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A DELAYED SIR EPIDEMIC MODEL WITH GENERAL INCIDENCE RATE

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Abstract. A delayed SIR epidemic model with a generalized incidence rate is studied. The time delay represents the incubation period. The threshold parameter, $R_0(\tau)$ is obtained which determines whether the disease is extinct or not. Throughout the paper, we mainly use the technique of Lyapunov functional to establish the global stability of both the disease-free and endemic equilibrium.

1. Introduction

Historically the mathematical modeling of epidemics has started since the time of Graunt [1]. In fact, Kermack and Mckendric [2] describe some classical deterministic mathematical models of epidemiology by considering the total population into three classes namely susceptible (S) individuals, infected (I) individuals and recovered (R) individuals which is known to us as SIR epidemic model. This SIR epidemic model is very important in today's analysis of diseases.

In recent years, epidemiological models have been studied by a number of authors [3, 4, 5, 6]. The basic and important research subjects for these systems are the existence of the threshold value which distinguishes whether the infectious disease will die out, the local and global stability of the disease-free equilibrium and the endemic equilibrium, the existence of periodic solutions, the persistence and extinction of the disease, etc. Many models in the literature represent the dynamics of disease by systems of ordinary differential equations without time delay. In order to reflect the real dynamical behaviors of models that depend on the past history of systems, it is reasonable to incorporate time delays into the systems [7]. In fact, inclusion of delays in epidemic models makes them more realistic by allowing the description of the effects of disease latency or immunity [8, 9, 12]. Most of the delay differential mathematical models are concerned with local stability of equilibria. The papers that are concerned with global stability of delayed differential models are relatively few.

Motivated by the works of Abta et al. [13], in the present paper, we are concerned with the effect of generalized incidence rate. To this end, we consider the following delay differential equation model.

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$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \mu S(t) - f(S(t), I(t))I(t), \\
\frac{dI}{dt} &= f(S(t-\tau), I(t-\tau))I(t-\tau)e^{-\mu\tau} - (\mu + d + r)I(t), \\
\frac{dR}{dt} &= rI(t) - \mu R(t),
\end{aligned} \tag{1.1}$$

where Λ is the recruitment rate of the population, μ is the natural death rate of the population, d is the death rate due to disease, r is the recovery rate of the infective individuals, $f(S, I)$ is the rate of transmission and τ is the incubation period. The term $e^{-\mu\tau}$ is the probability of surviving from time $t - \tau$ to time t. The goal of this paper is to study the global stability of delayed model (1.1). We present the construction of Lyapunov functionals for this model. This construction is based on ideas developed in [10, 11, 12].

As in [14], the incidence function $f(S, I)$ is assumed to be continuously differentiable in the interior of $\mathbf{R}^2_+$ and satisfies the following hypotheses:

$$f(0, I) = 0, \quad for\ all\ I \geq 0, \tag{H_1}$$

$$\frac{\partial f}{\partial S}(S, I) > 0, \quad for\ all\ S > 0\ and\ I \geq 0, \tag{H_2}$$

$$\frac{\partial f}{\partial I}(S, I) \leq 0\ , \quad for\ all\ S \geq 0\ and\ I \geq 0. \tag{H_3}$$

The first two equations in system (1.1) do not depend on the third equation, and therefore this equation can be omitted without loss of generality. System (1.1) can be rewritten as

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \mu S(t) - f(S(t), I(t))I(t), \\
\frac{dI}{dt} &= f(S(t-\tau), I(t-\tau))I(t-\tau)e^{-\mu\tau} - (\mu + d + r)I(t).
\end{aligned} \tag{1.2}$$

Let $C = C([-\tau, 0], \mathbf{R}^2)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^2$ with the topology of uniform convergence. By the fundamental theory of functional differential equations [15], it is easy to show that there exists a unique solution $(S(t), I(t))$ of system (1.2) with initial data $(S_0, I_0) \in C$. For ecological reasons, we assume that the initial conditions for system (1.2) satisfies:

$$S_0(\theta) \geq 0, \quad I_0(\theta) \geq 0, \quad \theta \in [-\tau, 0]. \tag{1.3}$$

The paper is organized as follows. In the next section, basic mathematical properties of the model are studied. In Section 3, the basic reproduction number is derived and the global asymptotic stability of the disease-free equilibrium is established. The global asymptotic stability of the endemic equilibrium is obtained in Section 4. Lastly, we give a brief discussion of our results in Section 5.

2. Positivity and boundedness of solutions

Since the model (1.2) represent population, it is important to prove that all solutions with nonnegative initial data will remain non-negative and bounded for all time.

Proposition 2.1. *Each component of the solution of system (1.2), subject to condition (1.3), remains non-negative and bounded for all $t \geq 0$.*

Proof. The solution $S(t)$ is positive for all $t \geq 0$. In fact, assuming the contrary, and letting $t_1 > 0$ be the first time such that $S(t_1) = 0$, then by the first equation of system (1.2) we have $\dot{S}(t_1) = \Lambda > 0$, and hence $S(t) < 0$ for $t \in]t_1 - \epsilon, t_1[$, where $\epsilon > 0$ is sufficiently small. This contradicts $S(t) > 0$ for $t \in [0, t_1[$. It follows that $S(t) > 0$ for $t > 0$. Now, we prove the positivity of solution $I(t)$. From (1.2), we get

$$I(t) = I(0)e^{-at} + e^{-\mu\tau} \int_0^t f\big(S(\theta - \tau), I(\theta - \tau)\big) I(\theta - \tau) e^{a(\theta - t)} d\theta, \tag{2.1}$$

with $a = \mu + d + r$.
Let $t \in [0, \tau]$, we have $\theta - \tau \in [-\tau, 0]$ for all $\theta \in [0, \tau]$. Using (1.3) and (2.1), we deduce that $S(t) \geq 0$ for $t \in [0, \tau]$. This method can now be repeated to deduce non-negativity of S on the interval $[\tau, 2\tau]$ and then on successive intervals $[n\tau, (n+1)\tau]$, $n \geq 2$, to include all positive times. This proves the positivity of solutions.

For boundedness of the solution, we define

$$T(t) = S(t) + e^{\mu\tau} I(t + \tau).$$

By non-negativity of the solution, it follows that

$$\begin{aligned} \frac{dT(t)}{dt} &= \Lambda - \mu S(t) - a e^{\mu\tau} I(t + \tau) \\ &\leq \Lambda - \mu T(t). \end{aligned}$$

This implies that $T(t)$ is bounded, and so are $S(t)$ and $I(t)$. This completes the proof of this proposition. ■

3. Reproductive number and stability analysis

The system (1.2) always has a disease-free steady state of the form $E_f(\frac{\Lambda}{\mu}, 0)$. Therefore, we define the basic reproduction number $R_0(\tau)$ of our model by

$$R_0 = \frac{f(\frac{\Lambda}{\mu}, 0) e^{-\mu\tau}}{\mu + d + r}, \tag{3.1}$$

where $\dfrac{1}{\mu + d + r}$ represents the average life expectancy of infectious individuals, $\dfrac{\Lambda}{\mu}$ represents the number of susceptible individuals at the beginning of the infectious process and $f(\frac{\Lambda}{\mu}, 0)$ represents the value of the function f when all individuals are susceptible. Hence, our R_0 is biologically well defined.

It is easy to see that if $R_0 \leq 1$, the disease-free steady state $E_f(\frac{\Lambda}{\mu}, 0)$ is the unique steady state, corresponding to the extinction of disease. The following theorem presents the existence and uniqueness of endemic equilibrium if $R_0 > 1$.

Theorem 3.1.

(1) *The disease-free equilibrium point of the system (1.2) is given by* $E_f(\frac{\Lambda}{\mu}, 0)$, *which exists for all parameter values.*
(2) *If* $R_0 > 1$, *then the system (1.2) has a unique endemic equilibrium of the form* $E^*(S^*, I^*)$ *with* $S^* \in]0, \frac{\Lambda}{\mu}[$ *and* $I^* > 0$.

Proof. The steady state of the system (1.2) satisfy the following system of equations

$$\Lambda - \mu S - f(S, I)I = 0, \tag{3.2}$$
$$f(S, I)Ie^{-\mu\tau} - (\mu + d + r)I = 0. \tag{3.3}$$

From (3.3) we get $I = 0$ or $f(S, I) = (\mu + d + r)e^{\mu\tau}$.

If $I = 0$, we obtain the disease-free equilibrium point $E_f(\frac{\Lambda}{\mu}, 0)$.

If $I \neq 0$, then using (3.2) and (3.3) we get the following equation

$$f(S, \frac{\Lambda - \mu S}{(\mu + d + r)e^{\mu\tau}}) = (\mu + d + r)e^{\mu\tau}. \tag{3.4}$$

We have $I = \dfrac{\Lambda - \mu S}{(\mu + d + r)e^{\mu\tau}} \geq 0$ implies that $S \leq \dfrac{\Lambda}{\mu}$. Hence, there is no positive equilibrium point if $S > \dfrac{\Lambda}{\mu}$.

Now, we consider the following function g defined on the interval $[0, \frac{\Lambda}{\mu}]$

$$g(S) = f(S, \frac{\Lambda - \mu S}{(\mu + d + r)e^{\mu\tau}}) - (\mu + d + r)e^{\mu\tau}.$$

Since, $g(0) = -(\mu+d+r)e^{\mu\tau} < 0$ and $g(\dfrac{\Lambda}{\mu})=(\mu+d+r)e^{\mu\tau}(R_0-1) > 0$ for $R_0 > 1$. Further, $g'(S) = \frac{\partial f}{\partial S} - \frac{\mu}{(\mu+d+r)e^{\mu\tau}}\frac{\partial f}{\partial I} > 0$. Hence, there exists a unique endemic equilibrium $E^*(S^*, I^*)$ with $S^* \in]0, \frac{\Lambda}{\mu}[$ and $I^* > 0$. ∎

3.1. **Global stability of the disease-free equilibrium.**

The following theorem discusses the global stability of the disease-free equilibrium.

Theorem 3.2. *The disease-free equilibrium* E_f *of the system (1.2) is globally asymptotically stable whenever* $R_0 \leq 1$, *and unstable otherwise.*

Proof. Consider the following Lyapunov functional

$$V(t) = S(t) - S_0 - \int_{S_0}^{S(t)} \frac{f(S_0, 0)}{f(X, 0)} dX + e^{\mu\tau} I(t) + \int_{t-\tau}^{t} f\big(S(\theta), I(\theta)\big) I(\theta) d\theta,$$

where $S_0 = \dfrac{\Lambda}{\mu}$. To simplify the presentation, we shall use the following notation: $X = X(t)$ and $X_\tau = X(t-\tau)$ for any $X \in \{S, I\}$. Calculating the time derivative of V along the positive solution of system (1.2), we get

$$\begin{aligned}
\dot{V}(t)|_{(1.2)} &= (1 - \frac{f(S_0,0)}{f(S,0)})\dot{S} + e^{\mu\tau}\dot{I} + f\big(S,I\big)I - f\big(S_\tau, I_\tau\big)I_\tau \\
&= (1 - \frac{f(S_0,0)}{f(S,0)})(\Lambda - \mu S) + \frac{f(S_0,0)}{f(S,0)} f(S,I)I - aIe^{\mu\tau} \\
&= \mu S_0\big(1 - \frac{S}{S_0}\big)\big(1 - \frac{f(S_0,0)}{f(S,0)}\big) + ae^{\mu\tau}I(\frac{f(S,I)}{f(S,0)}R_0 - 1) \\
&\leq \mu S_0(1 - \frac{S}{S_0})(1 - \frac{f(S_0,0)}{f(S,0)}) + ae^{\mu\tau}I(R_0 - 1).
\end{aligned}$$

Using the following trivial inequalities

$$1 - \frac{f(S_0,0)}{f(S,0)} \geq 0 \quad for \quad S \geq S_0,$$

$$1 - \frac{f(S_0,0)}{f(S,0)} < 0 \quad for \quad S < S_0.$$

Thus, we have

$$(1 - \frac{S}{S_0})(1 - \frac{f(S_0,0)}{f(S,0)}) \leq 0.$$

Since $R_0 \leq 1$, we have $\dot{V}|_{(1.2)} \leq 0$. Thus, the disease-free equilibrium E_f is stable, and $\dot{V}|_{(1.2)} = 0$ if and only if $S = S_0$ and $I(R_0 - 1) = 0$. We discuss two cases:

- If $R_0 < 1$, then $I = 0$.
- If $R_0 = 1$. From $S = S_0$ and the first equation of (1.2), we have

  $$\frac{dS}{dt} = \frac{dS_0}{dt} = \Lambda - \mu S_0 - f(S_0, I)I = 0.$$

  Then, $f(S_0, I)I = 0$. Since $S_0 > 0$, then $f(S_0, I) > f(0, I) = 0$ (use (H_1) and (H_2)). Hence, $I = 0$.

By the above discussion, we deduce that the largest compact invariant set in $\Gamma = \{(S, I) | \dot{V} = 0\}$ is just the singleton E_f. From LaSalle invariance principle [16], we conclude that E_f is globally asymptotically stable.

On the other hand, the characteristic equation at the disease-free equilibrium E_f is given by

$$(\xi + \mu)[\xi + a(1 - R_0 e^{-\xi\tau})] = 0. \tag{3.5}$$

Obviously, $\xi = -\mu$ is eigenvalue for (3.5), and hence, the stability of E_f is determined by the distribution of the roots of equation

$$\xi + a(1 - R_0 e^{-\xi\tau}) = 0. \tag{3.6}$$

It is easy to show that (3.6) has a real positive root when $R_0 > 1$. Indeed, we put

$$\varphi(\xi) = \xi + a(1 - R_0 e^{-\xi\tau}).$$

We have that $\varphi(0) = a(1 - R_0) < 0$, $\lim_{\xi\to+\infty}\varphi(\xi) = +\infty$ and the function φ is continuous on interval $[0, +\infty[$. Consequently, φ has a positive real root and the disease-free equilibrium is unstable when $R_0 > 1$. This proves the theorem. ■

4. Global stability of the endemic equilibrium

Note that the disease-free equilibrium E_f is unstable when $R_0 > 1$. Now, we establish a set of conditions which are sufficient for the global stability of the endemic equilibrium E^*.

For the global stability of E^*, we assume that $R_0 > 1$ and the function f satisfies the following condition:

$$(1 - \frac{f(S,I)}{f(S,I^*)})(\frac{f(S,I^*)}{f(S,I)} - \frac{I}{I^*}) \le 0, \ for\ all\ S,\ I > 0. \tag{4.1}$$

Theorem 4.1. *Assume $R_0 > 1$ and (4.1) hold. Then the endemic equilibrium E^* of the system (1.2) is globally asymptotically stable.*

Proof. Consider the following Lyapunov functional

$$\begin{aligned} W(t) &= S(t) - S^* - \int_{S^*}^{S(t)} \frac{f(S^*,I^*)}{f(X,I^*)}dX + e^{\mu\tau}I^*\phi(\frac{I(t)}{I^*}) \\ &\quad + f(S^*,I^*)I^*\int_{t-\tau}^{t}\phi(\frac{f\big(S(\theta),I(\theta)\big)I(\theta)}{f(S^*,I^*)I^*})d\theta, \end{aligned}$$

where $\phi(x) = x - 1 - \ln x$, $x \in \mathbf{R}^+$. Obviously, $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ attains its global minimum at $x = 1$ and $\phi(1) = 0$.

The function $\psi : x \mapsto x - x^* - \int_{x^*}^{x}\frac{f(S^*,I^*)}{f(X,I^*)}dX$ has the global minimum at $x = x^*$ and $\psi(x^*) = 0$. Then, $\psi(x) \ge 0$ for any $x > 0$.

Hence, $W(t) \ge 0$ with equality holding if and only if $\frac{S(t)}{S^*} = \frac{I(t)}{I^*} = 1$ for all $t \ge 0$.

Finding the time derivative of $W(t)$ along the positive solution of system (1.2) gives

$$\begin{aligned} \dot{W}(t)|_{(1.2)} &= (1 - \frac{f(S^*,I^*)}{f(S,I^*)})\dot{S} + e^{\mu\tau}(1 - \frac{I^*}{I})\dot{I} \\ &\quad + f(S^*,I^*)I^*\big(\phi(\frac{f\big(S,I\big)I}{f(S^*,I^*)I^*}) - \phi(\frac{f\big(S_\tau,I_\tau\big)I_\tau}{f(S^*,I^*)I^*})\big). \end{aligned}$$

Note that $\Lambda = \mu S^* + aI^*e^{\mu\tau}$ and $f(S^*,I^*) = ae^{\mu\tau}I^*$.

Hence,

$\dot{W}(t)|_{(1.2)} =$

$$\begin{aligned} &= \mu S^*(1 - \frac{S}{S^*})(1 - \frac{f(S^*,I^*)}{f(S,I^*)}) + ae^{\mu\tau}I^*\big(1 - \frac{f(S^*,I^*)}{f(S,I^*)} + \frac{I}{I^*}\frac{f(S,I)}{f(S,I^*)}\big) \\ &\quad + ae^{\mu\tau}I^*\big(1 - \frac{I^*}{I} - \frac{f\big(S_\tau,I_\tau\big)I_\tau}{f(S^*,I^*)I}\big) + f(S^*,I^*)I^*\ln\big(\frac{f\big(S_\tau,I_\tau\big)I_\tau}{f(S,I)I}\big) \\ &= \mu S^*(1 - \frac{S}{S^*})(1 - \frac{f(S^*,I^*)}{f(S,I^*)}) + ae^{\mu\tau}I^*\big(3 - \frac{f(S^*,I^*)}{f(S,I^*)} - \frac{f(S,I)}{f(S^*,I^*)} - \frac{f(S,I^*)}{f(S,I)}\big) \end{aligned}$$

$$
\begin{aligned}
&+ae^{\mu\tau}I^*\big(-1-\frac{I}{I^*}+\frac{f(S,I^*)}{f(S,I)}+\frac{I}{I^*}\frac{f(S,I)}{f(S,I^*)}\big)\\
&+ae^{\mu\tau}I^*\big(\frac{f(S,I)}{f(S^*,I^*)}-\frac{f(S_\tau,I_\tau)I_\tau}{f(S^*,I^*)I}+\ln\big(\frac{f(S_\tau,I_\tau)I_\tau}{f(S,I)I}\big)\big)\\
=\;&\mu S^*(1-\frac{S}{S^*})(1-\frac{f(S^*,I^*)}{f(S,I^*)})+ae^{\mu\tau}I^*\big(-1-\frac{I}{I^*}+\frac{f(S,I^*)}{f(S,I)}+\frac{I}{I^*}\frac{f(S,I)}{f(S,I^*)}\big)\\
&-ae^{\mu\tau}I^*\big[\frac{f(S^*,I^*)}{f(S,I^*)}+\frac{f(S,I)}{f(S^*,I^*)}+\frac{f(S,I^*)}{f(S,I)}-3-\frac{f(S,I)}{f(S^*,I^*)}\\
&+\frac{f(S_\tau,I_\tau)I_\tau}{f(S^*,I^*)I}-\ln\big(\frac{f(S_\tau,I_\tau)I_\tau}{f(S,I)I}\big)\big]\\
=\;&\mu S^*(1-\frac{S}{S^*})(1-\frac{f(S^*,I^*)}{f(S,I^*)})+ae^{\mu\tau}I^*\big(-1-\frac{I}{I^*}+\frac{f(S,I^*)}{f(S,I)}+\frac{I}{I^*}\frac{f(S,I)}{f(S,I^*)}\big)\\
&-ae^{\mu\tau}I^*\big[\phi\big(\frac{f(S^*,I^*)}{f(S,I^*)}\big)+\phi\big(\frac{f(S,I^*)}{f(S,I)}\big)+\phi\big(\frac{f(S_\tau,I_\tau)I_\tau}{f(S^*,I^*)I}\big)\big].
\end{aligned}
$$

Using the following trivial inequalities

$$1-\frac{f(S^*,I^*)}{f(S,I^*)}\geq 0 \quad for \quad S\geq S^*,$$

$$1-\frac{f(S^*,I^*)}{f(S,I^*)}< 0 \quad for \quad S< S^*.$$

Thus, we have

$$(1-\frac{S}{S^*})(1-\frac{f(S^*,I^*)}{f(S,I^*)})\leq 0.$$

From (4.1) we have

$$-1-\frac{I}{I^*}+\frac{f(S,I^*)}{f(S,I)}+\frac{I}{I^*}\frac{f(S,I)}{f(S,I^*)}=(1-\frac{f(S,I)}{f(S,I^*)})(\frac{f(S,I^*)}{f(S,I)}-\frac{I}{I^*})\leq 0.$$

Since $\phi(x)\geq 0$ for $x>0$, we have $\dot{W}|_{(1.2)}\leq 0$. Thus, E^* is stable, and $\dot{W}|_{(1.1)}=0$ if and only if $S=S^*$ and $I=I^*$. So, the largest compact invariant set in $\Gamma=\{(S,I)|\dot{W}=0\}$ is the singleton E^*. From LaSalle invariance principle [16], we conclude that E^* is globally asymptotically stable. ■

5. Conclusion

We have analytically studied a delayed model with a generalized incidence rate. By constructing two suitable Lyapunov functionals, we found the sufficient conditions of the global stability for the endemic and disease-free equilibrium of the model. When $R_0(\tau)\leq 1$, the disease-free steady state is globally asymptotically stable, and no other equilibria exist. When $R_0(\tau)>1$, the disease free steady state loses its stability, and a unique endemic equilibrium E^* appears. Using Lyapunov functional technique, we have been able to show that under certain restrictions on the parameter values, the endemic equilibrium is globally asymptotically stable. Our results show that, the time delay can affect the global stability if it becomes large enough to change the sign of $(R_0(\tau)-1)$.

On the other hand, the basic reproduction number $R_0(\tau)$ is a decreasing function of the delay. Hence, the delay prevents the disease by reducing the value of $R_0(\tau)$ to a level lower than one. Moreover, ignoring the delay in an epidemiological model will overestimate $R_0(\tau)$.

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