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The Cauchy problem for a class of fractional impulsive differential equations with delay*

Xiaozhi Zhang†**, Chuanxi Zhu, Zhaoqi Wu**

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

Abstract. In this work, the Cauchy initial value problem is discussed for a class of fractional impulsive differential equations with delay, and the criteria on existence and uniqueness are obtained. Finally, an example is also provided to illustrate the effectiveness of our main results.

Key Words: Caputo fractional derivative; existence and uniqueness; impulsive equations; time delay

MR(2010) Subject Classification: 34A08, 34A37

1. Introduction

In this paper, we consider the Cauchy initial value problem (IVP for short) of fractional impulsive differential equations with delay of the form

$$\begin{cases} D^\alpha x(t) = f(t, x_t), & t \neq t_k,\ t \in [0, T]; \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p; \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where D^α is the Caputo's fractional derivative of order $0 < \alpha < 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $I_k \in C(\mathbb{R}, \mathbb{R})$ are given functions satisfying some assumptions that will be specified later. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, and they satisfy that $x(t_k^-) = x(t_k)$. If

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†Corresponding author.

E-mail addresses: xzzhang@yahoo.com.cn (X. Zhang), chuanxizhu@126.com (C. Zhu), wuzhaoqi_conquer@163.com (Z. Wu).

$x \in C([-\tau, T], \mathbb{R})$, then for any $t \in [0, T]$, define x_t by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$, here x_t represents the history of the state from time $t-\tau$ to the present time t. $\phi \in C([-\tau, 0], \mathbb{R})$ and $\phi(0) = 0$.

Recently, differential equations of fractional order have been proved to be valid tools in the modeling of many phenomena in various fields of engineering and science such as physics, electrochemistry, electromagnetics, control theory, viscoelasticity, porous media and so forth. On the other hand, fractional differential equations also serve as an excellent tool for the description of memory and hereditary properties of various materials and processes. With these advantages, the models of fractional order become more and more practical and realistic than the classical models of integer order, such effects in the latter are not taken into account. As a result, the subject of fractional differential equations is gaining much attention and importance. For more details on this theory and on its applications, we refer to the recent monographs of Miller and Ross [19], Kilbas et al. [13, 23], Oldham and Spanier [20], Hilfer [12], Metzler et al. [17], the researches of Agrawal et al. [2, 22], and the papers of [1, 3-11, 16, 27-30].

There are significant developments in the theory of impulses especially in the area of impulsive differential equations with fixed moments (see e.g., [15] and [24]), which provided a natural description of observed evolution processes, regarding as important tools for better understanding several real word phenomena in applied sciences. In [3, 7], Benchohra et al. established sufficient conditions for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative of order $0 < q < 1$ and $1 < q < 2$, respectively. The authors of [8, 27, 28] studied the three-point boundary value problem, the anti-periodic boundary value problem and the mixed boundary conditions of fractional differential equations with impulses involving Caputo derivative, respectively, by means of Banach's fixed point theorem and Schauder's fixed point theorem (see [14, 26, 31, 32]).

It is well known that the control systems subject to delay have been extensively studied and the delay differential equations are large and important class of dynamic systems, which often arise in either natural or technological control problems. Time delay, always existing in real systems, usually results in oscillations around the discontinuity surface ([18, 21, 25]). Zhou et al. [30] studied Cauchy initial value problem of fractional neutral functional differential equations with infinite delay, obtaining various criteria on existence and uniqueness. Benchohra [6] and Deng [9] discussed the solutions for the same fractional differential equations with infinite delay by using different methods, respectively. Bahakhani [5] considered the nonlinear fractional differential equations with delay in two-dimensional case, and Zhang et al. [29] investigated the fractional equations with infinite delay and nonlocal conditions.

To the best of our knowledge, there are few papers that consider the fractional differential equations with both impulses and delays ([1, 16, 29]). In consequence, motivated by the works mentioned above, the aim of this paper is to discuss the existence and uniqueness of solutions of fractional differential equations with delay and impulses in (1.1). By using the idea of successive approximations, our main results can be seen as a generalization of the work in [2].

The organization of this paper is as follows. In Section 2, we present some required definitions, notations, and a lemma that will be used to prove our main results. In Section 3, the existence and uniqueness of the solution for the problem (1.1) are obtained in Theorem 3.1. And finally, an example is given to illustrate the effectiveness and feasibility of our main results in Section 4.

2. Preliminaries

In this section, we recall some definitions and propositions of fractional calculus and solution operator ([13,19]).

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L_1([a,b],\mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \tag{2.1}$$

where Γ is the gamma function.

Definition 2.2. For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by

$$(D_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds, \tag{2.2}$$

where $n = [\alpha] + 1$.

Definition 2.3. For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds, \tag{2.3}$$

where $n = [\alpha] + 1$.

Remark 2.1. The Caputo fractional derivative operator ${}^cD_t$ is a left inverse of integral operator I_t^α but in general is not a right inverse,

$$^cD_t^\alpha(I_t^\alpha u(t)) = u(t), \tag{2.4}$$

and the following holds

$$I_t(^cD_t^\alpha u(t)) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} u^{(k)}(a), t \in [a,b]. \tag{2.5}$$

For the sake of the readers' convenience, we introduce the following notations.

Let $J = [0,T]$, $J_0 = [0,t_1]$, $J_i = [t_i, t_{i+1}], i = 1,2,\ldots,p-1$, $J_p = [t_p, T]$, and $J' = J \backslash \{t_1, t_2, \ldots, t_p\}$. We denote $PC(J) = \{u : [0,T] \to \mathbb{R} | u \in C(J', \mathbb{R}),\ u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k^-) = u(t_k), k = 1,2,\ldots,p\}$. Obviously, $PC(J)$ is a Banach space with the norm $||u|| = \sup\limits_{t\in J} |u(t)|$.

Lemma 2.1. *Assume that $h \in C([0,T],\mathbb{R}), T > 0$. A function $x \in PC(J)$ is a solution of the initial value problem*

$$\begin{cases} D^\alpha x(t) = h(t), & t \neq t_k,\ t \in [0,T]; \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\ldots,p; \\ x(t) = \phi(t), & t \in [-\tau, 0] \end{cases} \tag{2.6}$$

if and only if x satisfies the following integral equation

$$x(t) = \begin{cases} \phi(t), & t \in [-\tau, 0]; \\ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds + \sum\limits_{j=1}^k I_j(x(t_j)) & \\ \quad + \sum\limits_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} h(s) ds, & t \in (t_k, t_{k+1}], k = 0,1,\ldots,p. \end{cases} \tag{2.7}$$

Proof. Assume that x satisfies the problem (2.6). One can see, from Remark 2.1 and $\phi(0) = 0$, that

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \quad \text{for } t \in J_0 = [t_0, t_1].$$

In view of $x(t_1^+) - x(t_1^-) = I_1(x(t_1))$, we obtain that

$$x(t_1^+) = I_1(x(t_1)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds.$$

It follows that for $t \in (t_1, t_2]$,

$$\begin{aligned} x(t) &= x(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} h(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds + I_1(x(t_1)). \end{aligned}$$

In consequence, we can see, by means of $x(t_2^+) = x(t_2^-) + I_2(x(t_2))$, that

$$x(t_2^+) = \sum_{i=0}^{1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} h(s) ds + \sum_{j=1}^{2} I_j(x(t_j)),$$

which implies that for $t \in (t_2, t_3]$,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds + \sum_{i=0}^{1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} h(s) ds + \sum_{j=1}^{2} I_j(x(t_j)).$$

Repeating the above process, the solution $x(t)$ for $t \in (t_k, t_{k+1}]$ can be written as

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s) ds + \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} h(s) ds + \sum_{j=1}^{k} I_j(x(t_j)).$$

Conversely, if x is a solution of (2.7), one can obtain, by a direct computation, that $D^\alpha x(t) = h(t),\ t \neq t_k, t \in [0,T]$, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k))$, where

$$x(t_k^+) = \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} h(s) ds + \sum_{j=1}^{k} I_j(x(t_j)),$$

and

$$x(t_k^-) = \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} h(s) ds + \sum_{i=0}^{k-2} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} h(s) ds + \sum_{j=1}^{k-1} I_j(x(t_j)).$$

This completes the proof. $\square$

3. Main results

Firstly, set $C_0 = \{y | y \in C([0,T], \mathbb{R}), y(0) = 0\}$. For each $y \in C_0$, we denote by $\overline{y}$ the function defined by

$$\overline{y}(t) = y(t),\ 0 \leq t \leq T,\ \text{and}\ \overline{y}(t) = 0,\ -\tau \leq t \leq 0. \tag{3.1}$$

If x is a solution of (1.1), then $x(\cdot)$ can be decomposed as $x(t) = \overline{y}(t) + \varphi(t)$ for $-\tau \leq t \leq T$, which implies that $x_t = \overline{y}_t + \varphi_t$ for $0 \leq t \leq T$, where

$$\varphi(t) = 0,\ 0 \leq t \leq T,\ \text{and}\ \varphi(t) = \phi(t),\ -\tau \leq t \leq 0. \tag{3.2}$$

Therefore, the problem (1.1) can be transformed into the following fixed point problem of the operator $F : C_0 \to \mathbb{R}$,

$$\begin{aligned} Fy(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,\overline{y}_s+\varphi_s)ds \\ &+\sum_{i=0}^{k-1}\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}f(s,\overline{y}_s+\varphi_s)ds \\ &+\sum_{j=1}^{k}I_j(\overline{y}(t_j)),\ t\in(t_k,t_{k+1}],k=0,1,\dots,p. \end{aligned} \tag{3.3}$$

Now, let us present our main result.

Theorem 3.1. *For the functions $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ and $I_k : \mathbb{R}\to\mathbb{R}$, assume the following conditions hold.*

(H_1) *there exists a continuous function $a : [0,T] \to \mathbb{R}^+$ satisfying*

$$|f(t,u_t)-f(t,v_t)| \le a(t)\sup_{s\in[0,t]}|u(s)-v(s)|,\ u,v\in\mathbb{R},\ t\in[0,T];$$

(H_2) *there exists a constant $L_k > 0$ such that $|I_k(u)-I_k(v)| \le L_k|u-v|,\ k=1,2,\dots,p$;*

(H_3) $\sum\limits_{i=1}^{p+1}\frac{a_iT^\alpha}{\Gamma(\alpha+1)}+\sum\limits_{j=1}^{p}L_j<1$, *where* $a_k=\sup\limits_{t\in(t_k,t_{k+1})}a(t)$;

(H_4) *there exists a constant $M > 0$ such that $|f(t,\varphi_t)| \le M$,* where φ is defined in (3.2).

Then the problem (1.1) has a unique solution on J.

Proof. To complete the proof, we shall use the method of successive approximations. Define a sequence of functions $y_n : [0,T]\to\mathbb{R}, n=0,1,2,\dots$ as follows:

$$y_0(t)=0,\quad y_n(t)=Fy_{n-1}(t). \tag{3.4}$$

Since $y_0(t)=0$, it is easy to see from (3.1) that $(\overline{y}_0)_s=0$ for $s\in[0,T]$. Thus we have

$$\begin{aligned} |y_1(t)-y_0(t)| &\le \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,\varphi_s)|ds+\sum_{j=1}^{k}I_j(0) \\ &+\sum_{i=0}^{k-1}\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}|f(s,\varphi_s)|ds \\ &\le \frac{M(t-t_k)^\alpha}{\Gamma(\alpha+1)}+\sum_{i=1}^{k}\frac{M(t_i-t_{i-1})^\alpha}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}|I_j(0)| \\ &\le \sum_{i=1}^{p+1}\frac{M(t_i-t_{i-1})^\alpha}{\Gamma(\alpha+1)}+\sum_{j=1}^{p}|I_j(0)| := N_0,\ k=1,2,\dots,p, \end{aligned}$$

it follows that $||y_1 - y_0|| \leq N_0$. Furthermore,

$$\begin{aligned}
& |y_n(t) - y_{n-1}(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left| f(s, (\overline{y}_{n-1})_s + \varphi_s) - f(s, (\overline{y}_{n-2})_s + \varphi_s) \right| ds \\
& + \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} \left| f(s, (\overline{y}_{n-1})_s + \varphi_s) - f(s, (\overline{y}_{n-2})_s + \varphi_s) \right| ds \\
& + \sum_{j=1}^{k} \left| I_j(\overline{y}_{n-1}(t_j)) - I_j(\overline{y}_{n-2}(t_j)) \right| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} a(s) \sup_{r\in[0,s]} \left| \overline{y}_{n-1}(r) - \overline{y}_{n-2}(r) \right| ds \\
& + \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} a(s) \sup_{r\in[0,s]} \left| \overline{y}_{n-1}(r) - \overline{y}_{n-2}(r) \right| ds \\
& + \sum_{j=1}^{k} L_j \left| \overline{y}_{n-1}(t_j) - \overline{y}_{n-2}(t_j) \right| \\
\leq\ & \left(a_k \frac{(t-t_k)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^{k} a_i \frac{(t_i - t_{i-1})^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{k} L_j \right) \cdot ||y_{n-1} - y_{n-2}|| \\
\leq\ & \left(\sum_{i=1}^{p+1} a_i \frac{T^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{p} L_j \right) \cdot ||y_{n-1} - y_{n-2}|| \\
:=\ & N ||y_{n-1} - y_{n-2}||, \qquad (3.5)
\end{aligned}$$

which implies that $||y_n - y_{n-1}|| \leq N||y_{n-1} - y_{n-2}||$ with $N < 1$. Note that for any $m > n > 0$, we have

$$\begin{aligned}
||y_m - y_n|| &\leq ||y_{n+1} - y_n|| + ||y_{n+2} - y_{n+1}|| + \cdots + ||y_m - y_{m-1}|| \\
&\leq (N^n + N^{n+1} + \cdots + N^{m-1}) \cdot ||y_1 - y_0|| \\
&\leq \frac{N^n}{1-N} ||y_1 - y_0||. \qquad (3.6)
\end{aligned}$$

For sufficiently large numbers m, n, it follows from the above inequalities with $N < 1$ that $||y_m - y_n|| \to 0$. Thus, $\{y_n(t)\}$ is a Cauchy sequence in $PC(J)$. Since $PC(J)$ is a complete Banach space, then $||y_n - y|| \to 0$ $(n \to \infty)$ for some $y \in PC(J)$, which means that $y_n(t)$ is uniformly convergent to $y(t)$ with respect to t.

In what follows, we shall show that $y(t)$ is a solution of the equation (1.1). Observe that

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,(\overline{y}_n)_s+\varphi_s)ds-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,\overline{y}_s+\varphi_s)ds\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,(\overline{y}_n)_s+\varphi_s)-f(s,\overline{y}_s+\varphi_s)\right|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}a(t)(t-s)^{\alpha-1}\sup_{r\in[0,s]}\left|\overline{y}_n(r)-\overline{y}(r)\right|ds \\
=\ & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}a(t)(t-s)^{\alpha-1}\sup_{r\in[0,s]}\left|y_n(r)-y(r)\right|ds.
\end{aligned}$$

Since $y_n(t)\to y(t)$ as $n\to+\infty$, for any $\epsilon>0$, there exists a sufficiently large number $n_0>0$ such that for all $n>n_0$, we have

$$|y_n(r)-y(r)|<\min\{\frac{\Gamma(\alpha+1)}{\sum\limits_{i=0}^{p}a_iT^\alpha}\epsilon,\ \frac{1}{\sum\limits_{j=1}^{p}L_j}\epsilon\}.$$

Therefore,

$$\left|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,(\overline{y}_n)_s+\varphi_s)ds-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,\overline{y}_s+\varphi_s)ds\right|<\epsilon, \tag{3.7}$$

$$\begin{aligned}
&\left|\sum_{i=0}^{k-1}\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}f(s,(\overline{y}_n)_s+\varphi_s)ds\right. \\
&\left.-\sum_{i=0}^{k-1}\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}f(s,\overline{y}_s+\varphi_s)ds\right| \\
\leq\ & \sum_{i=0}^{k-1}\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\left|f(s,(\overline{y}_n)_s+\varphi_s)-f(s,\overline{y}_s+\varphi_s)\right|ds \\
\leq\ & \sum_{i=0}^{k-1}a(t_i)\frac{(t_i-t_{i-1})^\alpha}{\Gamma(\alpha+1)}\sup_{r\in[0,s]}\left|y_n(r)-y(r)\right|ds<\epsilon,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\left|\sum_{j=1}^{k}|I_j(\overline{y}_n(t_j))-\sum_{j=1}^{k}|I_j(\overline{y}(t_j))\right| \\
\leq\ & \sum_{j=1}^{k}L_j|\overline{y}_n(t_j)-\overline{y}(t_j)| \\
=\ & \sum_{j=1}^{k}L_j|y_n(t_j)-y(t_j)|<\epsilon.
\end{aligned} \tag{3.9}$$

In consequence, we can see that for a sufficiently large number $n > n_0$,

$$\begin{aligned}
& |y(t) - Fy(t)| \\
\leq\ & |y(t) - y_{n+1}(t)| + |y_{n+1}(t) - Fy_n(t)| + |Fy_n(t) - Fy(t)| \\
\leq\ & |y(t) - y_{n+1}(t)| + \Bigg| y_{n+1}(t) - \Bigg[\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, (\overline{y}_n)_s + \varphi_s) ds \\
& + \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} f(s, (\overline{y}_n)_s + \varphi_s) ds + \sum_{j=1}^{k} I_j(\overline{y}_n(t_j)) \Bigg] \Bigg| \\
& + \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, \overline{y}_s + \varphi_s) ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, (\overline{y}_n)_s + \varphi_s) ds \right| \\
& + \Bigg| \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} f(s, \overline{y}_s + \varphi_s) ds \\
& - \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} f(s, (\overline{y}_n)_s + \varphi_s) ds \Bigg| \\
& + \left| \sum_{j=1}^{k} I_j(\overline{y}_n(t_j)) - \sum_{j=1}^{k} I_j(\overline{y}(t_j)) \right|.
\end{aligned}$$

Thus, in view of the convergence of the two previous and (3.7)-(3.9), one obtains that $|y(t) - Fy(t)| \to 0$, which implies that y is a solution of (1.1).

Finally, we prove the uniqueness of the solution. Assume that $y, z : [0, T] \to \mathbb{R}$ are two solutions of (1.1). Note that

$$\begin{aligned}
& |y(t) - z(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} a(s) \sup_{r \in [0,s]} |\overline{y}(r) - \overline{z}(r)| ds \\
& + \sum_{i=0}^{k-1} \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} a(s) \sup_{r \in [0,s]} |\overline{y}(r) - \overline{z}(r)| ds + \sum_{j=1}^{k} L_j |\overline{y}(t_j) - \overline{z}(t_j)| \\
\leq\ & \left(\sum_{i=1}^{p+1} \frac{a_i T^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{p} L_j \right) \cdot ||y - z||.
\end{aligned}$$

According to the condition (H_3), the uniqueness of the problem (1.1) follows immediately, which completes the proof. □

Remark 3.1. Notice that by setting $\tau = 0$, which means that the time delay vanished, and in the case of $a_i = a_j, L_i = L_j (i \leq j, i, j = 1, 2, \ldots, p)$, then Theorem 3.1 reduces to Theorem 4.3 of Agarwal [2]. In consequence, we extend the results in [2] in many aspects.

4. An example

Consider the following fractional impulsive differential equations with time delay

$$\begin{cases} D^{\alpha}x(t) = \frac{e^{-t}|x_t|}{(9+e^t)(1+|x_t|)}, & t \in [0,1], t \neq \frac{1}{2}, 0 < \alpha < 1; \\ \Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2}^-)|}{3+|x(\frac{1}{2}^-)|}, & \\ x(t) = \phi(t) = \frac{e^{-t}-1}{2}, & -\tau \leq t \leq 0, \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$, $\Gamma(\alpha+1) > \frac{3}{10}$, τ is a nonnegative constant. $x_t(\theta) = x(t+\theta)$ for $-\tau \leq \theta \leq 0$ and $0 \leq t \leq 1$.

Set

$$f(t,x) = \frac{e^{-t}x}{(9+e^t)(1+x)}, \quad I(x) = \frac{x}{3+x}, \quad \text{for } (t,x) \in [0,1] \times [0,+\infty).$$

Now, we can see that

$$\begin{aligned} |f(t,u_t) - f(t,v_t)| &= \frac{e^{-t}}{(9+e^t)} \frac{\Big| |u_t| - |v_t| \Big|}{(1+|u_t|)(1+|v_t|)} \\ &\leq \frac{e^{-t}}{(9+e^t)} |u_t - v_t| \\ &\leq a(t) \sup_{s\in[0,t]} |u(s) - v(s)|, \end{aligned}$$

where $a(t) = \frac{e^{-t}}{(9+e^t)}$ and $a = \sup\limits_{t\in[0,1]} a(t) = \frac{1}{10}$, so the condition ($\mathrm{H}_1$) is satisfied.

On the other hand, we get that

$$|I(u) - I(v)| = \frac{3|u-v|}{(3+u)(3+v)} \leq \frac{1}{3}|u-v|, \ \ u,v > 0,$$

which satisfies the condition (H_2) of Theorem 3.1 with $L = \frac{1}{3}$.

By a direct computation, one obtains that

$$\sum_{i=1}^{p+1} \frac{a_i T^{\alpha}}{\Gamma(\alpha+1)} + \sum_{j=1}^{p} L_j = \frac{2}{10} \frac{1}{\Gamma(\alpha+1)} + \frac{1}{3} < 1$$

and

$$|f(t,x_t)| = \frac{e^{-t}}{(9+e^t)} \frac{|x_t|}{(1+|x_t|)} \leq \frac{e^{-t}}{9+e^t} \leq \frac{1}{10}, \ \ t \in [0,1].$$

As a result, the equations in (4.1) satisfy all the hypotheses in Theorem 3.1, which guarantees that (4.1) has a unique solution.

Remark 4.1 In the case of $\tau = 0$ with the time delay vanishing, one deduces the slightly generalized form of the equations of (130)-(132) in [2]. However, the method for verifying the existence and uniqueness introduced by the author of [2] is not valid here to consider the equations with time delay.

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