Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **80**, 1-9; **http://www.math.u-szeged.hu/ejqtde/**

Existence and uniqueness of almost periodic solutions for a class of nonlinear Duffing system with time-varying delays*

Yanli Xu†

Department of Mathematics, Xiangnan College, Chenzhou, Hunan 423000, P.R. China

Abstract: In this paper, we investigate the existence and uniqueness of almost periodic solutions for a class of nonlinear Duffing system with time-varying delays. By using theory of exponential dichotomies and contraction mapping principle, we establish some new results and give an example to illustrate the theoretical analysis in this work.

Keywords: Almost periodic solution; Duffing system; exponential dichotomy; contraction mapping principle.

MR(2000) Subject Classification: 34C25; 34K13; 34K25.

1. Introduction

In recent years, the dynamic behaviors of nonlinear Duffing equations have been widely investigated in [1-4] due to the application in many fields such as physics, mechanics, engineering, other scientific fields. In such applications, it is important to know the existence of the almost periodic solutions for nonlinear Duffing equations. Some results on existence of the almost periodic solutions were obtained in the literature. We refer the reader to [5−8] and the references cited therein.

Recently, L. Q. Peng and W. T. Wang [9] considered the following model for nonlinear Duffing equation with a deviating argument

$$x''(t) + cx'(t) - ax(t) + bx^m(t - \tau(t)) = p(t), \tag{1.1}$$

*Supported by the Construct Program of the Key Discipline in Hunan Province, the Science and Technology Planning project of Chenzhou City in Hunan Province (grant no.[2011]29), and the Natural Scientific Research Fund of Hunan Provincial Education Department of PR China (grant no. 11C1186).

†Corresponding author. Tel.:+86 07352653203; fax: +86 07352653203. E-mail:xuyanliling@yahoo.cn.

where $\tau(t)$ and $p(t)$ are almost periodic functions on R, $m > 1$, a, b and c are constants. Let $Q_1(t)$ be a continuous and differentiable function on R. Define

$$y = \frac{dx}{dt} + \xi x - Q_1(t),\ Q_2(t) = p(t) + (\xi - c)Q_1(t) - Q_1'(t), \tag{1.2}$$

where $\xi > 1$ is a constant, then L. Q. Peng and W. T. Wang [9] transformed (1.1) into the following system

$$\begin{cases} \dfrac{dx(t)}{dt} = -\xi x(t) + y(t) + Q_1(t), \\ \dfrac{dy(t)}{dt} = -(c-\xi)y(t) + (a - \xi(\xi - c))x(t) - bx^m(t - \tau(t)) + Q_2(t). \end{cases} \tag{1.3}$$

Consequently, some sufficient conditions for the existence of positive almost periodic solutions of (1.1) and (1.3) were established in [9]. On the other hand, in the real scientific fields, the coefficients a, b and c in (1.1) and (1.3) are usually time-varying. Hence, the system (1.3) can be naturally extended to the following Duffing system with time-varying coefficients and delays

$$\begin{cases} \frac{dx(t)}{dt} = -\delta_1(t)x(t) + y(t) + Q_1(t) \\ \frac{dy(t)}{dt} = \delta_2(t)y(t) + [\alpha(t) - \delta_2^2(t)]x(t) - \beta(t)x^m(t - \tau(t)) + Q_2(t), \end{cases} \tag{1.4}$$

where $\alpha(t)$, $\beta(t)$, $\tau(t)$, $\delta_1(t)$, $\delta_2(t)$, $Q_1(t)$ and $Q_2(t)$ are almost periodic functions on R, $m > 1$ is an integer, $\alpha(t) > 0$, $\beta(t) \neq 0$. However, to the best of our knowledge, few authors have considered the problem for almost periodic solutions of system (1.4). Motivated by the above arguments, the main purpose of this present paper is to give the conditions to guarantee the existence of almost periodic solutions of system (1.4).

For convenience, we introduce some notations. Throughout this paper adopt the following notations:$X = (x_1, x_2)^T \in R^2$ to denote a column vector, in which the symbol $(^T)$ denote the transpose of a vector. We let $|X|$ denote the absolute-value vector given by $|X| = (|x_1|, |x_2|)^T$, and define $||X|| = \max\limits_{1\leq i\leq 2} |x_i|$. A vector $X \geq 0$ means that all x_i are greater than or equal to zero. $X > 0$ is defined similarly. For vectors X and Y, $X \geq Y$ (resp. $X > Y$) means that $X - Y \geq 0$(resp. $X - Y > 0$). Let $\underline{\delta}_1$, $\underline{\delta}_2$, δ^*, l, θ and q be defined as

$$\underline{\delta}_1 = \inf_{t\in R} |\delta_1(t)|, \quad \underline{\delta}_2 = \inf_{t\in R} |\delta_2(t)|, \quad \delta^* = \min\{\underline{\delta}_1, \underline{\delta}_2\}, \tag{1.5}$$

$$l = \max\{\frac{\sup\limits_{t\in R}|Q_1(t)|}{\delta^*}, \frac{\sup\limits_{t\in R}|Q_2(t)|}{\delta^*}\}, \quad \theta = \max\{\frac{1}{\delta^*}, \frac{\sup\limits_{t\in R}[|\alpha(t) - \delta_2^2(t)| + |\beta(t)|]}{\delta^*}\}, \tag{1.6}$$

$$q = \max\{\frac{1}{\delta^*}, \frac{\sup\limits_{t\in R}[|\alpha(t) - \delta_2^2(t)| + |\beta(t)|m(\frac{2l}{1-\theta})^{m-1}]}{\delta^*}\}. \tag{1.7}$$

Set

$$B = \{\varphi | \varphi = (\varphi_1(t), \varphi_2(t))^T\},$$

where φ is an almost periodic function on R. For $\forall \varphi \in B$, we define the induced modulus $||\varphi||_B = \sup_{t \in R} ||\varphi(t)||$, then B is a Banach space.

Definition 1.[10,11] Let $u(t) : R \to R^n$ be continuous in t. $u(t)$ is said to be almost periodic on R if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\lambda : ||u(t + \lambda) - u(t)|| < \varepsilon, \forall t \in R\}$ is relatively dense, i.e., for $\varepsilon > 0$, it is possible to find a real number $L = L(\varepsilon) > 0$ where, for any interval with length $L(\varepsilon)$, there exists a number $\lambda = \lambda(\varepsilon)$ in this interval such that $||u(t + \lambda) - u(t)|| < \varepsilon$, for all $t \in R$.

Definition 2.[10,11] Let $z \in R^n$ and $Q(t)$ be an $n \times n$ continuous matrix defined on R. The linear system

$$\frac{dz(t)}{dt} = Q(t)z(t), \tag{1.8}$$

is said to admit an exponential dichotomy on R if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (1.8) satisfying

$$||X(t)PX^{-1}(s)|| \leq ke^{-\alpha(t-s)}, \quad \text{for all} \quad t \geq s,$$

$$||X(t)(I - P)X^{-1}(s)|| \leq ke^{-\alpha(s-t)}, \quad \text{for all} \quad t \leq s.$$

Lemma 1.1. [10,11] Let $Q(t) = (q_{ij})$ be an $n \times n$ almost periodic matrix defined on R and let there exist a positive constant ν such that

$$|q_{ii}(t)| - \sum_{j=1, j \neq i}^{n} |q_{ij}(t)| \geq \nu, i = 1, 2, \ldots, n.$$

Then the linear system (1.8) admits an exponential dichotomy on R.

Lemma 1.2. [10,11] If the linear system (1.8) admits an exponential dichotomy, then the almost periodic system

$$\frac{dz(t)}{dt} = Q(t)z(t) + g(t), \tag{1.9}$$

has a unique almost periodic solution $z(t)$ and

$$z(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(s)g(s)ds, \tag{1.10}$$

2. Existence and uniqueness of almost periodic solutions

Theorem 2.1. Assume $\delta_i(t) > 0, i = 1, 2$, and let positive constants l, θ and q satisfy

$$\theta < 1, \quad \frac{l}{1-\theta} < 1, \quad q < 1. \tag{2.1}$$

Then there exists a unique almost periodic solution of system (1.4) in the region

$$B^* = \{\varphi | \varphi \in B, ||\varphi - \varphi_0|| \le \frac{\theta l}{1-\theta}\},$$

where

$$\varphi_0 = (\int_{-\infty}^{t} e^{-\int_s^t \delta_1(w)dw} Q_1(s)ds, -\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} Q_2(s)ds)^T.$$

Proof. For any $\varphi \in B$, we consider the almost periodic solution of the nonlinear almost periodic two-dimensional system with time-varying delays

$$\begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\delta_1(t) & 0 \\ 0 & \delta_2(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} \varphi_2(t) + Q_1(t) \\ \tilde{\varphi}_1(t) \end{pmatrix}, \tag{2.2}$$

where

$$\tilde{\varphi}_1(t) = (\alpha(t) - \delta_2^2(t))\varphi_1(t) - \beta(t)\varphi_1^m(t - \tau(t)) + Q_2(t).$$

Then, notice that the linear system

$$\begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\delta_1(t) & 0 \\ 0 & \delta_2(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{2.3}$$

admits an exponential dichotomy on R. Define a projection P by setting

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, by Lemma 1.2, we obtain that the system (2.2) has exactly one almost periodic solution:

$$\begin{pmatrix} x^\varphi(t) \\ y^\varphi(t) \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}(\varphi_2(s) + Q_1(s))ds \\ -\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s) - \delta_2^2(s))\varphi_1(s) - \beta(s)\varphi_1^m(s - \tau(s)) + Q_2(s)]ds \end{pmatrix}.$$

Define a mapping $T : B \to B$ by setting

$$T(\varphi)(t) = \begin{pmatrix} x^\varphi(t) \\ y^\varphi(t) \end{pmatrix}, \quad \forall \varphi \in B.$$

Since $B^* = \{\varphi | \varphi \in B, ||\varphi - \varphi_0|| \leq \frac{\theta l}{1-\theta}\}$, it is easy to see that B^* is a closed convex subset of B . According to the definition of the norm in Banach space B , we derive

$$\begin{aligned}
||\varphi_0||_B &= \sup_{t\in R} \max\{\textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} |Q_1(s)|ds, \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} |Q_2(s)|ds\} \\
&\leq \sup_{t\in R} \max\{\sup_{t\in R} |Q_1(t)| \textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} ds, \sup_{t\in R} |Q_2(t)| \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} ds\} \\
&\leq \max\{\frac{\sup_{t\in R} |Q_1(t)|}{\delta_1}, \quad \frac{\sup_{t\in R} |Q_2(t)|}{\delta_2}\} \\
&\leq \max\{\frac{\sup_{t\in R} |Q_1(t)|}{\delta^*}, \quad \frac{\sup_{t\in R} |Q_2(t)|}{\delta^*}\} = l.
\end{aligned}$$

Therefore, for any $\varphi \in B^*$, we have

$$||\varphi||_B \leq ||\varphi - \varphi_0||_B + ||\varphi_0||_B \leq \frac{\theta l}{1-\theta} + l = \frac{l}{1-\theta} < 1. \tag{2.4}$$

Now, we prove that the mapping T is a self-mapping from B^* to B^* . In fact, for any $\varphi \in B^*$, from (2.4), we obtain

$$\begin{aligned}
& ||T\varphi - \varphi_0||_B \\
= \; & \sup_{t\in R} \max\{|\textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} \varphi_2(s)ds|, |\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} [(\alpha(s) - \delta_2^2(s))\varphi_1(s) \\
& \quad - \beta(s)\varphi_1^m(s - \tau(s))]ds|\} \\
\leq \; & \sup_{t\in R} \max\{||\varphi||_B \textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} ds, \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} [|\alpha(s) - \delta_2^2(s)|||\varphi||_B \\
& \quad + |\beta(s)|||\varphi||_B^m]ds\} \\
\leq \; & \sup_{t\in R} \max\{\textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} ds, \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} [|\alpha(s) - \delta_2^2(s)| + |\beta(s)|]ds\}||\varphi||_B \\
\leq \; & \sup_{t\in R} \max\{\textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} ds, \sup_{t\in R}[|\alpha(t) - \delta_2^2(t)| + |\beta(t)|] \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} ds\}||\varphi||_B \\
\leq \; & \max\{\frac{1}{\delta^*}, \frac{\sup_{t\in R}[|\alpha(t)-\delta_2^2(t)|+|\beta(t)|]}{\delta^*}\}||\varphi||_B \\
= \; & \theta||\varphi||_B \\
\leq \; & \frac{\theta l}{1-\theta},
\end{aligned}$$

which implies that $T\varphi \in B^*$. So, the mapping T is a self-mapping from B^* to B^* .

Next, we prove that the mapping T is a contraction mapping of the B^*. In deed, in view of (1.5), (1.6), (1.7), (2.1), (2.4) and differential mean-value theorem, for all $\varphi, \psi \in B^*$, we have

$$\begin{aligned}
& |T(\varphi(t))-T(\psi(t))| \\
= & (|(T(\varphi(t))-T(\psi(t)))_1|, |(T(\varphi(t))-T(\psi(t)))_2|)^T \\
= & (|\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}[\varphi_2(s)-\psi_2(s)]ds|, |\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s)-\delta_2^2(s))(\varphi_1(s)-\psi_1(s)) \\
& -\beta(s)(\varphi_1^m(s-\tau(s))-\psi_1^m(s-\tau(s)))]ds|)^T \\
= & (|\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}[\varphi_2(s)-\psi_2(s)]ds|, |\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s)-\delta_2^2(s))(\varphi_1(s)-\psi_1(s)) \\
& -\beta(s)m(\psi_1(s-\tau(s))+h(s)(\varphi_1(s-\tau(s))-\psi_1(s-\tau(s)))^{m-1} \\
& (\varphi_1(s-\tau(s))-\psi_1(s-\tau(s)))]ds|)^T \\
= & (|\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}[\varphi_2(s)-\psi_2(s)]ds|, |\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s)-\delta_2^2(s))(\varphi_1(s)-\psi_1(s)) \\
& -\beta(s)m((1-h(s))\psi_1(s-\tau(s))+h(s)\varphi_1(s-\tau(s)))^{m-1} \\
& (\varphi_1(s-\tau(s))-\psi_1(s-\tau(s)))]ds|)^T,
\end{aligned}$$

where $h(s)\in(0,\ 1)$ is the mean point in Lagrange's mean value theorem. Then,

$$\begin{aligned}
& |T(\varphi(t))-T(\psi(t))| \\
\le & (\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} ds \sup_{s\in R}|\varphi_2(s)-\psi_2(s)|, \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}\{|\alpha(s)-\delta_2^2(t)|\cdot \\
& \sup_{s\in R}|\varphi_1(s)-\psi_1(s)| + |\beta(s)|m[\sup_{s\in R}|\psi_1(s-\tau(s))| + \sup_{s\in R}|\varphi_1(s-\tau(s))|]^{m-1}\cdot \\
& \sup_{s\in R}|\varphi_1(s-\tau(s))-\psi_1(s-\tau(s))|\}ds|)^T \\
\le & (\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw} ds||\varphi-\psi||_B, \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw} ds \sup_{s\in R}[|\alpha(s)-\delta_2^2(s)| \\
& +|\beta(s)|m(\tfrac{2l}{1-\theta})^{m-1}]||\varphi-\psi||_B)^T,
\end{aligned}$$

which yields that

$$||T(\varphi)-T(\psi)||_B \le \max\{\frac{1}{\delta^*}, \frac{\sup_{t\in R}[|\alpha(t)-\delta_2^2(t)|+|\beta(t)|m(\frac{2l}{1-\theta})^{m-1}]}{\delta^*}\}||\varphi-\psi||_B = q||\varphi-\psi||_B.$$

It follow from $q<1$ that the mapping T is a contraction. Therefore, the mapping T possesses a unique fixed point $z^*=(x^*(t), y^*(t))^T\in B^*$,$Tz^*=z^*$. By (2.2), z^* satisfies (1.4). So z^* is an almost periodic solution of system (1.4) in B^* . The proof of Theorem 2.1 is now completed.

If $\delta_i(t)<0, i=1,2$, we define a projection P by setting

$$P=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then using similar arguments to those above, we can show the following

Theorem 2.2. If $\delta_i(t) < 0, i = 1, 2$, and assume that (2.1) holds. Then, there exists a unique almost periodic solution of system (1.4) in the region

$$B_1^* = \{\varphi | \varphi \in B, ||\varphi - \varphi_1|| \leq \frac{\theta l}{1-\theta}\},$$

where

$$\varphi_1 = (-\int_t^{+\infty} e^{-\int_t^s \delta_1(w)dw} Q_1(s)ds, \int_{-\infty}^t e^{-\int_s^t \delta_2(w)dw} Q_2(s)ds)^T.$$

3. An example

Example 3.1. The nonlinear Duffing equation with a deviating argument

$$\begin{aligned} x''(t) &+ (\sin t - \cos t)x'(t) + (626 + 27\cos t + 25\sin t + \cos^2 t + \cos t \sin t)x(t) \\ &+ (1 + \sin^2 t)x^3(t - \cos t) = \cos\sqrt{2}t + \cos\sqrt{3}t - 50\cos t - 2\cos^2 t, \end{aligned} \tag{3.1}$$

has exactly one almost periodic solution $x^*(t)$ such that

$$\max\{\sup_{t\in R} |x^*(t)|, \sup_{t\in R} |\frac{dx^*(t)}{dt} + (25 + \sin t)x^*(t) - 2\cos t|\} \leq \frac{1}{11}.$$

Proof. Set

$$\delta_1(t) = 25 + \sin t, \quad \delta_2(t) = 25 + \cos t, \quad y(t) = \frac{dx(t)}{dt} + (25 + \sin t)x(t) - 2\cos t, \tag{3.2}$$

we can transform (3.1) into the following system:

$$\begin{cases} \frac{dx(t)}{dt} = -(25 + \sin t)x(t) + y(t) + 2\cos t, \\ \frac{dy(t)}{dt} = (25 + \cos t)y(t) + (2 - \sin^2 t)x(t)\sin^2 t - (2 - \cos^2 t)x^3(t - \cos t) \\ \qquad + \cos\sqrt{2}t + \cos\sqrt{3}t. \end{cases} \tag{3.3}$$

Since $\alpha(t) = 626 + 50\cos t + 2\cos^2 t$, $\beta(t) = 2 - \cos^2 t$, $Q_1(t) = -2\cos t$, $Q_2(t) = \cos\sqrt{2}t + \cos\sqrt{3}t$, $m = 3$, $\delta^* = 25$, then

$$l = \max\{\frac{\sup_{t\in R} |Q_1(t)|}{\delta^*}, \frac{\sup_{t\in R} |Q_2(t)|}{\delta^*}\} = \frac{2}{25}$$

$$\theta = \max\{\frac{1}{\delta^*}, \frac{\sup_{t\in R}[|\alpha(t) - \delta_2^2(t)| + |\beta(t)|]}{\delta^*}\} = \frac{3}{25}$$

$$q = \max\{\frac{1}{\delta^*}, \frac{\sup_{t\in R}[|\alpha(t) - \delta_2^2(t)| + |\beta(t)|m(\frac{2l}{1-\theta})^{m-1}]}{\delta^*}\} = \frac{254}{3025}$$

It is straight forward to check that all assumptions needed in Theorem 2.1 are satisfied. Hence, equation (3.1) has exactly one almost periodic solution $x^*(t)$ such that

$$\max\{\sup_{t\in R}|x^*(t)|,\sup_{t\in R}|\frac{dx^*(t)}{dt}+(25+\sin t)x^*(t)-2\cos t|\}\leq \frac{l}{1-\theta}=\frac{1}{11}.$$

The proof of example 3.1 is now completed.

Remark 3.1. Since equation (3.1) is a very simple form of nonlinear Duffing equation with time-varying coefficient

$$c(t)=\sin t-\cos t, a(t)=-(626+27\cos t+25\sin t+\cos^2 t+\cos t\sin t) \text{ and } b(t)=1+\sin^2 t,$$

it is clear that the results obtained in [9] are invalid for the above example. Moreover, one can find that main theorem in [8] is a special one of Theorem 2.1 when $\delta_1(t)$ and $\delta_2(t)$ are constants. Thus, the results of this paper substantially extend and improve the main results of refs. [8-9].

Acknowledgment

The author would like to express the sincere appreciation to the reviewer for his/her helpful comments in improving the presentation and quality of this paper. Moreover, the author expresses her sincere gratitude to Professor Jianli Li and Doctor Jing Xiao for the helpful discussion when this work is carried out.

References

[1] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press, Orlando, FL, 1985.

[2] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[3] T. Yashizaw, Asymptotic behavior of solutions of differential equations, in: Differential Equation: Qualitative Theory (Szeged, 1984), in: Colloq. Math. Soc. János Bolyai, vol. 47, North-Holland,Amsterdam, 1987, pp. 1141-1172.

[4] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[5] M. S. Berger, Y. Y. Chen, Forced quasi periodic and almost periodic oscillations of nonlinear Duffing equations, Nonlinear Anal. 19 (3) (1992) 249-257.

[6] W. Y. Zeng, Almost periodic solutions for nonlinear Duffing equations, Acta Math. Sinica, New Ser. 13 (3) (1997) 373-380.

[7] Q. Y. Wang, The existence and uniqueness of almost periodic solutions for nonlinear differential equations with time lag, Acta Math. Sinica 42 (3) (1999) 511-518 (in Chinese).

[8] Q. Y. Zhou, B. W. Liu, New results on almost periodic solution for a class of nonlinear Duffing equations with a deviating argument, Applied Mathematics Letters. 22 (2009) 6-11.

[9] L. Q. Peng, W. T. Wang, Positive Almost Periodic Solutions for a Class of Nonlinear Duffing Equations with a Deviating Argument, Electronic Journal of Qualitative Theory of Differential Equations 2010, No. 6, 1-12; http://www.math.u-szeged.hu/ejqtde/.

[10] A. M. Fink, Almost periodic differential equations, in: Lecture Notes in Mathematics, vol. 377, Springer, Berlin, 1974, pp. 80-112.

[11] C. Y. He, Almost Periodic Differential Equation, Higher Education Publishing House, Beijing, 1992, pp. 90-100 (in Chinese).

(Received July 21, 2012)