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EXPLICIT CONDITIONS FOR THE NONOSCILLATION OF DIFFERENCE EQUATIONS WITH SEVERAL DELAYS*

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Abstract. We present a sharp explicit condition sufficient for the nonoscillation of solutions to a scalar linear nonautonomous difference equation with several delays.

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1. Introduction

Let $\mathbb{Z}$ be the set of all integers. Throughout the paper, for given $p \in \mathbb{Z}$ we put by definition $\mathbb{N}_p = \{n \in \mathbb{Z} \mid n \geq p\}$.

Consider a scalar difference equation

$$x(n+1) - x(n) + \sum_{k=0}^{N} a_k(n)x(n - h_k(n)) = 0, \quad n \in \mathbb{N}_0, \tag{1}$$

where $a_k(n) \geq 0$ and $h_k(n) \geq 0$ for all $n \in \mathbb{N}_0$ and $k = 0, \dots, N$. These conditions are supposed to be true until the inverse is stated.

We say that a function $x\colon \mathbb{Z} \to \mathbb{R}$ is a *solution* of equation (1), if the equality in (1) is true for each $n \in \mathbb{N}_0$. Clearly, any initial function $\varphi\colon \mathbb{Z} \setminus \mathbb{N}_1 \to \mathbb{R}$ defines a unique solution of (1) satisfying the condition $x(n) = \varphi(n)$, $n \in \mathbb{Z} \setminus \mathbb{N}_1$.

As is customary, we say that equation (1) is *nonoscillatory*, if there exists its solution x such that for some $p \in \mathbb{N}_0$ for all $n, m \in \mathbb{N}_p$ we have $x(n)x(m) > 0$.

In the present paper we obtain new sufficient conditions for the nonoscillation of equation (1). The conditions are sharp and expressed in terms of

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functions a_k and r_k. The main tool of the investigation is the fundamental function of equation (1).

The article is organized as follows. The second section is preliminary. In that one, we consider some known results on relation between the fundamental function of (1), and that of a certain delay differential equation. Then we prove a discrete analogue to some known proposition on a difference inequality.

In the third section we consider (1) in case $N = 1$, and obtain two theorems on conditions of the positiveness of the fundamental function. In view of them and some known results, we conjecture that some stronger theorem is valid.

The main theorem of the paper and its proof are the matter of the fourth section.

The last two sections contain some generalizations of the main theorem, and a discussion, where we compare the new results with known ones.

2. Preliminaries

2.1. **Fundamental function.** Put $\Delta_p = \{(n, m) \in \mathbb{Z}^2 \mid n \geq m \geq p\}$.

A function $X\colon \mathbb{Z} \times \mathbb{N}_0 \to \mathbb{R}$ that is the solution of a problem

$$X(n+1, m) - X(n, m) = -\sum_{k=0}^{N} a_k(n) X(n - h_k(n), m), \quad (n, m) \in \Delta_0;$$
$$X(n, m) = 0, \quad (n, m) \in \mathbb{Z} \times \mathbb{N}_0 \setminus \Delta_0; \quad X(m, m) = 1, \quad m \in \mathbb{N}_0,$$

is called *the fundamental function* of equation (1). This notion plays a role analogous to that of the fundamental function of a delay differential equation (see [2, 6]). In particular, if it is known whether the fundamental function of an equation is positive, or has different signs, or oscillates, then one can draw conclusions about analogous properties of solutions of the equation. E.g., in [4] it is shown that if $(n - h(n)) \to \infty$ as $n \to \infty$, then the nonoscillation of (1) is equivalent to the positiveness of the function X on Δ_p for some $p \in \mathbb{N}_0$.

From the definition of X, it is obvious that the problem of obtaining conditions for the positiveness of X on Δ_p, for given $p \in \mathbb{N}_0$, is reduced to the case $p = 0$. In this paper we obtain explicit sharp conditions for the positiveness of X on Δ_0.

We use some known relations between the fundamental function of (1) and that of a certain functional differential equation.

Denote by $[\cdot]$ the integer part of a number; put

$$(2) \qquad p_k(t) = a_k([t]), \quad r_k(t) = h_k([t]) + t - [t], \quad k = 0, \ldots, N, \quad t \geq 0,$$

and consider the equation

$$\dot{y}(t) + \sum_{k=0}^{N} p_k(t) y(t - r_k(t)) = 0, \quad t \geq 0. \tag{3}$$

Put $\Delta_+ = \{(t,s) \in \mathbb{R}^2 \mid t \geq s \geq 0\}$. The fundamental function $Y\colon \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ of equation (3) is the solution of a problem [1, p. 24]

$$\frac{\partial Y(t,s)}{\partial t} = -\sum_{k=0}^{N} p_k(n) Y(t - r_k(t), s), \quad (t,s) \in \Delta_+;$$
$$Y(t,s) = 0, \quad (t,s) \in \mathbb{R} \times [0, +\infty) \setminus \Delta_+; \quad Y(s,s) = 1, \quad s \in [0, +\infty).$$

The following two results are established in [10]. Suppose (2) holds. Then, first,

$$X(n,m) = Y(n,m), \quad (n,m) \in \Delta_0; \tag{4}$$

second, for coefficients of equations (1) and (3) we have

$$\sum_{k=0}^{N} \int_{t-r(t)}^{t} p_k(s) \mathrm{d}s \leq \sum_{i=[t]-h([t])}^{[t]} \sum_{k=0}^{N} a_k(i), \quad t \geq 0, \tag{5}$$

where $r(t) = \max_{k \in \{0,\ldots,N\}} r_k(t)$, $h(n) = \max_{k \in \{0,\ldots,N\}} h_k(n)$, $a_k(n) = 0$ for $n < 0$, and $p_k(t) = 0$ for $t < 0$.

Associate these results with the following one that is well known in theory of functional differential equations.

Theorem 1 ([1, p. 32])**.** *Suppose* $\sup_{t \geq 0} \int_{t-r(t)}^{t} \sum_{k=0}^{N} p_k(s) \mathrm{d}s \leq 1/e$. *Then* $Y(t,s) > 0$ *for all* $(t,s) \in \Delta_+$.

The next proposition is a consequence of (4), (5), and Theorem 1.

Theorem 2. *Suppose* $\sup_{n \in \mathbb{N}_0} \sum_{i=n-h(n)}^{n} \sum_{k=0}^{N} a_k(i) \leq 1/e$. *Then* $X(n,m) > 0$ *for all* $(n,m) \in \Delta_0$.

Remark 1. The correspondence between solutions of difference equations and that of differential equations with piecewise constant delay was first established in the paper [5] by K. L. Cooke and J. Wiener.

I. Győri and M. Pituk were apparently the first to apply the correspondence to obtain sufficient nonoscillation conditions for delay difference equations in [8] (Corollaries 3.6 and 3.7).

2.2. **Lemma.** In theory of functional differential equations there are methods based on the technique of functional differential inequalities, and used to estimate solutions of equations, in particular, to obtain conditions for the positiveness of a fundamental function. One can find many results of that kind in the new monograph [1].

The next proposition deals with a differential inequality corresponding to equation (3); it is a simple corollary of Lemma 2.4.3 from [3, p. 57].

Put $(Ky)(t) = \dot{y}(t) + \sum_{k=0}^{N} p_k(t)y(t - r_k(t))$.

Lemma 1. *If there exists a locally absolutely continuous function* $w\colon \mathbb{R} \to \mathbb{R}$ *such that* $w(t) \geq 0$ *for all* $t < 0$, $w(t) > 0$ *for all* $t \geq 0$, *and* $(Kw)(t) \leq 0$ *for almost all* $t \geq 0$, *then for the fundamental function* Y *of* (3) *the following estimate is valid:* $Y(t, s) \geq w(t)/w(s)$ *for all* $(t, s) \in \Delta_+$.

We will obtain now an analogous fact for equation (1).

Put $(Lx)(n) = x(n+1) - x(n) + \sum_{k=0}^{N} a_k(t)x(n - h_k(n))$.

Lemma 2. *If there exists a function* $v\colon \mathbb{Z} \to \mathbb{Z}$ *such that* $v(n) \geq 0$ *for all* $n < 0$, $v(n) > 0$ *for all* $n \geq 0$, *and* $(Lv)(n) \leq 0$ *for all* $n \geq 0$, *then for the fundamental function* X *of* (1) *the following estimate is valid:* $X(n, m) \geq v(n)/v(m)$ *for all* $(n, m) \in \Delta_0$.

Proof. Suppose v is a function satisfying the assumptions. Define a function $w\colon \mathbb{R} \to \mathbb{R}$ by the rule $w(t) = v([t]) + (v([t+1]) - v([t]))(t - [t])$. It is obvious that $w(n) = v(n)$ for all $n \in \mathbb{Z}$, and if $v(n) > 0$ for $n \geq 0$ then $w(t) = v([t])(1 - (t - [t])) + v([t+1])(t - [t]) > 0$ for $t \geq 0$.

Let the functions p_k and r_k be defined by (2).

Suppose $t \in (n, n+1)$, where $n \in \mathbb{N}_0$; then

$$
\begin{aligned}
(Kw)(t) &= \dot{w}(t) + \sum_{k=0}^{N} p_k(t)w(t - r_k(t)) \\
&= v([t+1]) - v([t]) + \sum_{k=0}^{N} a_k([t])w([t] - h_k([t])) \\
&= v(n+1) - v(n) + \sum_{k=0}^{N} a_k(n)w(n - h_k(n)) = (Lv)(n) \leq 0.
\end{aligned}
$$

At points $t = n \in \mathbb{N}_0$ the function w is nondifferentiable. However, since the Lebesgue measure of $\mathbb{N}_0$ is zero, the inequality $(Kw)(t) \leq 0$ is valid for almost all $t \geq 0$. Thus, for the function w all the suppositions of Lemma 1 are fulfilled. Hence, for the fundamental function of (3) the estimate $Y(t, s) \geq$

$w(t)/w(s)$ holds. Since $w(n) = v(n)$ for all $n \in \mathbb{N}_0$, and (4), it follows that $X(n,m) \geq v(n)/v(m)$ for all $(n,m) \in \Delta_0$. □

In fact, the efficiency of applying Lemmas 1 and 2 depends on the choice of the functions v and w. However, if qualitative behavior of solutions of an equation under consideration is known, then one can often choose the functions such as to obtain a strong result. Below we use Lemma 2 to get sharp conditions of the positiveness of the fundamental function of (1).

3. Equation with one delay

Consider an equation

$$x(n+1) - x(n) + a(n)x(n - h(n)) = 0, \quad n \in \mathbb{N}_0, \tag{6}$$

where $a(n) \geq 0$ and $h(n) \geq 0$ for all $n \in \mathbb{N}_0$.

The applying of Theorem 2 to (6) gives the following result.

Theorem 3. *Suppose* $\sup_{n\in\mathbb{N}_0} \sum_{i=n-h(n)}^{n} a(i) \leq 1/e$. *Then the fundamental function of equation* (6) *is positive on* Δ_0.

It is proved in [7] that in case $a(n) \equiv A$, $h(n) \equiv H$ the inequality $A(H+1) \leq \left(\frac{H}{H+1}\right)^H$ is a necessary and sufficient condition for the existence of nonoscillatory solutions of (6). Therefore, the constant $1/e$ in Theorem 3 is sharp, that is it can not be replaced by a greater one. Indeed, $\lim_{H\to\infty} \left(\frac{H}{H+1}\right)^H = \frac{1}{e}$, where the sequence is decreasing. Hence, for arbitrary $\varepsilon > 0$ one may take H and A such that $\frac{1}{e} < \left(\frac{H}{H+1}\right)^H < \frac{1}{e} + \varepsilon$ and $\frac{H^H}{(H+1)^{H+1}} < A \leq \frac{1/e+\varepsilon}{H+1}$. In this case, since $A(H+1) > \left(\frac{H}{H+1}\right)^H$, the fundamental function is not positive on Δ_0, however $A(H+1) \leq \frac{1}{e} + \varepsilon$.

Using Lemma 2, obtain another condition for the nonoscillation of (6).

Theorem 4. *Suppose* $a(n) \leq A$, $h(n) \leq H$, $n \in \mathbb{N}_0$. *If* $A(H+1) \leq \left(\frac{H}{H+1}\right)^H$, *then the fundamental function of* (6) *is positive on* Δ_0.

Proof. Fix $\lambda \in (0,1)$, and put $v(n) = \lambda^n$. By Lemma 2, the fundamental function of (6) is positive if $(Lv)(n) \leq 0$ for all $n \in \mathbb{N}_0$. We have

$$\begin{aligned}(Lv)(n) &= \lambda^{n+1} - \lambda^n + a(n)\lambda^{n-h(n)} = \lambda^{n-H}\left(\lambda^{H+1} - \lambda^H + a(n)\lambda^{H-h(n)}\right) \\ &\leq \lambda^{n-H}\left(\lambda^{H+1} - \lambda^H + A\right).\end{aligned}$$

Put $f(\lambda) = \lambda^{H+1} - \lambda^H + A$. Since $f'(\lambda) = (H+1)\lambda^H - H\lambda^{H-1}$, the function f has a minimum at the point $\lambda_0 = \frac{H}{H+1} \in (0,1)$. Hence, for $(Lv)(n) \leq 0$ it is sufficient that $f(\lambda_0) \le 0$. Since

$$f(\lambda_0) = \left(\frac{H}{H+1}\right)^{H+1} - \left(\frac{H}{H+1}\right)^{H} + A = -\left(\frac{H}{H+1}\right)^{H} \frac{1}{H+1} + A,$$

it follows that $(Lv)(n) \le 0$ provided that $A(H+1) \le \left(\frac{H}{H+1}\right)^H$. $\square$

Let us compare Theorems 3 and 4. We have $\left(\frac{H}{H+1}\right)^H > \frac{1}{e}$ for all $H \ge 1$. On the other hand, $A(H+1) \ge \sum_{i=n-H}^{n} a(i)$, and the difference between the left part of the inequality and the right one may be large. It would be of interest to combine the advantages of both the theorems.

There are some known results of that kind. In 1990, G. Ladas offered [9] the following problem: is it true that if $h(n) \equiv H$, and $\sum_{i=n-H}^{n-1} a(i) \le \left(\frac{H}{H+1}\right)^{H+1}$ for all sufficiently large n, then equation (6) has a nonoscillating solution? In paper [12] an example was presented showing that the answer is negative. Then, the sum of the form $\sum_{i=n-h(n)}^{n-1} a(i)$ was used in papers [13] and [4] for equations with varying delays. In the first of the papers it was proved that if $(n - h(n)) \to \infty$ as $n \to \infty$, and $\sup_{n\in\mathbb{N}_p} \sum_{i=n-h(n)}^{n-1} a(i) \le \frac{1}{4}$, then equation (6) has a nonoscillating solution. In the second one the result was generalized for equation (1). The constant $\frac{1}{4}$ is sharp, since the inequality $A \le \frac{1}{4}$ is necessary for the nonoscillation of the autonomous equation $x(n+1) - x(n) = Ax(n-1)$. Moreover, for each $H \in \mathbb{N}_1$ and $\varepsilon > 0$ there is an equation of the form (6) such that $h(n) \equiv H$, the fundamental solution is not positive, and $\sup_{n\in\mathbb{N}_0} \sum_{i=n-H}^{n-1} a(i) = \frac{1}{4} + \varepsilon > 0$ (see the discussion in the last section). Note that $\left(\frac{H}{H+1}\right)^{H+1} > \frac{1}{4}$ for $H > 1$.

In paper [11] Theorem 3 was proved for the case $h(n) \equiv H$. There was also indicated that if $h(n) \equiv H$, and $\sup_{n\in\mathbb{N}_p} \sum_{i=n-H}^{n} a(i) \le \left(\frac{H}{H+1}\right)^{H+1}$, then equation (6) has a nonoscillating solution. Indeed, it is sufficient to note that $\left(\frac{H}{H+1}\right)^{H+1} < \frac{1}{e}$, and apply Theorem 3.

The authors of [12] and [11] conclude that, since the answer to the Ladas problem is negative, the discrete analogues of the oscillation results for delay differential equations may be not true. However, we believe that the analogy generally exists, and it is the question to investigate what form it has. A partial case of the basic theorem of this paper is the following statement.

Suppose $h(n) \le H$, $n \in \mathbb{N}_0$. If $\sup_{n\in\mathbb{N}_0} \sum_{i=n-h(n)}^{n} a(i) \le \left(\frac{H}{H+1}\right)^H$, then the fundamental function of equation (6) *is positive on Δ_0.*

4. The main result

We will now establish an auxiliary inequality, which, as we hope, could be interesting by itself.

Let $a_1, a_2, \dots, a_m$ be nonnegative real numbers. For every $k = 1, \dots, m$ put

$$S_k = \sum_{i_1=1}^{m-k+1} \sum_{i_2=i_1+1}^{m-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{m} a_{i_1} a_{i_2} \dots a_{i_k}$$

(so that we have $S_1 = \sum_{i=1}^m a_i$, $S_2 = \sum_{i=1}^{m-1} \sum_{j=i+1}^m a_i a_j, \dots, S_m = a_1 a_2 \dots a_m$).

Let $\binom{m}{k} = \frac{m!}{(m-k)!k!}$ be binomial coefficients.

Lemma 3. *For all $a_1, \dots, a_m$ and $k = 1, \dots, m$ the inequality $S_k \le \binom{m}{k} \left(\frac{S_1}{m}\right)^k$ holds.*

Proof. The case that $a_i = 0$ for all $i = 1, \dots, m$ is trivial; hence, suppose that there is a nonzero a_i. We have $S_1 > 0$. Define new variables $\alpha_i = a_i / S_1$; we have $0 \le \alpha_i \le 1$, $\sum_{i=1}^m \alpha_i = 1$. Fix $k \in \{1, \dots, m\}$ arbitrarily, and consider the function

$$\varphi = \varphi(\alpha_1, \dots, \alpha_m) = \frac{S_k}{S_1^k} = \sum_{i_1=1}^{m-k+1} \sum_{i_2=i_1+1}^{m-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{m} \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_k}$$

defined on the set

$$E = \left\{ (\alpha_1, \dots, \alpha_m) \in [0,1]^m \mid \sum_{i=1}^m \alpha_i = 1 \right\}.$$

Since φ is continuous, and E is compact, by the Weierstrass theorem, φ reaches its greatest value at some point $\alpha^0 = (\alpha_1^0, \dots, \alpha_m^0) \in E$. Consider an arbitrary point $\alpha = (\alpha_1 \dots, \alpha_m) \in E$. Suppose that $\alpha_i \ne \alpha_j$ for some i and j. If we replace both the numbers α_i and α_j by their arithmetic mean $(\alpha_i + \alpha_j)/2$, then we get a new point that is again in E. Moreover, by virtue of the obvious inequality $\alpha_i \alpha_j < ((\alpha_i + \alpha_j)/2)^2$ the value of φ increases. Therefore, if $\alpha_i \ne \alpha_j$ for some i and j, then $\alpha \ne \alpha^0$. It follows that $\alpha_1^0 = \dots = \alpha_m^0 = \frac{1}{m}$. Now by the definition of the function φ, for all $\alpha_i, \dots, \alpha_m$ we obtain

$$\frac{S_k}{S_1^k} = \varphi(\alpha_1, \dots, \alpha_m) \le \varphi\left(\frac{1}{m}, \dots, \frac{1}{m}\right) = \binom{m}{k} \frac{1}{m^k}.$$

This implies the desired inequality. □

Remark 2. In case $k = m$ Lemma 3 gives $(S_m)^{\frac{1}{m}} \le \frac{S_1}{m}$, which is the classical Cauchy inequality.

The main result of the paper is the following theorem. It is a generalization, for equation (1), of the statement formulated above for equation (6).

Put $a(n) = \sum_{k=0}^{N} a_k(n)$, $h(n) = \max_{k\in\{0,\dots,N\}} h_k(n)$, $n \in \mathbb{N}_0$; $a(n) = 0$, $n \in \mathbb{Z}\setminus\mathbb{N}_0$.

Theorem 5. *Suppose $h(n) \le H$ for all $n \in \mathbb{N}_0$, and*

$$\sup_{n\in\mathbb{N}_0} \sum_{i=n-h(n)}^{n} a(i) \le \left(\frac{H}{H+1}\right)^{H}.$$

Then the fundamental function $X(n,m)$ of equation (1) is positive for all $(n,m) \in \Delta_0$.

Proof. Fix $\lambda > 0$. Put $v(n) = \frac{1}{\prod_{i=0}^{n-1}(1+\lambda a(i))}$, $n \in \mathbb{Z}$. Note that the assumptions of Lemma 2 are fulfilled for the function v. Since $v(n+1) = \frac{v(n)}{1+\lambda a(n)}$ for all $n \in \mathbb{N}_0$, we have $v(n+1) - v(n) = -\frac{\lambda a(n) v(n)}{1+\lambda a(n)} = -\frac{\lambda a(n)}{\prod_{i=0}^{n}(1+\lambda a(i))}$. Hence,

$$\begin{aligned}
(Lv)(n) &= v(n+1) - v(n) + \sum_{k=1}^{N} a_k(n) v(n-h_k(n)) \\
&= -\frac{\lambda a(n)}{\prod_{i=0}^{n}(1+\lambda a(i))} + \sum_{k=0}^{N} \frac{a_k(n)}{\prod_{i=0}^{n-h_k(n)-1}(1+\lambda a(i))} \\
&\le -\frac{\lambda a(n)}{\prod_{i=0}^{n}(1+\lambda a(i))} + \frac{a(n)}{\prod_{i=0}^{n-h(n)-1}(1+\lambda a(i))} \\
&= \frac{a(n)}{\prod_{i=0}^{n}(1+\lambda a(i))} \left(\prod_{i=n-h(n)}^{n} (1+\lambda a(i)) - \lambda \right).
\end{aligned}$$

Obviously, $(Lv)(n) \le 0$ provided that there exists $\lambda > 0$ such that

$$\prod_{i=n-h(n)}^{n} (1+\lambda a(i)) - \lambda \le 0.$$

This implies that to prove the theorem it is sufficient to find $\lambda > 0$ such that the inequality holds for all $n \in \mathbb{N}_0$.

Put $P(\lambda) = \prod_{i=n-h(n)}^{n}(1+\lambda a(i)) - \lambda$. We have

$$P(\lambda) = 1 + \lambda p_1(n) + \lambda p_2(n) + \cdots + \lambda^{h(n)+1} p_{h(n)+1}(n) - \lambda,$$

where

$$p_k(n) = \sum_{i_1=n-h(n)}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n} a(i_1)a(i_2)\dots a(i_k), \quad k=1,\dots,h(n)+1.$$

Applying Lemma 3, we obtain

$$p_k(n) \le \binom{h(n)+1}{k} \left(\frac{\lambda p_1(n)}{h(n)+1}\right)^k, \quad k=2,\dots,h(n)+1.$$

Since the function $(1+\alpha/x)^x$ of the variable x is increasing provided $x, \alpha > 0$,

$$\begin{aligned} P(\lambda) &= 1 + \lambda p_1(n) + \lambda^2 p_2(n) + \cdots + \lambda^{h(n)+1} p_{h(n)+1}(n) - \lambda \\ &\le 1 + \lambda p_1(n) + \sum_{k=2}^{h(n)+1} \binom{h(n)+1}{k} \left(\frac{\lambda p_1(n)}{h(n)+1}\right)^k - \lambda \\ &= \left(1 + \frac{\lambda p_1(n)}{h(n)+1}\right)^{h(n)+1} - \lambda. \end{aligned}$$

If the assumption of the theorem is true, then $p_1(n) = \sum_{i=n-h(n)}^{n} a(i) \le \left(\frac{H}{H+1}\right)^H$ for all $n \in \mathbb{N}_0$. Hence,

$$P(\lambda) \le \left(1 + \frac{\lambda H^H}{(H+1)^{H+1}}\right)^{H+1} - \lambda.$$

Now putting $\lambda = \left(\frac{H+1}{H}\right)^{H+1}$ we obtain

$$P\left(\left(\frac{H+1}{H}\right)^{H+1}\right) \le \left(1 + \frac{\left(\frac{H+1}{H}\right)^{H+1} H^H}{(H+1)^{H+1}}\right)^{H+1} - \left(\frac{H+1}{H}\right)^{H+1} = 0.$$

Therefore, the fundamental function of (1) is positive by Lemma 2. □

Remark 3. Although $\left(\frac{H}{H+1}\right)^H$ is not defined for $H = 0$, the case $h(n) \equiv 0$ is covered by Theorem 5, if we let H to be an arbitrary positive number. Moreover, the estimate $\sup_{n\in\mathbb{N}_0} a(n) \le \left(\frac{H}{H+1}\right)^H$ may be supposed as a sharp condition of nonoscillation. Indeed, the fundamental function of the equation

$$x(n+1) - x(n) + a_0(n)x(n) = 0, \quad n \in \mathbb{N}_0,$$

has the form $X(n+1,m) = \prod_{i=m}^{n}(1 - a_0(i))$, hence it is positive if and only if $a_0(n) < 1$ for all $n \in \mathbb{N}_0$. Since $\left(\frac{H}{H+1}\right)^H \to 1$ as $H \to +0$, and $\left(\frac{H}{H+1}\right)^H < 1$ for all $H > 0$, it follows that $a_0(n) \le \left(\frac{H}{H+1}\right)^H$ for some $H > 0$ if and only if $a_0(n) < 1$.

5. Some generalizations

Below the condition $a_k(n) \geq 0$ is not supposed to be true anymore.

First, we are to obtain a generalization of Theorem 5 for an equation with coefficients that may change sign.

Lemma 4 ([4])**.** *Suppose $b_k(n) \geq a_k(n)$, $k = 0, \dots, N$, $n \in \mathbb{N}_0$, and the fundamental function of the equation*

$$x(n+1) - x(n) + \sum_{k=0}^{N} b_k(n) x(n - h_k(n)) = 0, \quad n \in \mathbb{N}_0, \tag{7}$$

is positive on Δ_0. Then the fundamental function of equation (1) *is also positive on Δ_0.*

Put $a_k^+(n) = \max\{a_k(n), 0\}$, $k = 0, \dots, N$, $n \in \mathbb{N}_0$. The next statement follows from Lemma 4.

Lemma 5. *Suppose the fundamental function of the equation*

$$x(n+1) - x(n) + \sum_{k=0}^{N} a_k^+(n) x(n - h_k(n)) = 0, \quad n \in \mathbb{N}_0, \tag{8}$$

is positive on Δ_0. Then the fundamental function of equation (1) *is also positive on Δ_0.*

The next theorem is a corollary of Theorem 5 and Lemma 5.

Theorem 6. *Suppose $h(n) \leq H$ for all $n \in \mathbb{N}_0$, and*

$$\sup_{n \in \mathbb{N}_0} \sum_{i=n-h(n)}^{n} \sum_{k=0}^{N} a_k^+(i) \leq \left(\frac{H}{H+1} \right)^H .$$

Then the fundamental function $X(n,m)$ of equation (1) *is positive for all $(n,m) \in \Delta_0$.*

We obtain another generalization for an equation with zero delay. Suppose $h_0(n) \equiv 0$ in equation (1), and rewrite (1) in the form

$$x(n+1) - x(n) + a_0(n)x(n) + \sum_{k=1}^{N} a_k(n) x(n - h_k(n)) = 0, \quad n \in \mathbb{N}_0. \tag{9}$$

In this case we may get conditions for the positiveness of the fundamental function without restriction for sign of coefficient $a_0(n)$.

Let us put

$$h(n) = \max_{k\in\{1,\dots,N\}} h_k(n), \quad b(n) = \prod_{i=0}^{n-1}(1-a_0(i)),$$

$$q_k(n) = a_k(n)\frac{b(n-h_k(n))}{b(n+1)}, \quad k=1,\dots,N, n\in\mathbb{N}_0.$$

Theorem 7. *If $a_0(n)\in(-\infty,1)$ and $h(n)\le H$ for all $n\in\mathbb{N}_0$, and*

$$\sup_{n\in\mathbb{N}_0}\sum_{i=n-h(n)}^{n}\sum_{k=1}^{N} q_k^+(i) \le \left(\frac{H}{H+1}\right)^H,$$

then the fundamental function of equation (9) *is positive on Δ_0.*

Proof. By change of variables $x(n)=b(n)y(n)$, equation (9) is reduced to the equation

$$y(n+1)-y(n)+\sum_{k=1}^{N} q_k(n)y(n-h_k(n)), \quad n\in\mathbb{N}_0. \tag{10}$$

Suppose the assumptions are fulfilled. Then the fundamental function of equation (10) is positive by Theorem 6. Since $b(n)>0$, and the fundamental functions X and Y, of equations (9) and (10) respectively, are related by the equality $X(n,m)=b(n)Y(n,m)$, the fundamental function of equation (9) is also positive. □

Example 1. Let us find conditions of nonoscillation for the equation

$$x(n+1)-x(n)+a_0(n)+ax(n-H)=0, \quad n\in\mathbb{N}_0.$$

By Theorem 7, the inequalities $a_0<1$ and $a\le(1-a_0)^{H+1}\frac{H^H}{(H+1)^{H+1}}$ imply the positiveness of the fundamental function. Moreover, it may be shown that these conditions are necessary for the positiveness. It follows that the restriction $a_0(n)<1$ is essential in Theorem 7.

6. Discussion

The conditions of nonoscillation obtained in works [7] and [12] are covered by Theorem 7. On the contrary, the conditions from papers [4] and [13] are not covered by our results, and do not cover them. To illustrate this, consider equation (6). Suppose $a(n)\equiv A$, $h(n)\equiv H$. Then Theorem 7 gives the condition $A(H+1)\le\left(\frac{H}{H+1}\right)^H$ for the positiveness of the fundamental function. The condition from [13] is $AH\le 1/4$, which is more restrictive. On the other hand, if we put $h(n) \equiv H \geq 2$, $a(0) = A$, $a(n) = 0$ for $n = 1, \ldots, H-1$, and $a(n+H) = a(n)$ for all $n \in \mathbb{N}_0$, then $\sup_{n\in\mathbb{N}_0} \sum_{i=n-H}^{n-1} a(i) = A$ and $\sup_{n\in\mathbb{N}_0} \sum_{i=n-H}^{n} a(i) = 2A$. In this case one should apply the condition from [13] rather than Theorem 7. Note that if $A > \frac{1}{4}$, then the fundamental solution of the constructed equation is not positive on Δ_0.

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