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A characterization of exponential stability for periodic evolution families in terms of lower semicontinuous functionals [1]

Constantin Buşe

West University of Timişoara
Department of Mathematics
Bd. V. Pârvan No. 4, 300223 *Timişoara, România*
E-mail: buse@hilbert.math.uvt.ro

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Abstract

A characterization of exponential stability for a 1-periodic evolution family of bounded linear operators acting on a Banach space in terms of lower semicontinuous functionals is given.

1 Introduction

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X and $x \in X$. We denote by f_x the map $t \mapsto ||T(t)x||$. The Datko-Pazy theorem is well known. It says that the semigroup $\mathbf{T}$ is uniformly exponentially stable if and only if for each $x \in X$ the function f_x belongs to $L^p([0,\infty))$ for some $1 \leq p < \infty$. See [3] for $p = 2$ and [9] for the general case. Jan van Neerven has shown that a part of this result still works replacing $L^p([0,\infty))$ by any Banach function space as in [7, Theorem 3.1.5].

Moreover an autonomous variant of Rolewicz theorem says that if $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing function such that $\phi(t) > 0$ for every

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

$t > 0$ and if for each $x \in X$ the map $\phi \circ f_x$ belongs to $L^1([0,\infty))$, then $\mathbf{T}$ is exponentially stable. Recently, Jan van Neerven obtained characterizations of exponential stability for semigroups in terms of lower semicontinuous functionals. The aim of this note is to extend the Neerven result from semigroups to periodic evolution families.

A family $\mathcal{U} = \{U(t,s))\}_{t \geq s \geq 0}$ of bounded linear operators acting on a Banach space X, is called a 1-*periodic* evolution family if:

1. $U(t,t) = I$ (I is the identity operator on X);
2. $U(t,s)U(s,r) = U(t,r)$ for every $t \geq s \geq r \geq 0$;
3. $U(t+1, s+1) = U(t,s)$ for every $t \geq s \geq 0$;
4. $\sup_{0 \leq s \leq t \leq 1} ||U(t,s)|| = M < \infty$.

An 1-periodic evolution family $\mathcal{U}$ is called *measurable* if for each $x \in X$ the map $t \mapsto ||U(t,0)x||$ is measurable. This notion will be used in the Corollary 2.3 below.

Practically one parameter semigroups will always occur with $U(t,s) = T(t-s)$.

It is easy to see that such family $\mathcal{U}$ has a growth bound, that is, there exist $\omega \in \mathbf{R}$ and $M_\omega \geq 1$ such that

$$||U(t,s)|| \leq M_\omega e^{\omega(t-s)} \text{ for every } t \geq s \geq 0. \tag{1}$$

We say that the family $\mathcal{U}$ is *exponentially stable* if we can choose a negative ω such that the estimation (1) holds. Let $V := U(1,0)$ be the monodromy operator associated with the family $\mathcal{U}$. It is well-known, see e.g. [1], that the family $\mathcal{U}$ is exponentially stable if and only if the spectral radius of the operator V satisfies $r(V) < 1$.

We denote by $\mathcal{M}_{\text{loc}}[0,\infty)$ the space of all locally bounded functions on $[0,\infty)$ endowed with the topology of uniform convergence on bounded sets.

By $\mathcal{M}^+_{\text{loc}}[0,\infty)$ we denote the positive cone of $\mathcal{M}_{\text{loc}}[0,\infty)$. The following result holds:

Theorem 1.1 *Let $J : \mathcal{M}^+_{loc}[0,\infty) \to [0,\infty]$ be a map with the following properties:*

1. J is lower semicontinuous;

2. J is nondecreasing, i.e. if $f, g \in \mathcal{M}^+_{loc}[0,\infty)$ and $0 \leq f \leq g$ then $J(f) \leq J(g)$;

3. for each natural number k and any positive ρ there exists $t \geq 0$ such that $J(\rho \cdot 1_{[0,t]}) > k$.

If a 1-periodic evolution family $\mathcal{U}$ is not exponentially stable then the set

$$\{x \in X : J(||U(\cdot, 0)x||) = \infty\}$$

is residual, that is, its complement is of the first category.

Here $1_{[0,t]}$ is the characteristic function of the interval $[0, t]$.

The proof of Theorem 1.1 is based on the following operator theoretical result which was proved by Jan van Neerven in [8]:

Theorem 1.2 *Let T be a bounded linear operator on a Banach space X and assume that its spectral radius satisfies $r(T) \geq 1$. Then for all $x \in X$ and $\delta > 0$ there exists a positive constant C with the following property: for all $n \in \mathbf{N}$ ($\mathbf{N}$ is the set of all natural numbers) there exists $y \in X$ such that $||x - y|| < \delta$ and $||T^j y|| \geq C$ for all $j = 0, 1, \cdots, n$.*

A natural consequence of Theorem 1.1 is the following result which extends a similar one from [8]:

Theorem 1.3 *Let J as in the above Theorem 1.1 and $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a locally bounded semigroup on a Banach space X, i. e. a semigroup of operators for which $\sup\{||T(t)|| : t \in [0, 1]\} < \infty$. If $\mathbf{T}$ is not exponentially stable, i.e. its uniform growth bound $\omega_0(\mathbf{T}) := \inf\limits_{t>0} \frac{\ln ||T(t)||}{t}$ is nonnegative, then the set*

$$\{x \in X : J(||T(\cdot)x||) = \infty\}$$

is residual.

Particularly we can obtain the following generalization of a semigroup version of the Rolewicz theorem and of the Datko-Pazy theorem see for example [2], [10], [4], [7],[6], [5].

Theorem 1.4 *Let ϕ be a $[0, \infty)$-valued, nondecreasing function on $[0, \infty)$ such that $\phi(t) > 0$ for all $t > 0$ and $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a locally bounded semigroup on a Banach space X. If $\mathbf{T}$ is not exponentially stable, then the* *set*

$$\{x \in X : \int_0^\infty \phi(||T(t)x||)dt = \infty\}$$

is residual.

Proof. The fact that $\mathbf{T}$ is locally bounded implies that for each $x \in X$ the map $t \mapsto ||T(t)x||$ is measurable. Now we can apply the above Theorem 1.1 for the functional $J(f) := \int_0^\infty \phi(f(t))dt$ defined for any locally bounded and measurable function f on $[0, \infty)$.

2 Proof of Theorem 1.1 and other consequences

Using Theorem 1.2 we can establish the following simple Lemma.

Lemma 2.1 *If a* 1*-periodic evolution family* $\mathcal{U}$ *is not exponentially stable (or equivalently if* $r(V) \geq 1$*) then for each* $x \in X$ *and each positive constant* δ*, there exists a positive constant* L *with the following property: for all* $t \geq 0$ *there exists* $y \in X$ *such that* $||y - x|| < \delta$ *and*

$$||U(s,0)y|| \geq L \text{ for all } s \in [0,t]. \tag{2}$$

Proof of Lemma 2.1. Let C be as in Theorem 1.2, $t \geq 0$, $s \in [0,t]$ and $n = [s] + 1$, where $[s]$ denotes the integer part of s i.e. the biggest natural number which is less than or equal to s. We choose y as in Theorem 1.2 and we have:

$$\begin{aligned} C &\leq ||V^n y|| = ||U(n,0)y|| \\ &= ||U(n,s)U(s,0)y|| = ||U(1, s-[s])U(s,0)y|| \\ &\leq M||U(s,0)y||. \end{aligned}$$

This shows that (2) holds with $L = \frac{C}{M}$.

Proof of Theorem 1.1. We shall use the same ideas as in the proof of Theorem 4 in [8]. We may suppose that (1) is fulfilled with some positive ω. Thus from (1), using the fact that each operator $U(t,0)$ is linear, follows the continuity of the map $x \mapsto ||U(\cdot,0)x||$. The lower semicontinuity of J and the continuity of the map $x \mapsto ||U(\cdot,0)x||$ implies that for each $k = 1, 2, \cdots$, the set

$$X_k := \{x \in X : J(||U(\cdot,0)x||) > k\}$$

is open. Then is suffices to prove that each X_k is dense in X.

Let $x \in X$ and $\delta > 0$. We choose the constant L as in Lemma 2.1. Then for every $k = 1, 2, \cdots$ there exist $t_{L_k} \geq 0$

and $y_k \in X$ with $||y_k - x|| < \delta$ and $J(L \cdot 1_{[0,t_{L_k}]}) > k$ such that

$$||U(s,0)y_k|| \geq L \text{ for every } s \in [0, t_{L_k}].$$

Finally we obtain:

$$\begin{aligned} J(||U(\cdot,0)y_k||) &\geq J(||U(\cdot,0)y_k|| \cdot 1_{[0,t_{L_k}]}) \\ &\geq J(L \cdot 1_{[0,t_{L_k}]}) > k, \end{aligned}$$

that is, $y_k \in X_k$.

In the end of this note we mention the following result which characterizes the exponential stability of periodic evolution families in terms of Banach function spaces.

Theorem 2.2 *Let E be a Banach function space over* $[0,\infty)$ *such that*

$$\lim_{t\to\infty} ||1_{[0,t]}||_E = \infty$$

and $\mathcal{U}$ be a 1*-periodic evolution family which is not exponentially stable. Then the set*

$$\{x \in X : ||U(\cdot,0)x|| \notin E\}$$

is residual.

The details of the proof can be found in [8, page 485]. We remark here that Jan van Neerven have not used the measurability condition in his proof from [8].

Corollary 2.3. *Let E as in the Theorem* 2.2. We suppose that the norm of E has the Fatou property (see [8]) *and that $\mathcal{U}$ is a* 1*-periodic and measurable evolution family. If the set of all $x \in X$ for which the map $||U(\cdot,0)x||$ belongs to E is of second category in X, then $\mathcal{U}$ is exponentially stable.*

Proof. The proof is modelled after [8]. Using the Fatou property follows that for each natural number k the set

$$Y_k := \{x \in X : |||U(\cdot,0)x|||_E \leq k\},$$

is closed. By assumption follows that $\cup_{k\geq 1} Y_k$ is of the second category, so there exists k_0 such that Y_{k_0} has nonempty interior. Moreover, if the open ball with centre in x and radius $2\delta > 0$ is contained in Y_{k_0} then by the triangle inequality in E, the ball with centre in origin and radius 2δ is contained in Y_{2k_0}. Then for each nonzero $x \in X$, we have:

$$|||U(\cdot,0)x|||_E = \frac{||x||}{\delta}|||U(\cdot,0)(\frac{\delta}{||x||}x)|||_E < \infty.$$

In order to obtain the assertion we apply Theorem 4.5 from [1].

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