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Dirichlet boundary value problem for Duffing's equation

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Abstract

We use a direct variational method in order to investigate the dependence on parameter for the solution for a Duffing type equation with Dirichlet boundary value conditions.

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1 Introduction

Recently the classical variational problem for a Duffing type equation received again some attention. In [1, 2, 4, 8] some variational approaches were used in order to receive the existence of solutions for both periodic and Dirichlet type boundary value problems. Mainly direct method is applied under various conditions pertaining to at most quadratic growth imposed on the nonlinear term given in [1] and further relaxed in [8]. Dirichlet problems for such equations could also be considered by some other methods, for example Min-max Theorem due to Manashevich, [5]. In [6] the author gives some historical results concerning the Dirichlet problem for Duffing type equations and discusses the methods which are used in reaching the existence results which are different form the ones which we use and comprise the classical variational approach, the topological method. In most sources cited the authors assume the friction term $r \in C^1(0,1)$; $r(\tau) \geq 0$ for $\tau \in [0,1]$, and they require some further conditions on r. Mainly a type of monotonicity of r is assumed or else

$$\frac{1}{4}r^2(t) + \frac{1}{2}\frac{d}{dt}r(t) > 0$$

for all $t \in [0,1]$, see [4]. The standard procedure to treat Duffing's Equation with control function $u \in \mathrm{L}^\infty(0,1)$ and G satisfying some suitable assumptions:

$$\begin{gathered} \tfrac{d^2}{dt^2}x(t) + r(t)\tfrac{d}{dt}x(t) + G(t, x(t), u(t)) = 0 \\ x(0) = x(1) = 0 \end{gathered} \tag{ClassicDEq}$$

is as follows, see [8]. Denote $R(t) = e^{\int_0^t \frac{1}{2}r(\tau)d\tau}$. Since $r(\tau) \geq 0$ on $[0,1]$ we see that

$$R_{\max} = e^{\max\limits_{\tau\in[0,1]} r(\tau)} \geq R(t) \geq R(0) = 1.$$

Upon putting $y = R(t)x$ boundary problem (ClassicDEq) reads

$$\begin{gathered} -\tfrac{d^2}{dt^2}y(t) + \tilde{r}(t)y(t) = R(t)G(t, \tfrac{y(t)}{R(t)}, u(t)) \\ y(0) = y(1) = 0. \end{gathered}$$

In this paper we are concerned with the variational formulation for the Duffing Equation but we apply the different schema. Namely we consider (DEq)

$$\begin{gathered} \tfrac{d^2}{dt^2}x(t) + r(t)\tfrac{d}{dt}x(t) - F_x(t, x(t)) - f(t) = 0 \\ x(0) = x(1) = 0 \end{gathered} \tag{DEq}$$

under the assumptions that $r \in \mathrm{L}^\infty(0,1)$ and $f \in \mathrm{L}^1(0,1)$. Solutions to above are investigated in $\mathrm{H}_0^1(0,1)$ and these are the weak solutions. We shall show that by the Fundamental Lemma of the Calculus of Variations, any weak solutions to (DEq) is classical one, i.e.

$$x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1).$$

The equation (DEq) is not in a variational form i.e. there is no suitable functional J for which (DEq) corresponds to its critical points. Then by putting $h = \frac{dx}{dt}$, we may consider the following auxiliary problem

$$\frac{d^2}{dt^2}x(t) + r(t)h(t) - F_x(t, x(t)) - f(t) = 0. \tag{AuxEq}$$

We see that weak sense solutions to above are the critical points to J given by following integral

$$J(x) = \int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + [f(t) - r(t)h(t)]\, x(t) + F(t,x)dt. \tag{Func}$$

Under the assumptions that $h \in \mathrm{L}^\infty(0,1)$, $r \in \mathrm{L}^\infty(0,1)$, and some growth requirements on F we can prove that problem (AuxEq) has at least one solution. To prove this, its sufficient to show that

1. functional J is differentiable in sense of Gâteaux
2. functional J is coercive
3. functional J is weakly lower semi continuous

When solutions to (AuxEq) are obtained for any $h \in \mathrm{L}^\infty(0,1)$, we will apply the iterative procedure assuming that

$$\int_0^1 |F_x(t, x(t)) - F_x(t, y(t))|\, dt \le L \|x - y\|_{\mathrm{H}_0^1(0,1)}$$

for any x,$y \in \mathrm{H}_0^1(0,1)$, $L < 1$ independent of x,y and $\frac{\|r\|}{1-L} < 1$. This will provide solutions to (DEq).

Moreover functions $F, F_x : [0,1] \times \mathbb{R} \to \mathbb{R}$ will be a Caratheodory functions, satisfying the conditions below:

$$\begin{gathered} F(\cdot, 0) \in \mathrm{L}^1(0,1) \\ \forall_{d>0}\, \forall_{x\in[-d,d]}\, \exists_{f_d \in \mathrm{L}^1(0,1)}, \quad |F_x(t,x)| \le f_d(t). \end{gathered} \tag{H1}$$

When compared with existing results, see [8], our approach allows for a more general friction term, i.e. it belongs to L^∞. However in [8] where the friction is continuous the Author obtains results when $L = 1$ which is not possible with our approach.

1.1 Preliminaries

The following two remarks will be essential for our argument.

Remark 1.1. *Let $1 \le p < q, x \in \mathrm{L}^q(0,1), f \in \mathrm{L}^{\frac{q}{q-p}}(0,1)$. Then*

$$\int_0^1 |x(t)|^p |f(t)| dt \le \|x\|^p_{\mathrm{L}^q(0,1)} \cdot \|f\|_{\mathrm{L}^{\frac{q}{q-p}}(0,1)}. \tag{1.1}$$

Remark 1.2. *Let $1 \le p < q$ and $x \in \mathrm{L}^q(0,1)$. Then*

$$\|x\|_{\mathrm{L}^p(0,1)} \le \|x\|_{\mathrm{L}^q(0,1)}. \tag{1.2}$$

We shall also require Poincarè inequality in following form:

Lemma 1.3. ***Poincarè inequality****[3, prop. 8.13, p.218]*
Let $x \in \mathrm{H}^1_0(0,1)$

$$\|x\|_{\mathrm{L}^2(0,1)} \le \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}.$$

Since this formulation of Poincarè inequality does not match exactly version presented in [3], we shall present proof for this fact.
Proof.
Since $x \in \mathrm{H}^1_0(0,1)$, then $x(0) = 0$ then

$$|x(t)| = |x(t) - x(0)| = \left| \int_0^t \frac{dx}{dt} dt \right| \le \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^1(0,1)} \overset{(1.2)}{\le} \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}.$$

This leads us to very important estimation, that we will use a lot in this paper:

$$\|x\|_{\mathrm{L}^\infty(0,1)} \le \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}. \tag{1.3}$$

From (1.2) we have that for 2 and $n > 2$ the following holds

$$\|x\|_{\mathrm{L}^2(0,1)} \le \|x\|_{\mathrm{L}^n(0,1)}.$$

By taking $n \to \infty$ and using known property that $\|x\|_{\mathrm{L}^\infty(0,1)} = \lim_{p\to\infty} \|x\|_{\mathrm{L}^p(\Omega)}$, we obtain

$$\|x\|_{\mathrm{L}^2(0,1)} \le \|x\|_{\mathrm{L}^\infty(0,1)} \overset{(1.3)}{\le} \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}.$$

□

Since Poincarè inequality holds we shall use the following norm in $H_0^1(0,1)$ space

$$\|x\|^2_{H_0^1(0,1)} := \int_0^1 \left(\frac{dx}{dt}(t)\right)^2 dt.$$

2 Variational framework

We shall prove that solving (AuxEq) is equivalent with solving critical points problem for following functional J defined at $H_0^1(0,1)$.

We shall consider functional described by formula:

$$J(x) = \int_0^1 [\frac{1}{2}(\frac{dx}{dt})^2 + [f(t) - r(t)h(t)]x(t) + F(t,x)]dt \tag{Func}$$

for each $x \in H_0^1(0,1)$. We shall consider two versions of assumptions that will provide different results.

1. Convex version
$$x \to F(t,x) \quad \text{for a.e. } t \in [0,1] \tag{H2}$$
is convex
2. Bounded version. There exists constants $A \in \mathbb{R} \setminus \{0\}, B, C \in \mathbb{R}$ such that
$$F(t,x) \geq A|x|^2 + B|x| + C \tag{H3}$$
for all $x \in \mathbb{R}$, almost everywhere $t \in [0,1]$.

We shall prove that this functional is well defined, is Gâteaux differentiable and it's critical points are the weak solutions to (AuxEq). We will also prove that the regularity class of this solution is higher than $H_0^1(0,1)$.

We would like to compute Gâteaux derivatives, but first we have to ensure that we can differentiate under integration sign. We see the following properties:

Lemma 2.1. *Under assumption* (H1) *the following equality holds for any* $x \in H_0^1(0,1)$ *and* $g \in H_0^1(0,1)$

$$\lim_{h\to 0} \int_0^1 \frac{F(t,x+hg) - F(t,x)}{h} dt = \int_0^1 \lim_{h\to 0} \frac{F(t,x+hg) - F(t,x)}{h} dt. \tag{2.1}$$

In order to prove this, Dominated convergence Theorem is applied.

Lemma 2.2. *Functional* (Func)*,* $J : H_0^1(0,1) \to \mathbb{R}$ *is well defined under assumptions* (H1)*.Also the functional* (Func) *is differentiable in sense of Gâteaux and its derivative is equal to*

$$\delta J(x,g) = \int_0^1 \frac{dx}{dt}\frac{dg}{dt} + [f(t) - r(t)h(t) + F_x(t,x)]\, g(t)dt \tag{GD}$$

for each $g \in H_0^1(0,1)$*.*

Sketch of the proof.
We see that

$$\int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + (r(t)h(t) - f(t))x(t)dt$$

is well defined.

By (H1) for any x we see

$$\int_0^1 |F(t,x)|dt \leq \int_0^1 |F(t,0)|dt + \int_0^1 f_d(t)|x(t)|dt \leq$$

$$\leq \|F(\cdot,0)\|_{\mathrm{L}^1(0,1)} + \|f_d\|_{\mathrm{L}^1(0,1)} \cdot \|x\|_{\mathrm{H}_0^1(0,1)} < +\infty$$

The differentiability is a consequence of (2.1). □

Definition 2.3. *Every $x \in \mathrm{H}_0^1(0,1)$ for which that satisfies the following equality*

$$\forall g \in \mathrm{H}_0^1(0,1) \quad \delta J(x,g) = 0 \tag{WS}$$

shall be called a weak solution.

We shall now prove that weak solution (WS) for functional (Func) is a classical solution. Then we shall see that functional critical points to J are the weak solutions to (AuxEq)

Lemma 2.4. ***du Bois-Raymond Lemma****[7, p 31, sec 1.3, Lemma 1.1]*
Let $v \in \mathrm{L}^2(I,\mathbb{R}), I = [0,1]$, $w \in \mathrm{L}^1(I,\mathbb{R})$ be such functions that

$$\int_I v(x)h'(x)dx = -\int_I w(x)h(x)dx$$

for any $h \in \mathrm{H}_0^1(I)$. Then there exists constant $c \in \mathbb{R}$, such that

$$v(x) = \int_0^x w(s)ds + c$$

for almost every $x \in I$.

Lemma 2.5. *Let x be a solution to* (WS)*. If* (H1) *is satisfied, then this solution is classical solution to* (AuxEq)*.*

Proof.
Since in Theorem 2.2 we have proved that $f - r \cdot h + F_x(\cdot,x)$ is integrable. Applying du Bois - Raymond Lemma for $v = \frac{dx}{dt}$ and $w = f - r \cdot h + F_x(\cdot,x)$. Then the solution of $\delta J(x,g) = 0, g \in \mathrm{H}_0^1(0,1)$ is of a class $\mathrm{W}^{2,1}(0,1)$ and thus is a classical one. □

3 The existence of a solution

In this section we shall prove the existence of solution to (AuxEq).

Lemma 3.1. *The functional J given by formula* (Func) *is weakly lower semicontinious under* (H1).

Proof.
By norm continuity, the first part of (Func) is weakly lower semicontinuous.

$$\int_0^1 (f(t) - r(t)h(t))(\cdot)(t)dt$$

is linear and continuous thus it is w.l.s.c. For F function we need to apply some additional theory in order to prove its w.l.s.c.

Lets consider weakly converged sequence in $\mathrm{H}_0^1(0,1)$, $x_n \rightharpoonup x_0$. By Arzela-Ascoli Theorem there exists such subsequence that converge uniformly in $\mathrm{C}(0,1)$. Then for sufficiently large d, the below condition holds:

$$\max_{t\in(0,1)} |x_n(t)| \leq d$$

for sufficiently large n. By Lebesgue's dominated convergence Theorem we obtain:

$$\int_0^1 F(t, x_n)dt \to \int_0^1 F(t, x_0)dt, \quad n \to \infty$$

Then it is proved that functional (Func) is w.l.s.c. □

Lemma 3.2. *Functional J* (Func) *is coercive if* (H1) *and one of the assumptions holds*

1. *F satisfies* (H2)
2. *F satisfies* (H3) *with A satisfying: $|A| < \frac{1}{2}$.*

Proof.
First we observe that

$$\int_0^1 \frac{1}{2}(\frac{dx}{dt})^2 dt = \frac{1}{2}\|x\|^2_{\mathrm{H}_0^1(0,1)}. \tag{3.1}$$

We see that:

$$\int_0^1 (f(t) - r(t)h(t))x(t)dt \geq -\|f - r\cdot h\|_{\mathrm{L}^1(0,1)}\|x\|_{\mathrm{H}_0^1(0,1)}. \tag{3.2}$$

If F satisfies (H2) we obtain the following

$$\int_0^1 F(t,x)dt \geq \int_0^1 F(t,0)dt + F_x(t,0)xdt \geq$$
$$\geq \|F(\cdot,0)\|_{\mathrm{L}^1(0,1)} - \|F_x(\cdot,0)\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{H}_0^1(0,1)}. \tag{3.3}$$

Which proves the lemma in first case. If the other condition (H3) holds so there exists such $A \in \mathbb{R} \setminus \{0\}, B, C \in \mathbb{R}$ for which the following holds

$$F(t,x) \geq A|x|^2 + B|x| + C \geq A|x|^2 - |B||x| - |C|. \tag{3.4}$$

Integrating the both sides of (3.4) we get

$$\int_0^1 F(t,x(t))dt \geq \int_0^1 A|x(t)|^2 - |B||x(t)| - |C|dt \geq A\|x\|^2_{\mathrm{L}^2(0,1)} - |B|\|x\|_{\mathrm{L}^2(0,1)} - |C|.$$

We should consider two cases:

1. If sequence of norms x_n diverges in $\mathrm{H}_0^1(0,1)$ it may still converge in $\mathrm{L}^2(0,1)$. In such case
$$\|x_n\|_{\mathrm{L}^2(0,1)} \leq \|x_n\|_{\mathrm{H}_0^1(0,1)}.$$
2. In opposite case, the same inequality holds since Poincarè inequality is applicable.

Thus

$$\int_0^1 F(t,x) \geq -|A|\|x\|^2_{\mathrm{H}_0^1(0,1)} - (|B| + |C|)\|x\|_{\mathrm{H}_0^1(0,1)}$$

Then functional (Func) is coercive since $|A| < \frac{1}{2}$ for unbounded case. Together with bonded case this proves lemma in second case.

□

Above means that we have 2 cases in which we proved coerciveness.

1. F is convex
2. F is bounded from below with $|A| < \frac{1}{2}$

The theorem below proves the existence of solution.

Theorem 3.3. *Let E be reflexive Banach space and let the functional $f : E \to \mathbb{R}$ be w.l.s.c. and coercive. Then there exist a function that minimizes f.[7]*

Then we have the following

Theorem 3.4. *There exists at least one solution to* (AuxEq) *if* (H1) *is satisfied and one of the following holds:*

1. *F satisfies* (H2)
2. *F satisfies* (H3) *with A satisfying $|A| < \frac{1}{2}$.*

Proof.
By Lemmas 3.1 and 3.2, and reflexiveness of $H_0^1(0,1)$ we see that assumptions of Theorem 3.3 are satisfied. Then there exists solutions in functional critical points problem. By Lemma 2.5 this solution is a classical solution to (AuxEq).
□

4 Iterative scheme framework

In this section we shall prove that using equation (AuxEq) we may provide the solution of (DEq).

Theorem 4.1. *If* (H1) *is satisfied and if one of below conditions holds*

1. *F is convex* (H2)
2. *F is bounded* (H3) *and $|A| < \frac{1}{2}$.*

and moreover

$$\int_0^1 |F_x(t,x(t)) - F_x(t,y(t))|\, dt \le L\, \|x-y\|_{H_0^1(0,1)} \tag{4.1}$$

for any x,y and $L < 1$ independent of x,y. Then if $\frac{\|r\|_{L^\infty(0,1)}}{1-L} < 1$ then problem (DEq) *has at least one solution*

Proof.
Let h be an arbitrary taken function $h \in H^1(0,1)$. Lets then define a sequence $(x_n) \subset H_0^1(0,1) \cap W^{2,1}(0,1)$, $n \in \mathbb{N}$. We consider following formula

$$\begin{cases} \frac{d^2}{dt^2}x_n + r\frac{d}{dt}x_{n-1} - F_x - f = 0 \quad ,n \in \mathbb{N} \\ x_0 := h \in H^1(0,1). \end{cases} \tag{4.2}$$

By using Theorem 3.4 and induction with respect to n, it is easy to prove that such sequence is well defined.

We shall prove that (x_n) is Cauchy sequence in $H_0^1(0,1)$ with respect to norm. Since the solution in understood in weak sense, we do the following. Let $n, m \in \mathbb{N}$. Then the (4.2) for n and m is multiplied by $(x_n - x_m)$ and then integrated with respect to $t \in [0,1]$.

$$-\int_0^1 \frac{d^2x_n}{dt^2}(x_n - x_m)dt = \int_0^1 \left(r\frac{dx_{n-1}}{dt} - F_x(t,x_n) - f\right)(x_n - x_m)dt$$

$$-\int_0^1 \frac{d^2x_m}{dt^2}(x_n - x_m)dt = \int_0^1 \left(r\frac{dx_{m-1}}{dt} - F_x(t,x_m) - f\right)(x_n - x_m)dt$$

After subtracting the sides and integrating by parts

$$\|x_n - x_m\|^2_{H_0^1(0,1)} = \int_0^1 \left(r\frac{dx_{n-1}}{dt} - F_x(t,x_n) - f(t)\right)(x_n - x_m)dt + \\ -\int_0^1 \left(r\frac{dx_{m-1}}{dt} - F_x(t,x_m) - f(t)\right)(x_n - x_m)dt.$$

Thus by (1.3) for $0 \neq x_n - x_m \in \mathrm{H}_0^1(0,1)$, we have that

$$\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)} \leq \int_0^1 \left| r\frac{dx_{n-1}}{dt} - F_x(t,x_n) - f(t) - r\frac{dx_{m-1}}{dt} + F_x(t,x_m) + f(t) \right| dt.$$

By (4.1) we have that

$$\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)} \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|x_{n-1} - x_{m-1}\|_{\mathrm{H}_0^1(0,1)} + L\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)}.$$

Thus we have that:

$$\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)} \leq \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} \|x_{n-1} - x_{m-1}\|_{\mathrm{H}_0^1(0,1)}.$$

Since $\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} < 1$ we have that (x_n) is Cauchy sequence with respect to $\mathrm{H}_0^1(0,1)$ norm. □

5 Example

Example 5.1. *The above schema can be applied for the following equation*

$$\frac{d^2x}{dt^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}} \frac{dx}{dt}(t) - \frac{1}{2}e^{-t}x(t) = t + 1.$$

Indeed, (H1) *is confirmed since*

$$F(\cdot, x) := \frac{1}{2}e^{-(\cdot)}x^2 \in \mathrm{L}^1(0,1)$$

and for any $d > 0$ and $x \in [-d,d]$ we have that

$$F_x(t,x) = \frac{1}{2}e^{-t}x \leq \frac{1}{2}e^{-t}d \in \mathrm{L}^1(0,1)$$

Also (H2) *is satisfied since $F(\cdot,x) := \frac{1}{2}e^{-(\cdot)}x^2$ is convex with respect to its second variable.*

We can observe for F_x that

$$|F_x(t,x) - F_x(t,y)| = |\frac{1}{2}e^{-t}(x-y)|$$

After integrating sides with respect to $t \in [0,1]$, and knowing that $(x-y) \leq \|x-y\|_{\mathrm{L}^\infty(0,1)} \leq \|x-y\|_{\mathrm{H}_0^1(0,1)}$ we obtain:

$$\int_0^1 |F_x(t,x) - F_x(t,y)|dt \leq \|x-y\|_{\mathrm{H}_0^1(0,1)} \int_0^1 \frac{1}{2}e^{-t}dt = \frac{e-1}{2e}\|x-y\|_{\mathrm{H}_0^1(0,1)}$$

which jointly implies that:

$$\int_0^1 |F_x(t,x) - F_x(t,y)|dt \leq$$

$$\tfrac{e-1}{2e}\|x-y\|_{\mathrm{H}_0^1(0,1)} \leq 0.32\|x-y\|_{\mathrm{H}_0^1(0,1)}$$

with $L = 0.71 < 1$.

Since $\|r\|_{\mathrm{L}^\infty(0,1)} = \left\|0.25 \cdot e^{-\frac{t^2}{2}}\right\|_{\mathrm{L}^\infty(0,1)} = 0.25$ *and* $\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} = \frac{0.25}{1-0.32} < 0.37 < 1$ *then by Theorem 4.1 we conclude that problem* (5.1) *has at least one solution.*

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