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New periodic and quasi-periodic motions of a relativistic particle under a planar central force field with applications to scalar boundary periodic problems*

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Abstract

We consider a relativistic particle under the action of a time-periodic central force field in the plane. Assuming an attractive type condition on some neighborhood of infinity there are many subharmonic and quasiperiodic motions. Moreover, the obtained information allows to give applications for many scalar problems involving the relativistic operator.

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1 Introduction and main results

The motion of a particle subjected to the influence of an (autonomous) central force field in the plane may be mathematically modelled as a system of differential equations

$$\ddot{x} = f(|x|)\frac{x}{|x|} \qquad x \in \mathbb{R}^2 \setminus \{0\},$$

and it has had a great importance in Mechanics from the very beginning of this discipline in the seventeenth century. The central force field is determined by the function f and, from a physical point of view, we can have:

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1. **Attractive force fields:** f is called an attractive force field whenever $f(r) < 0$ for all $r > 0$. For instance, letting $f(r) = c/r^2$ for some negative constant $c < 0$, the gravitational force created by a point mass fixed at the origin is attractive. This is the case of the well-known Kepler problem.

2. **Repulsive force fields:** f is called a repulsive force field whenever $f(r) > 0$ for all $r > 0$. For example considering $f(r) = c/r^2$ for some positive constant $c > 0$ gives the Coulombian force field created by an electrical charged particle fixed at the origin which works with another one with the same charge.

3. **Mixed force fields:** In this case f is considered positive at some levels and negative in others. The central force field can model the force field created by a charged particle which changes the sign depending on the position with respect to the origin.

These force fields are autonomous, i.e., they depend on the position but not directly on the time. However, Newton [18], in his study about Kepler's 2nd law had already considered the motion of a particle subjected to a periodic sequence of discrete time impulses. On the other hand, many problems involve particles which vary depending on the time they have been considered.

For example, the Gylden-Meshcherskii problem

$$\ddot{x} = M(t)\frac{x}{|x|^3} \qquad x \in \mathbb{R}^2 \setminus \{0\},$$

even though it was originally proposed to explain the secular acceleration observed in the Moon's longitude, nowadays has many physical interpretations. Amongst others, it can be regarded as a Kepler problem with variable masses ($M(t) = -G \cdot m_1(t) \cdot m_2(t)$, G is the gravitational constant and $m_1(t), m_2(t)$ are the masses of the bodies) or a Coulombian problem with charged particles changing of sign ($M(t) = \kappa \cdot q_1(t) \cdot q_2(t)$, κ is the Coulomb constant and $q_1(t), q_2(t)$ are the charges of the particles). Mainly, the Gylden-Meshcherskii problem is used in the framework of the Kepler problem to describe a variety of phenomena including the evolution of binary stars, dynamics of particles around pulsating stars and many others (see [3, 10, 19, 20] and the references there); but also it could be used as a Coulombian problem to study phenomenons such as: the stabilization of matter-wave breather in Bose-Einstein condensates, the propagation of guided waves in planar optical fibers, the electromagnetic trapping of a neutral atom near a charged wire (see [7] for the scalar versions).

When dealing with particles moving at speed close to that of light it may be important to take into account the relativistic effects. Relativistic Dynamics is theoretically founded in the context of Special Relativity (see for instance [12, Chapter 33]), and the relativistic Kepler or Coulomb problem has been considered in previous works [1, 6, 17]. However, it seems that for the most general non-autonomous center field force is still very unexplored, in this line we can cite the recent paper [22]. When the mass of our particle at rest and the speed of light are normalized to one, we are led to consider the following family of second-order systems in the plane:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,|x|)\frac{x}{|x|}\,, \qquad\qquad x \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

Here, $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}$, $f = f(t,r)$ is assumed to be Caratheodory and T-periodic in the time variable t (i.e. $f(t,\cdot) : (0,\infty) \to \mathbb{R}$ is continuous for almost every $t \in \mathbb{R}$, $f(\cdot,r) : \mathbb{R} \to \mathbb{R}$ is $T-$periodic and medible for every $r > 0$ and for every fixed positive numbers a, b the function $\max_{r\in[a,b]} |f(t,r)|$ is locally integrable on $\mathbb{R}$). Notice however that it may be singular at $r = 0$. Solutions of (1) are understood in a classical sense, i.e., a C^1 function $x : \mathbb{R} \to \mathbb{R}^2$ is a solution provided that

$$x(t) \neq 0, \qquad |\dot{x}(t)| < 1\,, \qquad\qquad t \in \mathbb{R},$$

and the equality (1) holds on almost everywhere.

In this paper we are interested in finding a certain class of functions which include the $T-$periodic ones but also some subharmonic and quasi-periodic solutions. To introduce this class of functions it will be convenient to use the polar coordinates and rewrite in complex notation each continuous function $x : \mathbb{R} \to \mathbb{R}^2\setminus\{0\} \equiv \mathbb{C}\setminus\{0\}$ as $x(t) = r(t)e^{i\theta(t)}$, where $r(t) = |x(t)|$ and $\theta : \mathbb{R} \to \mathbb{R}$ is some continuous determination of the argument along of the function x. We say that x is $T-$radially periodic whenever $r(t)$ is $T-$periodic and there exists a real number ω such that $\theta(t) - \omega t$ is $T-$periodic. This number was introduced in [22] as the rotational (or angular velocity) of x and denoted as $\operatorname{rot}(x)$.

For instance, the T-radially periodic function $x : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ is T- periodic if and only if $\operatorname{rot} x$ is an integer multiple of $2\pi/T$. If $\operatorname{rot} x = (m/n)\,(2\pi/T)$ for some relatively prime integers $m \neq 0 \neq n$ then x will be subharmonic with minimal period nT.

Finally, if $\frac{\operatorname{rot} x}{2\pi/T}$ is irrational (and $|x|$ is not constant) then x will not be periodic of any period and instead it will be quasi-periodic with two frequencies $\omega_1 = \frac{2\pi}{T}$; $\omega_2 = \operatorname{rot} x$. This is easy to check, since $x(t) = r(t)e^{i\theta(t)}$ can be decomposed on the product of the T-periodic $r(t)e^{i(\theta(t)-\operatorname{rot}(x)t)}$ and the $2\pi/\operatorname{rot} x-$periodic $e^{i\operatorname{rot}(x)t}$.

Some illustrative examples in order to understand better the above concepts may be seen in [11] and [22], especially in this last one there is a graphic representation with the meaning of the concept (see [22, Figure 1]).

It is well-known that if our force field is globally repulsive, i.e.,

$$f(t,r) > 0 \qquad \text{for } (t,r) \in \mathbb{R} \times (0,\infty),$$

then (1) has no $T-$radially periodic solutions. This fact is easily checkable multiplying in (1) by x and integrating on $[0,T]$. However, when our force field is attractive at some level $r_* > 0$ and autonomous (it does not depend on t), i.e.,

$$f(r_*) < 0,$$

then an easy computation proves that there are $T-$radially periodic solutions of (1) with constant angular velocity equal to

$$|\omega| = \frac{\sqrt{2}}{r_* \sqrt{1 + \sqrt{1 + \left(\frac{2}{r_* f(r_*)}\right)^2}}}.$$

In addition to the above trivial results, recently we have proven that if we consider an attractive continuous (non-autonomous) force field at some level then there exists many infinitely $T-$radially periodic solutions of (1).

The above-mentioned results applied to the relativistic Gylden-Meshcherskii problem

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = M(t)\frac{x}{|x|^3}\,, \qquad x \in \mathbb{R}^2 \setminus \{0\}, \tag{2}$$

imply that M must be negative in all $\mathbb{R}$, which is too much restrictive for some type of physical problems above considered. The main objective in this paper is overcoming this restriction. For that we will prove the following statement.

Theorem 1 *Assume the existence of* $r_* > 0$ *such that*

$$\int_0^T \max_{r\in[\lambda,\lambda+T/2]} f(t,r)dt < 0 \qquad \forall \quad \lambda \geq r_*.$$

Then either

$$\left\{\min_{t\in\mathbb{R}} |x(t)| : x \textit{ is a } T-\textit{rad. periodic solution of } (1)\right\} = (0,\infty)$$

or there exist $T-$*rad. periodic solutions of* (1) *with angular velocity equal to 0.*

Now, according to Theorem 1 it can be proven that $\overline{M} := \int_0^T M(s)ds < 0$ implies the existence of $T-$rad. periodic solutions of (2). In fact, $\overline{M} < 0$ is a necessary and sufficient condition for the existence of non-constant $T-$rad. periodic solutions of (2).

Immediately the following questions arise from Theorem 1: what type of solutions are obtained in Theorem 1?, do $T-$periodic solutions of (1) exist? The following theorem gives such answers.

Theorem 2 *Under the assumption of Theorem 1 there exists* $\omega_* > 0$ *with the following property: for every* $\omega \in (-\omega_*, \omega_*) \setminus \{0\}$ *there is a* $T-$*rad. periodic solution* $x_\omega = x_\omega(t)$ *of* (1) *such that* $\mathrm{rot}(x_\omega) = \omega$.

In particular, taking $\omega = (2\pi)/(nT)$ for some natural number large enough we can find the existence of sub-harmonic solutions having as minimal period a multiple of T. On the other hand, putting $\omega = (2\pi/T)s$ for some irrational number s we obtain the existence of infinitely many quasi-periodic orbits of (1).

At this moment, we would like to analyze the advantages and disadvantages of our results with respect to the known ones in the actual literature, not only in the relativistic case but also in the classical one. The only analytical result known for us assuming relativistic effects is [22, Theorem 1.1]. Obviously, both results are independent and complementary. In both results, the singularity does not play any role, and the $T-$rad. periodic solutions are from the same nature, they rotate around the origin with very small angular velocity. Moreover, in both results an attractiveness condition is required on f. In the actual Theorems 1-2 we need something less than attractiveness on f, but, in contrast with the result proved in [22], such condition must be assumed not only at some fixed level but, at some neighborhood at infinity. However, the obtained information here is major and it will allow, since it will be seen, to get consequences even for scalar equations. With regard to the result in the classical case, we can find important differences. For example, on no account can be obtained analogous Theorems 1-2 for this case, because f must fulfil some sub-linear requirement in order to avoid the resonance phenomenon.

Finally, we point out that Theorems 1-2 also can become false if our particle is restricted to be on a line instead of on the plane. In this case problem (1) is reduced to

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \tag{3}$$

Now $T-$rad. periodic solutions of (1) are reduced to $T-$periodic solutions of (3) (understanding by $T-$periodic solution, in the scalar case, a positive $T-$periodic function r in C^1); for that, obviously, if our force field is globally attractive, by contradiction, integrating in (3) on the period of some possible $T-$periodic solution r, it is proven that (3) cannot have any $T-$periodic solution. This implies, according to Theorem 1, that under the global attractiveness of f, for every positive number r there exists a $T-$rad. periodic solution of (1) x_r such that $\min_{t\in\mathbb{R}} |x_r(t)| = r$. On the contrary, if one would know that there exists some level $r_0 > 0$ such that (1) has no any $T-$rad. periodic solutions then Theorem 1 provides the existence of $T-$periodic solutions of (3). On this idea is based our next theorem:

Theorem 3 *Under the assumption of Theorem 1. If there exists at level* $r_0 > 0$ *such that*

$$\int_0^T \min_{r\in[r_0, r_0+T/2]} f(t,r)dt > 0,$$

then (3) *has at least one (positive)* $T-$*periodic solution.*

Intuitively, in order to get some efficient conditions guaranteeing the existence of (positive) $T-$periodic solutions of (3), it is necessary that f is attractive in some neighborhood of infinity. Therefore, Theorem 3 can be applied to many equations, for example to

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = \frac{q_1(t)}{r^\gamma} - \frac{q_2(t)}{r^\delta} + e(t), \qquad r > 0,$$

where q_1, q_2 are non-negative locally integrable and periodic functions, e is only locally integrable and periodic and γ, δ are positive constants. This type of equations can be important from the physical point of view for the studying process as trapless 3D Bose-Einstein condensate taking into account relativistic effects (see for the classical case [16, Section 5]). In addition, we do not know any analytical result from literature on them. As a particular case it can be considered the classic equation of Lazer and Solimini (case $q_2 \equiv 0$) with a weak type singularity (i.e. $\gamma \in (0,1)$). In [4] it proved that, in this particular case, in addition of the necessary assumption $\int_0^T e(s)ds < 0$, more requirements must be assumed, but it was not possible to find them. Since it does not require any difficulty to apply Theorem 3 to study this type of equations, it will be convenient only to indicate that the above method works in order to avoid too many trivial arguments.

As a short and concrete application of Theorem 3, we will study the well-known Mathieu-Duffing equations with relativistic effects. The obtained results can be compared with [5], where we got independent conditions using a variational approach; but the solutions could be non-positive (for more results about Mathieu-Duffing equations in the classical case see [21] and the references therein).

Actually, it will be important to recall some basic concepts on the equation (1). Let $x = re^{i\theta(t)}$ be written in polar coordinates, it is a $T-$rad. periodic solution of (1) if and only if there exist a real number $\mu \in \mathbb{R}$ (the relativistic angular moment) and (r, p) a $T-$periodic solution of

$$\dot{r} = \frac{rp}{\sqrt{\mu^2 + r^2 + r^2p^2}}\,, \qquad \dot{p} = \frac{\mu^2}{r^2\sqrt{\mu^2 + r^2 + r^2p^2}} + f(t,r)\,, \tag{HS}$$

here $p = \dot{r}/\sqrt{1 - \dot{r}^2 - r^2\dot{\theta}^2}$ is called the relativistic linear momentum. Moreover, the rotational of any $T-$rad. periodic solution of (1) can be computed in terms of r, p and μ:

$$\operatorname{rot}(r, p; \mu) := \frac{\mu}{T}\int_0^T \frac{dt}{r(t)\sqrt{\mu^2 + r^2(t) + r^2(t)p^2(t)}}. \tag{4}$$

The above process is reversible, i.e., if $(r, p; \mu)$ is a $T-$periodic solution of (HS) then $x = re^{i\theta}$, where θ is any primitive of $\mu/(r(t)\sqrt{\mu^2 + r^2(t) + r^2(t)p^2(t)})$, is a $T-$rad. periodic solution of (1). Taking into account this property we will equivalently use the system (HS) in order to study the equation (1) (see [22] for more details).

The paper is structured as follows: In section 2 some a priori bounds for the $T-$periodic solutions of (HS) are obtained. In Section 3, by using the continuation arguments of Leray-Schauder degree, our main Theorems 1-2 will be proven. In the last section, Section 4, we will apply Theorem 1 to study the existence of periodic solutions in the scalar case. An important example will illustrate our results.

2 A priori bounds

It is well-known that the fundamental principle on which is based the Relativistic Mechanic is: the particles cannot travel faster than light (which in our model is assumed to be 1). This basic principle implies the existence of bounds on the variation of $T-$rad. periodic solutions of (1), both in the angular and the radial components. More precisely:

Lemma 1 *Let* $(r, p; \mu)$ *a* $T-$*periodic solution of (HS). Then,*

(a) $\max_{t\in\mathbb{R}} r(t) - \min_{t\in\mathbb{R}} r(t) < T/2$.

(b) $|\operatorname{rot}(r, p; \mu)| \leq \frac{1}{\min_{t\in\mathbb{R}} r(t)}$.

Proof. The first part of the proof is obtained using that the oscillation of any $T-$periodic and continuously differentiable function r is bounded by $\|\dot{r}\|_\infty T/2$, i.e., it fulfils $\max_{t\in\mathbb{R}} r(t) - \min_{t\in\mathbb{R}} r(t) \leq \|\dot{r}\|_\infty T/2$ (see [4, Lemma 6]). This elementary estimation will be frequently used in the presented paper. On the other hand, the definition (4) implies (b). ■

The main hypothesis of Theorems 1-2 is the attractiveness (in average) of the force field at someplace far from the origin, i.e., there exists $r_* > 0$ such that

$$\int_0^T \max_{r\in[\lambda,\lambda+T/2]} f(t, r)dt < 0 \qquad \forall \quad \lambda \geq r_*. \tag{5}$$

Under this assumption one easily checks that any $T-$periodic solution $(r, p; \mu)$ with $\min_{t\in\mathbb{R}} r(t) \geq r_*$ has (relativistic) angular momentum $\mu \neq 0$.

Lemma 2 *Assume* (5). *Then* $\operatorname{rot}(r, p; \mu) \neq 0$ *for any* $T-$*periodic solution* $(r, p; \mu)$ *of (HS) with* $\min_{t\in\mathbb{R}} r(t) \geq r_*$.

Proof. We use an argument by contradiction, we assume that there is $(r, p; \mu)$ a $T-$periodic solution of (HS) with $\min_{t\in\mathbb{R}} r(t) \geq r_*$ and $\mu = 0$ (in view of (4) it is equivalent to assume that $\operatorname{rot}(r, p; \mu) = 0$). In particular, from (HS) we deduce that the $T-$periodic function r fulfils the equation (3). By integrating on the period of r and using (5) we get a contradiction. ■

The next goal will be to find some a priori estimates on the (relativistic) linear and angular momentum (p and μ) for every $T-$periodic solution $(r, p; \mu)$ of (HS) contained on some annular region.

Lemma 3 *For every* $a > 0$ *there exist* $P > 0$ *and* $M > 0$ *(depending only on* a *and* f*) such that*

$$\|p\|_\infty \leq P, \qquad |\mu| < M,$$

for any $T-$*periodic solution* $(r, p; \mu)$ *of (HS) such that* $r(t) \in [a, a + T]$.

Proof. Let $(r, p; \mu)$ be any $T-$periodic solution of (HS) with $r(t) \in [a, a+T]$.

We define $t^* \in [0, T]$, $t_* \in [t^*, t^* + T]$ the points where p attains its global maximum and minimum respectively. Therefore, by integrating on $[t^*, t_*]$ the second equation of (HS) we can prove the first part of the statement, i.e.,

$$\max_{t\in\mathbb{R}} p(t) - \min_{t\in\mathbb{R}} p(t) \leq \int_0^T \max_{r\in[a,a+T]} |f(t,r)|dt =: P.$$

In order to proof the second part we integrate on the period of p the second equation of (HS) obtaining

$$\mu^2 = -\frac{1}{\int_0^T \frac{dt}{r^2(t)\sqrt{\mu^2+r^2(t)+r^2(t)p^2(t)}}} \int_0^T f(t, r(t))dt.$$

Assuming that $|\mu| \geq 1$ (on otherwise the proof is completed), the above identity allows to check

$$|\mu| < (1+a)^3 + \frac{(a+T)^2\sqrt{1+(a+T)^2+(a+T)^2P^2}}{T} \int_0^T \max_{r\in[a,a+T]} |f(t,r)|dt =: M.$$

■

Fixed any positive number a, according to Lemma 3 we can define $P(a) := \int_0^T \max_{r\in[a,a+T]} |f(t,r)|dt$ and the function

$$M(a) := (1+a)^3 + \frac{(a+T)^2\sqrt{1+(a+T)^2+(a+T)^2P^2(a)}}{T} \int_0^T \max_{r\in[a,a+T]} |f(t,r)|dt$$

in the way that $\|p\|_\infty \leq P(a)$ and $|\mu| < M(a)$ for any $T-$periodic solution $(r, p; \mu)$ of (HS) with $r(t) \in [a, a+T]$. Moreover, there exists $\lambda_2 > 0$ large enough such that

$$\frac{M(a)}{(a+T/2)^2\sqrt{2}} + \frac{1}{T}\int_0^T f(t, a+T/2)dt > 0 \qquad \forall \quad a \geq \lambda_2. \tag{6}$$

This inequality will be used in the next section, so that it will be convenient to recall it.

3 Continuation of solutions with nonzero degree

Theorems 1-2 were formulated with respect to system (1). However, in view of the equivalence between finding $T-$rad. periodic solutions of (1) and finding $T-$periodic solutions of (HS), we can equivalently state:

Theorem 1bis. Assume (5). Then, either

$$\left\{\min_{t\in\mathbb{R}} r(t) : (r, p; \mu) \text{ is a } T\text{−periodic solution of (HS)}\right\} = (0, \infty),$$

or there are $T-$periodic solutions $(r, p; \mu)$ of (HS) with $\mu = 0$.

Theorem 2bis. Under the assumption of Theorem 1bis., there exists $\omega_* > 0$ with the following property: for every $\omega \in (-\omega_*, \omega_*) \setminus \{0\}$ there is some $T-$periodic solution $(r, p; \mu)$ of (HS) with $\mathrm{rot}(r, p; \mu) = \omega$.

This section will be destined to prove these results. For that, it will be useful to consider the following change of variable:

$$r(t) = \lambda(1 + \widetilde{r}(t)), \quad p(t) = \lambda\widetilde{p}(t) \qquad (\lambda \in (0, \infty)). \tag{7}$$

Here $\widetilde{r}$, $\widetilde{p}$ belong to the Banach spaces $C_0(\mathbb{R}/T\mathbb{Z}) := \{\widetilde{r} \in C(\mathbb{R}/T\mathbb{Z}) : \widetilde{r}(0) = 0\}$, $C(\mathbb{R}/T\mathbb{Z})$ (is the set of the continuous $T-$periodic functions on $\mathbb{R}$) respectively. To simplify the notation, we will write $Y := C_0(\mathbb{R}/T\mathbb{Z}) \times C(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}$, the Banach space induced by the classical norm, i.e., if $y = (\widetilde{r}, \widetilde{p}; \mu) \in Y$ we will write $\|y\| = \|\widetilde{r}\|_\infty + \|\widetilde{p}\|_\infty + |\mu|$.

With regard to the new variables ($\widetilde{r}$, $\widetilde{p}$ and μ), (HS) can be rewritten as

$$\dot{\widetilde{r}} = N_1[\lambda, \widetilde{r}, \widetilde{p}; \mu]\,, \qquad \dot{\widetilde{p}} = N_2[\lambda, \widetilde{r}, \widetilde{p}; \mu]\,, \tag{$\widetilde{\mathrm{HS}}$}$$

where $N_i : \Omega \subset \mathbb{R} \times Y \to C(\mathbb{R}/T\mathbb{Z})$ $(i = 1, 2)$ are the Nemitskii operators defined by

$$N_1[\lambda, \widetilde{r}, \widetilde{p}; \mu] := \frac{\lambda(1 + \widetilde{r})\widetilde{p}}{\sqrt{\mu^2 + \lambda^2(1 + \widetilde{r})^2 + \lambda^4(1 + \widetilde{r})^2\widetilde{p}^2}},$$

$$N_2[\lambda, \widetilde{r}, \widetilde{p}; \mu] := \frac{\mu^2}{\lambda^3(1 + \widetilde{r})^2\sqrt{\mu^2 + \lambda^2(1 + \widetilde{r})^2 + \lambda^4(1 + \widetilde{r})^2\widetilde{p}^2}} + \frac{f(t, \lambda(1 + \widetilde{r}))}{\lambda},$$

and Ω is the natural open subset of $\mathbb{R} \times Y$ for which everything is well-defined, in this case $\Omega := \{(\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathbb{R} \times Y : \lambda > 0, \quad \min_{t \in \mathbb{R}} \widetilde{r}(t) > -1\}$.

We point out that whenever $(r, p; \mu)$ is a $T-$periodic solution of (HS) then, taking $\lambda = r(0)$ and defining $\widetilde{r}$, $\widetilde{p}$ by (7), $(\widetilde{r}, \widetilde{p}; \mu)$ will be a $T-$periodic solution of ($\widetilde{\mathrm{HS}}$); and vice versa, if there exist $\lambda > 0$ and $(\widetilde{r}, \widetilde{p}; \mu)$ a $T-$periodic solution of ($\widetilde{\mathrm{HS}}$) then $(r, p; \mu)$ defined by (7) will be a $T-$periodic solution of (HS) with $r(0) = \lambda$.

The key to prove Theorems 1bis-2bis will be the next result.

Proposition 1 *Assume* (5). *Then there exists a connected subset $\mathcal{C}$ of the $T-$periodic solutions of ($\widetilde{HS}$) verifying*

(i) $\{\lambda : (\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C})\} \supset [r_* + T/2, \infty)$

and one of the following conditions:

($\mathbf{i_a}$) $\{\lambda(1 + \min_{t\in\mathbb{R}} \widetilde{r}(t)) : (\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C}\} = (0, \infty)$

($\mathbf{i_b}$) $\mathcal{C} \cap [\mathbb{R} \times C_0(\mathbb{R}/T\mathbb{Z}) \times C(\mathbb{R}/T\mathbb{Z}) \times \{0\}] \neq \emptyset$.

The connection of $\mathcal{C}$ refers to the topology of $\mathbb{R} \times Y$.

We postpone the proof of Proposition 1 to the end of the section; at this moment let us see how it can be used in order to obtain Theorems 1bis-2bis.

Let $\mathcal{C}$ be the connected set given by Proposition 1. Notice that the set $\mathcal{C}_1 := \{(\lambda(1+\widetilde{r}), \lambda\widetilde{p}; \mu) : (\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C}\}$ is a subset of the $T-$periodic solutions of (HS) when $\lambda > 0$, because of the change of variable done in (7). Moreover, $\mathcal{C}_1$ is a connected subset on $C(\mathbb{R}/T\mathbb{Z}) \times C(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}$.

Proof of Theorem 1bis. The proof is concluded rewriting Proposition 1 using the change of variable (7) and the connected set $\mathcal{C}_1$. ■

It is clear from (HS) that whenever $(r, p; \mu)$ is a $T-$periodic solution of this system, $(r, p; -\mu)$ is another one; furthermore $\operatorname{rot}(r, p; \mu) = -\operatorname{rot}(r, p; -\mu)$. This fact has the following consequence: for studying Theorem 2bis is sufficient to check that there exists $\omega_* > 0$ with the property that for any $\omega \in (0, \omega_*)$ there is a $T-$periodic solution $(r, p; \mu)$ of (HS) with $\operatorname{rot}(r, p; \mu) = \omega$. This will be our next goal.

Proof of Theorem 2bis. We pick up an element $(r_1, p_1; \mu_1)$ of $\mathcal{C}_1$ such that $\min_{t\in\mathbb{R}} r_1(t) \geq r_*$. Since $(r_1, p_1; \mu_1)$ is a $T-$periodic solution of (HS), Lemma 2 implies that $\omega_* = \operatorname{rot}(r_1, p_1; \mu_1) > 0$ (the positiveness of ω_* can be assumed by the previous discussion). Let $\omega \in (0, \omega_*)$ be, we chose other element $(r_2, p_2; \mu_2)$ of $\mathcal{C}_1$ such that $\min_{t\in\mathbb{R}} r_2(t) > 1/\omega$. According to Lemma 1(b), $|\operatorname{rot}(r_2, p_2; \mu_2)| < \omega$. From the connectedness of $\mathcal{C}_1$ follows the existence of $(r, p; \mu) \in \mathcal{C}_1$ such that $\operatorname{rot}(r, p; \mu) = \omega$. ■

At this moment it only remains to show Proposition 1. With this aim we rewrite $(\widetilde{\text{HS}})$ in an abstract way. The 1-dimensional subspace of $C(\mathbb{R}/T\mathbb{Z})$ composed by the constant functions will be identified with $\mathbb{R}$; we use this identification in order to define the projections on this subspace $\Pi, Q : C(\mathbb{R}/T\mathbb{Z}) \to C(\mathbb{R}/T\mathbb{Z})$:

$$\Pi x := x(0) = x(T), \qquad Qx := \frac{1}{T}\int_0^T x(s)ds.$$

For any $x \in \operatorname{Ker} Q$ we denote by Kx to the primitive of x vanishing at $t = 0, T$, and the linear operator $K : \operatorname{Ker} Q \to \operatorname{Ker} \Pi$ defined in this way is compact. Taking into account the definitions of Nemitskii operator we can rewrite $(\widetilde{\text{HS}})$ as a fixed point problem (depending on a parameter) defined on suitable open set of Y

$$y = F[\lambda; y],$$

the (non-linear) operator $F : \Omega \to Y$ is given by

$$F[\lambda; y] := (K(I-Q)N_1[\lambda; y], \Pi\widetilde{p} + QN_1[\lambda; y] + K(I-Q)N_2[\lambda; y], \mu + QN_2[\lambda; y]),$$

(we denote I to the identity operator of $C(\mathbb{R}/T\mathbb{Z})$). We point out that F is completely continuous, i.e., it is continuous and maps bounded sets of $\mathbb{R} \times Y$ whose closure is contained in Ω into relatively compact subsets of Y.

Let us define $\lambda_2 > r_*$ in such a way that (6) holds and let consider the family of bounded open subset of Y

$$U_\lambda := \left\{ y \in Y : \|\widetilde{r}\|_\infty < \frac{T}{2\lambda}, \quad \|\widetilde{p}\|_\infty < \frac{P(\lambda - T/2) + 1}{\lambda}, \quad 0 < \mu < M(\lambda - T/2) \right\}.$$

Before we prove Proposition 1, we compute the Leray-Schauder degree of $F[\lambda; \cdot]$ on U_λ when $\lambda - T/2 \geq \lambda_2$.

Lemma 4 *Assume the conditions of Proposition 1 and U_λ an open set chosen before. Then $F[\lambda, \cdot]$ has no fixed point on ∂U_λ and*

$$d_{LS}(I - F[\lambda; \cdot], U_\lambda, 0) = 1.$$

Proof. First, we check that $F[\lambda; y] \neq y$ for any $y \in \partial U_\lambda$. Indeed, this set can be divided in three parts non-disjoints: the set of the elements $(\widetilde{r}, \widetilde{p}; \mu) \in Y$ such that $\|\widetilde{r}\|_\infty = T/(2\lambda)$, the set of elements $(\widetilde{r}, \widetilde{p}, \mu) \in Y$ such that $\|\widetilde{p}\|_\infty = (P(\lambda - T/2) + 1)/\lambda$ or the set of elements $(\widetilde{r}, \widetilde{p}; \mu)$ such that either $\mu = 0$ or $\mu = M(\lambda - T/2)$. Recalling that whenever we define r, p as in (7), then $F[\lambda; y] = y$ implies that $(r, p; \mu)$ is a $T-$periodic solution of (HS) with $r(0) = \lambda \geq \lambda_2$. According to Lemma 1(a), Lemma 3 (with $a = \lambda - T/2$), neither $\|\widetilde{r}\|_\infty = T/(2\lambda)$ nor $\|\widetilde{p}\|_\infty = (P(\lambda - T/2) + 1)/\lambda$ can be happened. Moreover, Lemma 2 and Lemma 3 (with $a = \lambda - T/2$) prevent that $\mu = 0$ and $\mu = M(\lambda - T/2)$.

In order to prove the result we consider a homotopy to some fixed point problem whose associated operator has degree known and different from zero. Let us define $H : [0, 1] \times \overline{U_\lambda} \to Y$ as

$$H[s; y] := (sK(I - Q)N_1^s[\lambda; y], \Pi\widetilde{p} + QN_1^s[\lambda, y] + sK(I - Q)N_2^s[\lambda, y], \mu + QN_2^s[\lambda, y]),$$

where $N_1^s[\lambda, y] = N_1[\lambda, s\widetilde{r}, \widetilde{p}, s\mu]$ and $N_2^s[\lambda, y] = N_2[\lambda, s\widetilde{r}, s\widetilde{p}, \mu]$. Notice that H is completely continuous and $H[1; y] = F[\lambda; y]$. In addition, $H[s; y] = y$ if and only if $QN_1^s[\lambda; y] = 0$, $QN_2^s[\lambda; y] = 0$ and

$$\dot{r} = \frac{s\alpha(s, r)p}{\sqrt{s^2\mu^2 + \alpha^2(s, r) + \alpha^2(s, r)p^2}},$$

$$\dot{p} = s\left[\frac{\mu^2}{\alpha^2(s, r)\sqrt{\mu^2 + \alpha^2(s, r) + s^2\alpha^2(s, r)p^2}} + f(t, \alpha(s, r))\right],$$

where r, p are defined by (7) and $\alpha(s, r) := \lambda(1 - s) + sr$. Taking into account that Lemma 1(a) implies that $|\alpha(s, r)| < \lambda + T/2$, a similar argument to the one already used to show that $F[\lambda; \cdot]$ does not have fixed points on ∂U proves now that $H[s; y] \neq y$ for any $s \in [0, 1)$ and $y \in \partial U$. When $s = 0$, r and p must be a constants. Since $QN_1[\lambda; 0, \widetilde{p}; 0] = 0$ then $\widetilde{p} = 0$. Moreover, since $QN_2[\lambda; 0, 0; \mu] = 0$, in a similar way like in Lemma 2 and Lemma 3 it follows that $0 < |\mu| < M(\lambda - T/2)$. Therefore the homotopy H is admissible, this means that

$$\deg_{LS}(I - F[\lambda; \cdot], U_\lambda, 0) = \deg_{LS}(I - H[1; \cdot], U_\lambda, 0) = \deg_{LS}(I - H[0; \cdot], U_\lambda, 0). \tag{8}$$

Since the image of $\overline{U_\lambda}$ by $H[0;\cdot]$ is contained on $\mathbb{R}^3$. By linking of (8) with Theorem 8.7 in the page 59 of [9] we see that

$$\deg_{LS}(I - F[\lambda;\cdot], U_\lambda, 0) = \deg_B(I_{\mathbb{R}^3} - H[0;\cdot]\big|_{\mathbb{R}^3}, U_\lambda \cap \mathbb{R}^3, 0),$$

where $\deg_B$ denotes the Brouwer degree. Observe that $\overline{U_\lambda} \cap \mathbb{R}^3 = [-a_1, a_1] \times [-a_2, a_2] \times [0, M(\lambda - T/2)]$ for some suitable positive numbers a_1, a_2, and

$$I_{\mathbb{R}^3} - H[0;\cdot]\big|_{\mathbb{R}^3}(\widetilde{r}, \widetilde{p}; \mu) = (\widetilde{r}, -QN_1[\lambda; 0, \widetilde{p}, 0], -QN_2[\lambda; 0, 0; \mu]).$$

We are led to consider the functions $\varphi : [-a_2, a_2] \to \mathbb{R},\ \psi : [0, M(\lambda - T/2)] \to \mathbb{R}$ defined by

$$\varphi(\widetilde{p}) := \frac{\widetilde{p}}{\sqrt{1 + \lambda^2 \widetilde{p}^2}}, \quad \psi(\mu) := \frac{\mu^2}{\lambda^3 \sqrt{\mu^2 + \lambda^2}} + \frac{1}{T} \int_0^T \frac{f(t, \lambda)}{\lambda} dt\,,$$

and their cartesian product

$$\varphi \times \psi : [-a_2, a_2] \times [0, M(\lambda - T/2)] \to \mathbb{R}^2, \qquad (\widetilde{p}, \mu) \mapsto (\varphi(\widetilde{p}), \psi(\mu))\,.$$

The usual properties of the degree imply

$$\begin{aligned} \deg_B\left(I_{\mathbb{R}^3} - H[0;\cdot]\big|_{\mathbb{R}^3}, U \cap \mathbb{R}^3, 0\right) &= \deg_B\left(\varphi \times \psi, (-a_2, a_2) \times (0, M), 0\right) \\ &= \deg_B\left(\varphi, (-a_2, a_2), 0\right) \deg_B\left(\psi, (0, M), 0\right) \\ &= 1. \end{aligned}$$

since φ and ψ change from negative to positive on their domain (see (6) with $a = \lambda - T/2$). It concludes the proof. ∎

Proof of Proposition 1. We choose λ_* a fixed number such that the bounded open set $U_{\lambda_*} \subset Y$ is well define and is possible to apply Lemma 4, i.e., the degree is well defined and it fulfils $\deg_{LS}(I - F[\lambda_*, \cdot], U_{\lambda_*}, 0) \neq 0$.

Under these assumptions the classical Leray-Schauder continuation theorem ([13], see also [2, 8, 14, 15]) provides the existence of a connected set $\mathcal{C}$ composed by the elements $(\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathbb{R} \times Y$ such that $(\widetilde{r}, \widetilde{p}; \mu)$ is a $T-$periodic solution of $(\widetilde{\mathrm{HS}})$ for that λ. Using the change of variable (7), Lemmas 1, 2 and 3 imply (i) and one of the following conditions:

1. $\{\lambda : (\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C}\} = (0, \infty)$.

2. $\inf\{\min_{t\in\mathbb{R}} \widetilde{r} : (\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C},\ \lambda \in (0, r_* + T/2)\} = -1$.

3. $\{(\widetilde{r}, \widetilde{p}; \mu) : (\lambda; \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C},\ \lambda \in (0, r_* + T/2)\}$ is not bounded.

4. $\mathcal{C} \cap (\mathbb{R} \times C_0(\mathbb{R}/T\mathbb{Z}) \times C(\mathbb{R}/T\mathbb{Z}) \times \{0\}) \neq \emptyset$.

In order to finish the proof we will see that if $(\mathrm{i_a})$ does not happen then $(\mathrm{i_b})$ happens. Indeed, that $(\mathrm{i_a})$ does not happen implies the existence of some constant $k > 0$ such that $\min_{t\in\mathbb{R}} r(t) \geq k$ for any $(r, p; \mu) \in \mathcal{C}_1$; in particular

$$\lambda = r(0) \geq \min_{t\in\mathbb{R}} r(t) \geq k \qquad \forall \quad (r, p; \mu) \in \mathcal{C}_1,$$

contradicting 1.. Other consequence of the before one is

$$\widetilde{r} \geq \frac{k}{\lambda} - 1 > \frac{k}{r_* + T/2} - 1 > -1 \qquad \forall \quad (\lambda, \widetilde{r}, \widetilde{p}; \mu) \in \mathcal{C},$$

thus 2. cannot be fulfilled. On the other hand, 3. do not hold. Indeed, $\widetilde{r}$ is clearly bounded; in addition, since $k \leq \min_{t\in\mathbb{R}} r(t) < r_* + T/2$, with the same argument of Lemma 3 is proven that $p = \lambda\widetilde{p}$ and $|\mu|$ are uniformly bounded (they depend only on k and $r_* + T/2$), so that 3. cannot be fulfilled. The only possibility is 4., but it is exactly the same that ($\mathrm{i_b}$).

■

4 Applications to the existence of periodic solutions for the scalar case

In this section, we will see how our Theorem 1 can be used in order to guarantee the existence of at least one $T-$periodic solution for the equation (3).

Since it has been mentioned in the introduction, our hypothesis (5) is not sufficient to guarantee the $T-$periodic solvability of (3). This is explained if one considers a global attractive function f, it fulfils hypothesis (5) but (3) has no periodic solutions. Therefore, in order to study the $T-$periodic solvability of (3) will be necessary, in some sense, that f is repulsive on somewhere, i.e., we will assume that there exists $r_0 > 0$ such that

$$\int_0^T \min_{r\in[r_0, r_0+T/2]} f(t,r)dt > 0. \tag{9}$$

Now, under the hypothesis (5) and (9) we can guarantee the existence of at least one $T-$periodic solution for (3) (see Theorem 3).

Proof of Theorem 3 Let us consider the system equation (1). Under hypothesis (5), Theorem 1 provides of two possibilities:

1) $\{\min_{t\in\mathbb{R}} |x(t)| : x$ is a $T-$rad. periodic solution of (1)$\} = (0,\infty)$

2) there exist $T-$rad. periodic solutions of (1) with angular momentum equal to 0.

Notice that if 2) happens, such solutions are $T-$periodic solutions of (HS) with the form $(r, p; \mu = 0)$, in particular r will be a $T-$periodic solution of (3). On the contrary, if (1) happens, we can choose a $T-$rad. periodic solution of (1) with $\min_{t\in\mathbb{R}} |x(t)| = r_0$. According to Cauchy-Schwartz inequality and integrating by parts, one can easily check

$$\int_0^T < \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right), \frac{x}{|x|} > dt \leq 0,$$

($<,>$ denotes the scalar product on $\mathbb{R}^2$). Since x fulfils (1), from the previous inequality we arrive to contradiction with (9). ■

In order to prove the applicability of Theorem 3 we could consider many types of equations, even equations whose solvability is still unknown. In order to compare the results we will only concern the Mathieu-Duffing type equations, i.e., the equations like

$$\frac{d}{dt}\left(\frac{\dot r}{\sqrt{1-\dot r^2}}\right) = q(t)r - p(t)r^3, \tag{10}$$

where q, p are locally integrable functions, with positive range for p, and both are $T-$periodic ones.

It will be convenient to introduce a new notation which will be only used for this part. We denote

$$P := \int_0^T p(s)ds, \quad Q_+ := \int_0^T q^+(s)ds, \quad Q_- = \int_0^T q^-(s)ds,$$

and $Q := Q_+ - Q_-$; here $q^+(s) := \max\{q(s), 0\}$, $q^-(s) = \min\{-q(s), 0\}$. With the previous notation we present the following results.

Corollary 1 *Assume* $\overline{q} := \int_0^T q > 0$. *If*

$$Q_+ < \frac{4}{T\sqrt{P}}\left(\frac{Q}{3}\right)^{\frac{3}{2}}, \tag{11}$$

there exists at least one (positive) $T-$periodic solution of (10). *Moreover, by symmetry of* (10) *there is another (negative) $T-$periodic solution.*

Proof. From (11) we can define $r_0 = \sqrt{Q/(3P)} - T/2 > 0$. Indeed,

$$\begin{aligned}
\frac{4}{\sqrt{P}}\left(\frac{Q}{3}\right)^{\frac{3}{2}} - TQ_+ &= 2\left(\frac{Q}{3P}\right)^{\frac{1}{2}}\left[Q_+ - Q_- - \frac{Q}{3}\right] - TQ_+ \\
&= 2\left(r_0 Q_+ - Q_-\left(\frac{Q}{3P}\right)^{\frac{1}{2}} - \left(\frac{Q}{3P^{\frac{1}{3}}}\right)^{\frac{3}{2}}\right);
\end{aligned}$$

since (11) implies that the left side is positive, the right side implies that $r_0 > 0$. Let us define $f(t,r) = q(t)r - p(t)r^3$. Because of $P > 0$ we can prove (5). On the other hand

$$\int_0^T \min_{r\in[r_0, r_0+T/2]} f(t,r)dt \geq r_0 Q - \left(r_0 + \frac{T}{2}\right)^3 P - \frac{TQ_-}{2}.$$

Notice that the function $\xi(r) = rQ - (r+\frac{T}{2})^3 P - \frac{TQ_-}{2}$ will attain its global maximum at r_0, so that r_0 is the best election. Moreover, due to (11) it follows that $\xi(r_0) > 0$, i.e., (9) holds. Now, the results is proven applying Theorem 3. ∎

In most of the cases, in literature is considered a particular case of (10), usually the considered equation is

$$\frac{d}{dt}\left(\frac{\dot{r}}{1-\dot{r}^2}\right) = (b_1 + b_2\cos t)r + cr^3, \tag{12}$$

where b_1, b_2 and c are real numbers; i.e., the particular case of (10) when $q(t) = b_1 + b_2\cos t$ and $p(t) = c$. In this type of problems is usual to find $2\pi-$periodic solutions (i.e. $T = 2\pi$). This will be done as example of applicableness of Corollary 1 (see [5, 21]).

Example 1 *Assume $b_1 > 0$ and $c > 0$. If*

$$\pi b_1 + b_2 < \left(\frac{b_1}{3c^{\frac{1}{3}}}\right)^{\frac{3}{2}},$$

the equation (12) *has at least one (positive) $2\pi-$periodic solution. Obviously, by symmetry, it has another solution with different sign.*

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