{F}$.

Thus all the hypotheses of the Schauder-Tychonoff fixed point thereom are fulfilled for $\mathcal{F}$, and so there exists an element $x(t) \in \mathcal{X}$ such that $x(t) = \mathcal{F}x(t)$, $t \ge T_1$, which satisfies the integral equation

$$x(t) = m\,\Phi_k(T_2) + \int_{T_1}^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r)x(r)^\gamma dr ds, \quad t \ge T_1\,. \tag{3.41}$$

Differentiating the above $2n+1$ times we conclude that $x(t)$ is a solution of (A) on $[T_2, \infty)$ satisfying

$$m\,\Phi_k(t) \le x(t) \le M\,\Phi_k(t), \qquad \text{for } \ t \ge T_2\,. \tag{3.42}$$

Therefore,

$$0 < \liminf_{t\to\infty} \frac{x(t)}{\Phi_k(t)} \le \limsup_{t\to\infty} \frac{x(t)}{\Phi_k(t)} < \infty\,. \tag{3.43}$$

Since the function $\Phi_k(t)$ satisfies (3.4), denoting

$$\phi_k(t) = \int_a^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{2n-k}}{(2n-k)!} q(r)\Phi_k(r)^\gamma dr ds, \tag{3.44}$$

we have that $\Phi_k(t) \sim \phi_k(t)$, $t \to \infty$, so that from (4.12)

$$0 < \liminf_{t\to\infty} \frac{x(t)}{\phi_k(t)} \le \limsup_{t\to\infty} \frac{x(t)}{\phi_k(t)} < \infty\,.$$

By application of Lemma 3.1 $2n+1$ times, using (3.44) we have

$$\begin{aligned} L &= \limsup_{t\to\infty} \frac{x(t)}{\phi_k(t)} \le \limsup_{t\to\infty} \frac{x'(t)}{\phi_k'(t)} \le \ldots \le \limsup_{t\to\infty} \frac{x^{(2n+1)}(t)}{\phi_k^{(2n+1)}(t)} \\ &= \limsup_{t\to\infty} \frac{q(t)x(t)^\gamma}{q(t)\Phi_k(t)^\gamma} = \left(\limsup_{t\to\infty} \frac{x(t)}{\Phi_k(t)}\right)^\gamma = \left(\limsup_{t\to\infty} \frac{x(t)}{\phi_k(t)}\right)^\gamma = L^\gamma . \end{aligned}$$

Since $\gamma < 1$, from above we conclude that

$$0 < L \le 1 . \tag{3.45}$$

Similary, we can see that

$$l = \liminf_{t\to\infty} \frac{x(t)}{\phi_k(t)}$$

satisfies

$$1 \le l < \infty . \tag{3.46}$$

From (3.45) and (3.46) we obtain that $l = L = 1$, which means that

$$x(t) \sim \phi_k(t) \sim \Phi_k(t), \quad t \to \infty ,$$

and ensures that $x(t)$ is a regularly varying solution of (A) with requested regularity index and the asymptotic behavior (3.28), (3.29), (3.30) corresponding to the case (3.5), (3.6), (3.7). ⊠

4 Decaying regularly varying solutions of (A)

This section is devoted to the study of regularly varying solutions belonging to $\mathcal{P}(\mathrm{II}_0)$, that is those solutions which decay to 0 as $t \to \infty$. It is assumed that coefficient $q(t)$ is regularly varying of index σ and expressed as in (3.1). Let $x(t) \in \mathcal{P}(\mathrm{II}_0)$ be a regularly varying solution of (A) expressed as $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$. In view of (1.3) there exists positive constants c and $T > a$ such that

$$x(t) \le c, \qquad t \ge T ,$$

so that the regularity index ρ of $x(t)$ clearly satisfies $\rho \le 0$, while if $\rho = 0$ slowly varying part $\xi(t)$ must satisfy $\xi(t) \to 0$ as $t \to \infty$. We will prove two theorems stated below which show that the totality of decaying regularly varying solutions of (A) always consists of only one class:

$$\text{ntr-RV}(0) \quad \text{or} \quad \mathrm{RV}(\rho) \ \text{ for some } \ \rho < 0. \tag{4.1}$$

Theorem 4.1 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$. Equation* (A) *possesses nontrivial slowly varying solutions if and only if*

$$\sigma = -2n-1 \quad \textit{and} \quad Q_0 = \int_a^\infty t^{2n} q(t)dt < \infty, \tag{4.2}$$

in which case the asymptotic behavior of any such solution $x(t)$ is governed by the unique asymptotic formula

$$x(t) \sim \left[\frac{1-\gamma}{(2n)!} \int_t^\infty s^{2n} q(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{4.3}$$

PROOF. **The "only if" part:** Let $x(t) \in \mathrm{ntr}-\mathrm{SV}$ be a solution of (A) on $[t_0,\infty)$ belonging to $\mathcal{P}(\mathrm{II}_0)$. Due to (1.5), $w_i = x^{(i)}(\infty) = 0$ for $i = 0,1,\dots,2n$, so we may integrate (A) $(2n+1)-$times from t to ∞ to get

$$x(t) = \int_t^\infty \frac{(s-t)^{2n}}{(2n)!}\, q(s)x(s)^\gamma ds, \quad t \geq t_0. \tag{4.4}$$

This means that function $q(t)x(t)^\gamma = t^\sigma l(t)x(t)^\gamma$ is $2n+1$ times integrable on $[t_0,\infty)$, so that σ must satisfy $\sigma \leq -2n-1$. If $\sigma < -2n-1$, then repeated application of Karamata's integration theorem yields

$$x(t) \sim \frac{t^{\sigma+2n+1}l(t)x(t)^\gamma}{S_k} \in \mathrm{RV}(\sigma+3), \quad S_k = \prod_{j=1}^{2n+1}[-(\sigma+i)]$$

which means that $x(t)$ is regularly varying of index $\sigma+2n+1<0$, contradicting the assumption that $x(t) \in \mathrm{ntr}-\mathrm{SV}$. Thus, $\sigma = -2n-1$ and using Proposition 2.3 (ii) and (iii) we obtain from (4.4)

$$x(t) \sim \frac{1}{(2n)!}\int_t^\infty s^{-1}l(s)x(s)^\gamma ds, \quad t\to\infty. \tag{4.5}$$

Let $y(t)$ denote the right-hand side of (4.5), which is transformed into differential relation

$$-y(t)^\gamma y'(t) \sim \frac{t^{-1}l(t)}{(2n)!} = \frac{t^{2n}q(t)}{(2n)!}, \quad t\to\infty\,. \tag{4.6}$$

Because in view of (4.5) $y(t)\to 0$, $t\to\infty$, (4.6) is integrable on $[t,\infty)$, implying $Q_0<\infty$ and establishing the asymptotic formula for $y(t)$

$$y(t) \sim \left[\frac{1-\gamma}{(2n)!}\int_t^\infty s^{2n}q(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t\to\infty\,,$$

which ensures the validity of the asymptotic formula (4.3) for $x(t)$.

The "if" part: Suppose that (4.2) holds and define the function

$$X_0(t) = \left[\frac{1-\gamma}{(2n)!}\int_t^\infty s^{2n}q(s)ds\right]^{\frac{1}{1-\gamma}} \in \mathrm{SV}\,. \tag{4.7}$$

We first show that $X_0(t)$ satisfies the integral asymptotic relation

$$x(t) \sim \int_t^\infty \frac{(s-t)^{2n}}{(2n)!}\, q(s)x(s)^\gamma ds, \quad t\to\infty\,. \tag{4.8}$$

In fact, $q(t)X_0(t)^\gamma = t^{-2n-1}l(t)X_0(t)^\gamma$, so integration $2n$ times on $[t,\infty)$ gives

$$\begin{aligned}\int_t^\infty \frac{(s-t)^{2n}}{(2n)!}\, q(s)X_0(s)^\gamma ds \quad &\sim \quad \int_t^\infty \frac{s^{-1}l(s)X_0(s)^\gamma}{(2n)!}\, ds \\ &= \frac{1}{(2n)!}\int_t^\infty s^{2n}q(s)X_0(s)^\gamma\, ds = X_0(t), \quad t\to\infty.\end{aligned}$$

Thus, there exists $T > a$ such that

$$\frac{X_0(t)}{2} \le \int_t^\infty \frac{(s-t)^{2n}}{(2n)!} q(s) X_0(s)^\gamma ds \le 2X_0(t), \quad t \ge T\,.$$

Considering the integral operator

$$\mathcal{G}x(t) = \int_t^\infty \frac{(s-t)^{2n}}{(2n)!} q(s) x(s)^\gamma ds \tag{4.9}$$

we may verify that $\mathcal{G}$ is a continuous self-map on the set $\mathcal{Y}$ consisting of continuous functions $x(t)$ on $[T,\infty)$ satisfying

$$kX_0(t) \le x(t) \le KX_0(t), \quad t \ge T, \tag{4.10}$$

where $k < 1$ and $K > 1$ are positive constants satisfying

$$k^{1-\gamma} \le \frac{1}{2}, \qquad K^{1-\gamma} \ge 2\,, \tag{4.11}$$

and that $\mathcal{G}$ sends $\mathcal{Y}$ into relatively compact subset of $C[T,\infty)$. Thus, $\mathcal{G}$ has a fixed point $x(t) \in \mathcal{Y}$, which is a solution of (A) satisfying (4.10), implying that

$$0 < \liminf_{t\to\infty} \frac{x(t)}{X_0(t)} \le \limsup_{t\to\infty} \frac{x(t)}{X_0(t)} < \infty\,. \tag{4.12}$$

Denoting

$$\phi_0(t) = \int_t^\infty \frac{(s-t)^{2n}}{(2n)!} q(s) X_0(s)^\gamma ds$$

and using that $X_0(t) \sim \phi_0(t)$, $t \to \infty$, we get

$$0 < \liminf_{t\to\infty} \frac{x(t)}{\phi_0(t)} \le \limsup_{t\to\infty} \frac{x(t)}{\phi_0(t)} < \infty\,.$$

Then, proceeding exactly as in the proof of the "if" part of Theorems 3.1-3.3, with application of Lemma 3.1, we conclude that $x(t) \sim \phi_0(t) \sim X_0(t)$, $t \to \infty$. This completes the proof of Theorem 4.1. ⊠

Theorem 4.2 *Suppose that* $q(t) \in \mathrm{RV}(\sigma)$*. Equation* (A) *possesses regularly varying solutions of index* $\rho < 0$ *if and only if*

$$\sigma < -2n-1, \tag{4.13}$$

in which case ρ *is determined by* (3.17) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique asymptotic formula*

$$x(t) \sim \left[\frac{t^{2n+1}q(t)}{D(\rho)}\right]^{\frac{1}{1-\gamma}}, \qquad t \to \infty, \tag{4.14}$$

where

$$D(\rho) = \prod_{i=0}^{2n} (i-\rho)\,. \tag{4.15}$$

Proof. **The "only if" part:** Let $x(t) \in \mathrm{RV}(\rho)$, $\rho < 0$ be a solution of (A) on $[t_0, \infty)$. Integration of (A) $(2n+1)-$times from t to ∞ gives

$$x(t) = \int_t^\infty \frac{(s-t)^{2n}}{(2n)!} q(s) x(s)^\gamma ds = \int_t^\infty \frac{(s-t)^{2n}}{(2n)!} s^{\sigma+\rho\gamma} l(s) \xi(s)^\gamma ds, \quad t \geq t_0. \tag{4.16}$$

Convergence of the last integral in (4.16) implies $\sigma+\rho\gamma \leq -2n-1$. The possibility $\sigma+\rho\gamma = -2n-1$ is excluded, because otherwise from Proposition 2.3 applied to (4.16) we would have (4.5), which means that $x(t) \in \mathrm{SV}$, contradicting the assumption that $x(t)$ is regularly varying of negative index. Therefore, $\sigma+\rho\gamma < -2n-1$ and repeated application of Karamata's integration theorem to the right-hand side of (4.16) yields

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2n+1} l(t) \xi(t)^\gamma}{T_k}, \quad t \to \infty, \quad \text{where } T_k = \prod_{j=1}^{2n+1} [-(\sigma+\rho\gamma+i)], \tag{4.17}$$

which shows that regularity index of $x(t)$ is $\rho = \sigma+\rho\gamma+2n+1 < 0$. Consequently, ρ is given by (3.17) and σ must satisfy $\sigma < -2n-1$. Using that

$$t^{\sigma+\rho\gamma+2n+1} l(t) \xi(t)^\gamma = t^{2n+1} q(t) x(t)^\gamma \quad \text{and} \quad T_k = \prod_{j=1}^{2n+1} (2n+1-i-\rho) = \prod_{i=0}^{2n} (j-\rho) = D(\rho),$$

(4.17) can be transformed into

$$x(t) \sim \frac{t^{2n+1} q(t) x(t)^\gamma}{D(\rho)}, \qquad t \to \infty,$$

which implies the asymptotic relation (4.14).

The "if" part: Suppose that (4.13) holds and define the function

$$Y_0(t) = \left[\frac{t^{2n+1} q(t)}{D(\rho)} \right]^{\frac{1}{1-\gamma}} \in \mathrm{RV}(\rho)\,. \tag{4.18}$$

We verify that $Y_0(t)$ satisfies the integral asymptotic relation (4.8). Expressing $Y_0(t)$ as

$$Y_0(t) = t^\rho l(t)^{\frac{1}{1-\gamma}} D(\rho)^{-\frac{1}{1-\gamma}},$$

we have

$$q(t) Y_0(t)^\gamma = t^{\sigma+\rho\gamma} l(t)^{\frac{1}{1-\gamma}} D(\rho)^{-\frac{\gamma}{1-\gamma}} = t^{\rho-2n-1} l(t)^{\frac{1}{1-\gamma}} D(\rho)^{-\frac{\gamma}{1-\gamma}}. \tag{4.19}$$

Integration of (4.19) $2n+1$ times on $[t,\infty)$ with application of Proposition 2.3-(ii) then implies the integral asymptotic relation for $Y_0(t)$:

$$\int_t^\infty \frac{(s-t)^{2n}}{(2n)!} q(s) Y_0(s)^\gamma ds \sim D(\rho)^{-\frac{\gamma}{1-\gamma}} \frac{t^\rho l(t)^{\frac{1}{1-\gamma}}}{D(\rho)} = Y_0(t), \quad t \to \infty\,.$$

Then, proceeding exactly as in the proof of the "if" part of Theorem 4.1, replacing $X_0(t)$ with $Y_0(t)$, solution $x(t)$ of equation (A) such that

$$m Y_0(t) \leq x(t) \leq M Y_0(t), \qquad \text{for large } t,$$

is obtained by the application of the Schauder-Tychonoff theorem, while application of Lemma 3.1 proves, afterwards, that the solution $x(t)$ is regularly varying of index $\rho < 0$ and enjoys the precise asymptotic behavior (4.14). ⊠

5 Overall structure of regularly varying solutions of equation (A)

On the basis of our main results presented in Section 3 and 4, combined with Theorems 1.1, 1.2, the structure of regularly varying solutions of equation (A) with regularly varying coefficient $q(t) \in \mathrm{RV}(\sigma)$ is determined in full detail.

Denote with $\mathcal{R}$ the set of all regularly varying solutions of (A), and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{RV}(\rho), \quad \mathrm{tr} - \mathcal{R}(\rho) = \mathcal{R} \cap \text{tr-RV}(\rho), \quad \mathrm{ntr} - \mathcal{R}(\rho) = \mathcal{R} \cap \text{ntr-RV}(\rho).$$

Using notation $\gamma_m = m(1-\gamma) - 2n - 1$, $m \in \{0, 1, 2, \dots, 2n\}$, to make the full analysis we separately consider case $\sigma < \gamma_0 = -2n - 1$ together with central cases

$$\sigma \in (\gamma_0, \gamma_1) \cup (\gamma_2, \gamma_3) \cup \ldots \cup (\gamma_{2n-2}, \gamma_{2n-1}) \quad \text{or} \quad \sigma \in (\gamma_1, \gamma_2) \cup (\gamma_3, \gamma_4) \cup \ldots \cup (\gamma_{2n-1}, \gamma_{2n}),$$

and border cases

$$\sigma \in \{\gamma_0, \gamma_2, \dots, \gamma_{2n}\} \quad \text{or} \quad \sigma \in \{\gamma_1, \gamma_3, \dots, \gamma_{2n-1}\}.$$

Structure of regularly varying solutions of equation (A):

(1) If $\sigma < \gamma_0$, then

$$\mathcal{R} = \bigcup_{j=0}^{2n} \mathrm{tr} - \mathcal{R}(j) \cup \mathcal{R}\left(\frac{\sigma + 2n + 1}{1 - \gamma}\right);$$

(2) If $\sigma = \gamma_0$ and $Q_0 < \infty$, then

$$\mathcal{R} = \bigcup_{j=0}^{2n} \mathrm{tr} - \mathcal{R}(j) \cup \mathrm{ntr} - \mathcal{R}(0);$$

(3) If $\sigma = \gamma_0$ and $Q_0 = \infty$, then

$$\mathcal{R} = \bigcup_{j=1}^{2n} \mathrm{tr} - \mathcal{R}(j);$$

(4) If $\sigma = \gamma_m$ for some $m \in \{1,\ 3, \dots, 2n-1\}$ and $Q_m < \infty$, then

$$\mathcal{R} = \bigcup_{j=m}^{2n} \mathrm{tr} - \mathcal{R}(j);$$

(5) If $\sigma = \gamma_m$ for some $m \in \{1,\ 3, \dots, 2n-1\}$ and $Q_m = \infty$, then

$$\mathcal{R} = \bigcup_{j=m+1}^{2n} \mathrm{tr} - \mathcal{R}(j) \cup \mathrm{ntr} - \mathcal{R}(m);$$

(6) If $\sigma \in (\gamma_{m-1}, \gamma_m)$ for some $m \in \{2,\ 4, \ldots, 2n-2\}$, then

$$\mathcal{R} = \bigcup_{j=m}^{2n} \mathrm{tr} - \mathcal{R}(j) \cup \mathcal{R}\left(\frac{\sigma + 2n + 1}{1-\gamma}\right);$$

(7) If $\sigma \in (\gamma_m, \gamma_{m+1})$ for some $m \in \{0,\ 2, \ldots, 2n-2\}$, then

$$\mathcal{R} = \bigcup_{j=m+1}^{2n} \mathrm{tr} - \mathcal{R}(j);$$

(8) If $\sigma = \gamma_m$ for some $m \in \{2, 4, \ldots, 2n\}$ and $Q_m < \infty$, then

$$\mathcal{R} = \bigcup_{j=m}^{2n} \mathrm{tr} - \mathcal{R}(j) \cup \mathrm{ntr} - \mathcal{R}(m);$$

(9) If $\sigma = \gamma_m$ for some $m \in \{2,\ 4, \ldots, 2n-2\}$ and $Q_m = \infty$, then

$$\mathcal{R} = \bigcup_{j=m+1}^{2n} \mathrm{tr} - \mathcal{R}(j);$$

(10) If $\sigma = \gamma_{2n}$ and $Q_{2n} = \infty$, then $\mathcal{R} = \emptyset$;

(11) If $\sigma > \gamma_{2n} = -1 - 2n\gamma$, then $\mathcal{R} = \emptyset$.

In connection with Theorem A stated in the Introduction a question is raised as to the existence of decaying positive solutions for the sublinear equation (A) in the case condition (1.2) is satisfied. Suppose that $q(t) \in \mathrm{RV}(\sigma)$ satisfies condition . Then, since $\sigma \geq -(2n-1)\gamma - 1 > \gamma_{2n}$, from the above observations it follows that equation (A) admits no regularly varying solutions. We are tempted to conjecture that Theorem A could be paraphrased to assert that all solutions of (A) are oscillatory if and only if (1.2) holds.

We close this section by comparing our results with ones in [2]. We will carry out a comparison based on the fact that in this paper, less general case of equation (1.10) is considered, with $\alpha_0 = -1$, $m = 2n+1$, $\varphi(x) = |x|^\gamma \mathrm{sgn}\, x$, $\omega = +\infty$, $Y_0 = +\infty$ or $Y_0 = 0$ (for similar results for $\alpha_0 = 1$, $m = 2n$ see [9, 10]) . Although not specifically emphasized in [2], Evtukhov and Samoilenko actually restricted their attention on the equation (1.10) with regularly varying coefficient and focused their attention only on its regularly varying solutions. Namely, all conditions imposed in [2] on function $q(t)$ in (1.10) (see (3.3), (3.7) and second conditions in (3.14), (3.18)), means, due to converse half of Karamata's theorem (see [1, Theorem 1.6.1]), that $q(t)$ is of regular variation. That fact is neither mentioned nor used by Evtukhov and Samoilenko. Moreover, while they clearly emphasized that $P_{+\infty}(Y_0, \lambda_0)-$solutions with $\lambda_0 \in \mathbb{R} \setminus \{0, 1/2, 2/3, \ldots, (n-2)/(n-1), 1\}$ and $\lambda_0 = +\infty$ are functions of regularly variation, such an assertion for $P_{+\infty}(Y_0, (n-i-1)/(n-i))-$solutions with $i \in \{1, 2, \ldots, n-1\}$ is missing. Also, by virtue of the representations of such solutions obtained in [2] such a conclusion is almost impossible to make. But, using the fact that $q(t)$ is of regular variation, it becomes quite clear each $P_{+\infty}(Y_0, \lambda_0)-$solution, with $\lambda_0 \neq 1$, is regularly varying, while assuming that $q(t)$ is of rapid variation, $P_{+\infty}(Y_0, 1)-$solutions are rapidly varying.

Moreover, we claim that the converse is also true, due to the fact that $q(t) \in \mathrm{RV}(\sigma)$. Denote with $\mathcal{P}(Y_0, \lambda_0)$ the set of all $P_{+\infty}(Y_0, \lambda_0)-$solutions of (A). Then, assuming that $y(t) \in \mathrm{RV}(\rho)$, from equation (A) we may conclude that $y^{(n)}(t) \in \mathrm{RV}(\sigma+\rho\gamma)$, which by application of Karamata's integration theorem implies that

$$y^{(n-1)}(t) \in \mathrm{RV}(\sigma + \rho\gamma + 1) \quad \Rightarrow \quad \lim_{t\to\infty} t\, \frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \sigma + \rho\gamma + 1,$$

$$y^{(n-2)}(t) \in \mathrm{RV}(\sigma + \rho\gamma + 2) \quad \Rightarrow \quad \lim_{t\to\infty} t\, \frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \sigma + \rho\gamma + 2\,.$$

Therefore, the last condition (1.11) in a definition of $P_{+\infty}(Y_0, \lambda_0)-$solution becomes

$$\frac{\sigma + \rho\gamma + 2}{\sigma + \rho\gamma + 1} = \lim_{t\to\infty} \frac{t\, \frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}}{t\, \frac{y^{(n)}(t)}{y^{(n-1)}(t)}} = \lim_{t\to\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t) y^{(n-2)}(t)},$$

so that

$$y(t) \in \mathcal{P}\left(Y_0, \frac{\sigma + \rho\gamma + 2}{\sigma + \rho\gamma + 1}\right) \quad \text{i.e} \quad \mathcal{R}(\rho) \subseteq \mathcal{P}\left(Y_0, \frac{\sigma + \rho\gamma + 2}{\sigma + \rho\gamma + 1}\right).$$

More specifically, it is not difficult to see that, in fact

$$\begin{aligned}
\mathcal{R}(\rho) &= \mathcal{P}\left(Y_0, \frac{2n-1-\rho}{2n-\rho}\right) \\
&\qquad\qquad \text{for } \rho \in (-\infty, 2n) \setminus \{0, 1, 2, \ldots, 2n-1\},\ Y_0 = \begin{cases} 0, & \text{if } \rho < 0 \\ +\infty, & \text{if } \rho > 0 \end{cases}, \\
\mathcal{R}(0) &= \mathcal{P}\left(0, \frac{2n-1}{2n}\right), \\
\mathcal{R}(k) &= \mathcal{P}\left(+\infty, \frac{2n+1-k}{2n-k}\right) \quad \text{for } k \in \{1, 2, \ldots, 2n-1\}, \\
\mathcal{R}(2n) &= \mathcal{P}\left(+\infty, +\infty\right),
\end{aligned}$$

which makes our results closely connected with ones in [2]. However, extensive use of theory of regular variation, particularly Karamata's integration theorem, combined with fixed point techniques, gives us the opportunity to fully describe the overall structure of regularly varying solutions of (A) on the basis of behavior and the regularity index of the regularly varying coefficient $q(t)$ and to formulate asymptotic formula for such solutions more precisely and accurate than in [2].

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