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LOCALIZED SOLUTIONS OF ELLIPTIC EQUATIONS: LOITERING AT THE HILLTOP

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Abstract. We find an infinite number of smooth, localized, radial solutions of $\Delta_p u + f(u) = 0$ in $\mathbb{R}^N$ - one with each prescribed number of zeros - where $\Delta_p u$ is the p-Laplacian of the function u.

1. Introduction

In this paper we will prove the existence of smooth, radial solutions with any prescribed number of zeros to:

$$\Delta_p u + f(u) = 0 \text{ in } \mathbb{R}^N, \tag{1.1}$$

$$u(x) \to 0 \text{ as } |x| \to \infty, \tag{1.2}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ $(p > 1)$ is the *p-Laplacian* of the function u (note that $p = 2$ is the usual Laplacian operator), f is the nonlinearity described below, and $N \geq 2$,.

Solutions of (1.1)-(1.2) arise as critical points of the functional $J : S \to \mathbb{R}$ defined by:

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{p} |\nabla u|^p - F(u)\, dx$$

where $F(u) = \int_0^u f(t)\, dt$ and $S = \{u \in W^{1,p}(\mathbb{R}^N) \mid F(u) \in L^1(\mathbb{R}^N)\}$.
Setting $r = |x|$ and assuming that u is a radial function so that $u(x) = u(|x|) = u(r)$ then:

$$\Delta_p u = |u'|^{p-2}[(p-1)u'' + \frac{N-1}{r} u'] = \frac{1}{r^{N-1}} (r^{N-1} |u'|^{p-2} u')'$$

where $'$ denotes differentiation with respect to the variable r.

We consider therefore looking for solutions of:

$$|u'|^{p-2}[(p-1)u'' + \frac{N-1}{r} u'] + f(u) = \frac{1}{r^{N-1}} (r^{N-1} |u'|^{p-2} u')' + f(u) = 0 \tag{1.3}$$

$$\lim_{r \to 0^+} u'(r) = 0, \tag{1.4}$$

$$\lim_{r \to \infty} u(r) = 0. \tag{1.5}$$

Remark: The case $p = 2$ was examined in [2]. There the authors proved the existence of an infinite number of solutions of (1.3)-(1.5) - one with each precribed number of zeros - for nonlinearities f similar to the ones examined in this paper. In this paper we have weaker assumptions than those in [2] and we also have only

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that $p > 1$. Existence of ground states of (1.3)-(1.5) for quite general nonlinearities f was established in [1]. Our extra assumptions on f allow us to prove the existence of an infinite number of solutions of (1.3)-(1.5).

For $p \neq 2$, equation (1.3) is degenerate at points where $u' = 0$ and we will see later that in some instances this prevents u from being twice differentiable at some points. We see however that by multiplying (1.3) by r^{N-1}, integrating on $(0, r)$, and using (1.4) we obtain:

$$-r^{N-1}|u'(r)|^{p-2}u'(r) = \int_0^r t^{N-1} f(u(t))\, dt. \tag{1.6}$$

Therefore, instead of seeking solutions of (1.3)-(1.5) in $C^2[0, \infty)$ we will attempt to find $u \in C^1[0, \infty)$ satisfying (1.4)-(1.6).

The type of nonlinearity we are interested in is one for which $F(u) \equiv \int_0^u f(t)\, dt$ has the shape of a "hilltop." We require that $f : [-\delta, \delta] \to \mathbb{R}$ and:

$$f \text{ is odd, there exists } K > 0 \text{ such that } |f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in [-\delta, \delta] \text{ and} \tag{1.7}$$

$$\text{there exists } \beta, \delta \text{ such that } 0 < \beta < \delta \text{ with } f < 0 \text{ on } (0, \beta), f > 0 \text{ on } (\beta, \delta), \text{ and } f(\delta) = 0. \tag{1.8}$$

We also require:

$$\text{there exists } \gamma \text{ with } \beta < \gamma < \delta \text{ such that } F < 0 \text{ on } (0, \gamma) \text{ and } F > 0 \text{ on } (\gamma, \delta). \tag{1.9}$$

Finally we assume:

$$\int_0 \frac{1}{\sqrt[p]{|F(t)|}}\, dt = \infty \text{ if } p > 2 \tag{1.10}$$

and:

$$\int^\delta \frac{1}{\sqrt[p]{F(\delta) - F(t)}}\, dt = \infty \text{ if } p > 2. \tag{1.11}$$

Main Theorem. *Let f be a function satisfying (1.7)-(1.11). Then there exist an infinite number of solutions of (1.4)-(1.6), at least one with each prescribed number of zeros.*

Remark: Assumption (1.8) can be weakened to allow f to have a finite number of zeros, $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \delta$ where $f < 0$ on $(0, \beta_1)$, $f > 0$ on (β_{n-1}, β_n) and we still require assumption (1.9). A key fact that we would then need to prove is that the solution of a certain initial value problem is unique. Sufficient conditions to assure this are (1.10)-(1.11) and the following:

$$\int^{\beta_{l+1}} \frac{1}{\sqrt[p]{F(\beta_{l+1}) - F(t)}}\, dt = \infty \text{ if } p > 2 \text{ and if } f > 0 \text{ on } (\beta_l, \beta_{l+1})$$

and

$$\int_{\beta_l} \frac{1}{\sqrt[p]{F(\beta_l) - F(t)}}\, dt = \infty \text{ if } p > 2 \text{ and if } f < 0 \text{ on } (\beta_l, \beta_{l+1}).$$

Remark: Let $0 < \beta < \delta$ and suppose $q_i \geq 1$ for $i = 1, 2, 3$. If $p > 2$ then also suppose $q_1 \geq p - 1$ and $q_3 \geq p - 1$. Let f be an odd function such that $f(u) = u^{q_1}|u - \beta|^{q_2 - 1}(u - \beta)(\delta - u)^{q_3}$ for $0 < u < \delta$ and suppose $F(\delta) > 0$. Then (1.7)-(1.11) are satisfied and the Main Theorem applies to all such functions f.

Remark: If $1 < p \leq 2$ then it follows from the fact that f is locally Lipschitz that (1.10) and (1.11) are satisfied. Since f is locally Lipschitz at $u = 0$, it follows that $|F(u)| \leq Cu^2$ in some neighborhood of $u = 0$ for some $C > 0$. Then since $1 < p \leq 2$:

$$\int_0 \frac{1}{\sqrt[p]{|F(t)|}}\, dt \geq \frac{1}{C^{\frac{1}{p}}} \int_0 \frac{1}{t^{\frac{2}{p}}} = \infty.$$

A similar argument shows that (1.11) also holds for $1 < p \leq 2$.

2. Existence, Uniqueness, and Continuity

We denote $C(S) = \{f : S \to \mathbb{R} \,|\, f \text{ is continuous on } S.\}$

Let f be locally Lipschitz and let $d \in \mathbb{R}$ with $|d| \le \delta$. Denote $u(r,d)$ as a solution of the initial value problem:

$$-r^{N-1}|u'(r)|^{p-2}u'(r) = \int_0^r t^{N-1} f(u(t))\,dt. \tag{2.1}$$

$$u(0) = d. \tag{2.2}$$

We will show using the contraction mapping principle that a solution of (2.1)-(2.2) exists.

For $p > 1$ we denote $\Phi_p(x) = |x|^{p-2}x$. Note that Φ_p is continuous for $p > 1$ and $\Phi_p^{-1} = \Phi_{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. For future reference we note that $\Phi_p'(x) = (p-1)|x|^{p-2}$ and $|\Phi_p(x)| = |x|^{p-1}$.

We rewrite (2.1) as:

$$-u' = \frac{1}{r^{\frac{N-1}{p-1}}} \Phi_{p'}[\int_0^r t^{N-1} f(u(t))\,dt]. \tag{2.3}$$

Integrating on $(0, r)$ and using (2.2) gives:

$$u = d - \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} \Phi_{p'}[\int_0^t s^{N-1} f(u(s))\,ds]\,dt. \tag{2.4}$$

Thus we see that solutions of (2.1)-(2.2) are fixed points of the mapping:

$$Tu = d - \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} \Phi_{p'}[\int_0^t s^{N-1} f(u(s))\,ds]\,dt. \tag{2.5}$$

Lemma 2.1. *Let f be locally Lipschitz and let d be a real number such that $|d| \le \delta$. Then there exists a solution $u \in C^1[0,\epsilon)$ of (2.1)-(2.2) for some $\epsilon > 0$. In addition, $u'(0) = 0$.*

Proof.

First, if $f(d) = 0$ then $u \equiv d$ is a solution of (2.1)-(2.2) and $u'(0) = 0$.

So we now assume that $f(d) \ne 0$. Denote $B_R^{\epsilon}(d) = \{u \in C[0,\epsilon) \text{ such that } \|u - d\| < R\}$ where $\|\cdot\|$ is the supremum norm. We will now show that if $\epsilon > 0$ and $R > 0$ are small enough then $T : B_R^{\epsilon}(d) \to B_R^{\epsilon}(d)$ and that T is a contraction mapping. Since f is bounded on $[\frac{|d|}{2}, \frac{|d|+\delta}{2}]$, say by M, it follows from (2.5) that:

$$|Tu - d| \le \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} (\frac{Mt^N}{N})^{\frac{1}{p-1}} = (\frac{p-1}{p})(\frac{M}{N})^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \le (\frac{p-1}{p})(\frac{M}{N})^{\frac{1}{p-1}} \epsilon^{\frac{p}{p-1}}.$$

Therefore we see that $\|Tu - d\| < R$ if ϵ is chosen small enough and hence $T : B_R^{\epsilon}(d) \to B_R^{\epsilon}(d)$ for ϵ small enough.

Next by the mean value theorem we see that for some h with $0 < h < 1$ we have:

$$|\Phi_{p'}[\int_0^t s^{N-1} f(u(s))\,ds] - \Phi_{p'}[\int_0^t s^{N-1} f(v(s))\,ds]| =$$

$$\frac{1}{p-1} |\int_0^t s^{N-1}[hf(u) + (1-h)f(v)]\,ds|^{\frac{2-p}{p-1}} |\int_0^t s^{N-1}[f(u) - f(v)]\,ds|. \tag{2.6}$$

Case 1: $1 < p \le 2$

Using again that f is bounded on $[d-1, d+1]$ by M and that the local Lipschitz constant is K (i.e. for $u, v \in B_1^\epsilon(d)$ we have $|f(u) - f(v)| \leq K|u - v|$) we obtain by (2.5)-(2.6):

$$\begin{aligned}\|Tu - Tv\| &\leq \frac{K}{p-1}\|u - v\| \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} M^{\frac{2-p}{p-1}} (\frac{t^N}{N})^{\frac{2-p}{p-1}} \frac{t^N}{N} \\ &= C_1 \|u - v\| \int_0^r t^{\frac{1}{p-1}}\, dt \leq C_2 \epsilon^{\frac{p}{p-1}} \|u - v\|\end{aligned}$$

where C_1, C_2 are constants depending only on $p, N, K,$ and M.

Case 2: $p > 2$

Since $f(d) \neq 0$ and f is continuous we may choose R small enough so that:

$$L \equiv \min_{[d-R, d+R]} |f| > 0.$$

Therefore,

$$|\int_0^t s^{N-1}[hf(u) + (1-h)f(v)]\, ds| \geq \frac{Lt^N}{N}. \tag{2.7}$$

Thus, by (2.5)-(2.7) we have

$$\begin{aligned}\|Tu - Tv\| &\leq \frac{K}{p-1}(\frac{L}{N})^{\frac{2-p}{p-1}} \|u - v\| \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} t^{\frac{N(2-p)}{p-1}} \frac{t^N}{N}\, dt \\ &= \frac{K}{(p-1)} \frac{1}{N^{\frac{1}{p-1}} L^{\frac{p-2}{p-1}}} \|u - v\| \int_0^r t^{\frac{1}{p-1}}\, dt \leq C_3 \epsilon^{\frac{p}{p-1}} \|u - v\|\end{aligned}$$

where C_3 depends only on $p, N, K,$ and M.

Therefore in both cases we see that T is a contraction for R and ϵ small enough. Thus by the contraction mapping principle, there is a *unique* $u \in C[0, \epsilon_1)$ such that $Tu = u$. That is, there is a continuous function u such that u satisfies (2.4) on $[0, \epsilon_1)$ for some $\epsilon_1 > 0$. In addition, since $f(d) \neq 0$ we see that the right hand side of (2.4) is continuously differentiable on $(0, \epsilon)$ for some ϵ with $0 < \epsilon \leq \epsilon_1$ and therefore $u \in C^1(0, \epsilon)$. Also, subtracting d from (2.4), dividing by r, and taking the limit as $r \to 0^+$ gives $u'(0) = 0$. Finally, dividing (2.1) by r^{N-1} and taking the limit as $r \to 0^+$ we see that $\lim_{r \to 0^+} u'(r) = 0$. Therefore, $u \in C^1[0, \epsilon)$. $\square$

Note we see from (2.3) that $u \in C^2$ at all points where $u' \neq 0$.

If $u'(r_0) = 0$ then using (2.1) we obtain:

$$-|u'(r)|^{p-2}u'(r) = \frac{1}{r^{N-1}} \int_{r_0}^r t^{N-1} f(u(t))\, dt.$$

It then follows that:

$$\lim_{r \to r_0} \frac{|u'(r)|^{p-2}u'(r)}{r - r_0} = \begin{cases} -\frac{f(u(r_0))}{N} & \text{if } r_0 = 0 \\ -f(u(r_0)) & \text{if } r_0 > 0. \end{cases} \tag{2.8}$$

Remark: If $1 < p \leq 2$ then we see from (2.8) that $u''(r_0)$ exists and rewriting (1.3) as:

$$(p-1)u'' + \frac{N-1}{r}u' + |u'|^{2-p} f(u) = 0,$$

we see that $u \in C^2[0, \epsilon)$.

Remark: If $p > 2$ then u *might not* be twice differentiable at points where $u' = 0$. In fact if $u'(r_0) = 0$ and $f(u(r_0)) \neq 0$ then by (2.8) we see that $\lim_{r \to r_0} |\frac{u'(r)}{r - r_0}| = \infty$ and so u is *not* twice differentiable at r_0.

Lemma 2.2. *Let f satisfy (1.7)-(1.9). If u is a solution of the initial value problem (2.1)-(2.2) with $|d| \le \delta$ on some interval $(0, R)$ with $R \le \infty$, then:*

$$F(u) \le F(d) \text{ on } (0, R) \tag{2.9}$$

and

$$\frac{p-1}{p}|u'|^p \le F(d) + |F(\beta)| \le F(\delta) + |F(\beta)| \text{ on } (0, R). \tag{2.10}$$

Proof.

We define the "energy" of a solution as:

$$E = \frac{p-1}{p}|u'|^p + F(u). \tag{2.11}$$

Differentiating E and using (2.1) gives:

$$E' = -\frac{N-1}{r}|u'|^p \le 0. \tag{2.12}$$

Integrating this on $(0, r)$ and using (1.8) gives:

$$\frac{p-1}{p}|u'|^p + F(u) = E \le E(0) = F(d) \le F(\delta) \text{ for } r > 0. \tag{2.13}$$

Inequalities (2.9)-(2.10) follow from (1.8)-(1.9) and (2.13).

Now by (1.9) we know that F is negative on $(0, \gamma)$ and by (1.8) we know that F is increasing on (β, δ). Therefore if $|d| < \delta$ then $F(d) < F(\delta)$. On the other hand if $|u(r_0)| = \delta$ for some $r_0 > 0$ then by (2.9) $F(\delta) \le F(d)$ - a contradiction. Hence if $|d| < \delta$ then $|u| < \delta$. $\square$

Lemma 2.3. *Let f satisfy (1.7)-(1.9). Let d be a real number such that $|d| \le \delta$. Then a solution of (2.1)-(2.2) exists on $[0, \infty)$.*

Proof.

If $|d| = \delta$ then $u \equiv d$ is a solution on $[0, \infty)$ and so we now suppose that $|d| < \delta$.

Let $[0, R)$ be the maximal interval of existence for a solution of (2.1)-(2.2). From lemma 2.1 we know that $R > 0$. Now suppose that $R < \infty$. By lemma 2.2, it follows that u and u' are uniformly bounded by $M = \delta + F(\delta) + |F(\beta)|$ on $[0, R)$. Therefore by the mean value theorem $|u(x) - u(y)| \le M|x - y|$ for all $x, y \in [0, R)$.
Thus, there exists $b_1 \in \mathbb{R}$ such that:

$$\lim_{r \to R^-} u(r) = b_1.$$

By (2.3) there exists $b_2 \in \mathbb{R}$ such that:

$$\lim_{r \to R^-} u'(r) = b_2.$$

If $b_2 \ne 0$ we can apply the standard existence theorem for ordinary differential equations and extend our solution of (2.1)-(2.2) to $[0, R + \epsilon)$ for some $\epsilon > 0$ contradicting the maximality of $[0, R)$.

If $b_2 = 0$ and $f(b_1) \ne 0$ we can again apply the contraction mapping principle as we did in lemma 2.1 to extend our solution of (2.1)-(2.2) to $[0, R + \epsilon)$ for some $\epsilon > 0$ contradicting the maximality of $[0, R)$.

Finally, if $b_2 = 0$ and $f(b_1) = 0$, we can extend our solution by defining $u(r) \equiv b_1$ for $r > R$ contradicting the maximality of $[0, R)$.

Thus in each of these cases we see that R cannot be finite and so a solution of (2.1)-(2.2) exists on $[0, \infty)$. $\square$

Lemma 2.4. *Let f satisfy (1.7)-(1.10). Let d be a real number such that $|d| < \delta$. Then there is a unique solution of (2.1)-(2.2) on $[0, \infty)$.*

Proof.

Case 1: $d = \pm\beta$

In this case we have $E(0) = F(\beta)$ (recall that F is even) and since $E' \leq 0$ (by (2.12)) we have $E(r) \leq E(0) = F(\beta)$ for $r \geq 0$. On the other hand, F has a minimum at $u = \pm\beta$ and so we see that $E(r) = \frac{p-1}{p}|u'|^p + F(u) \geq F(\beta)$. Thus $E \equiv F(\beta)$. Thus, $-\frac{N-1}{r}|u'|^p = E' \equiv 0$ and hence $u(r) \equiv \pm\beta$.

Case 2: $d = 0$.

Here we have $E(0) = 0$ and since $E' \leq 0$ we have $E(r) \leq 0$ for $r \geq 0$.

Let $r_1 = \sup\{\, r \geq 0 \,|\, E(r) = 0\}$. If $r_1 = \infty$ then $u(r) \equiv 0$.

So suppose $r_1 < \infty$. If $r_1 = 0$ then we have $u(r_1) = 0$ and $u'(r_1) = 0$.

If $r_1 > 0$ then since $E' \leq 0$ we have $E(r) \equiv 0$ on $[0, r_1]$ hence $-\frac{N-1}{r}|u'|^p = E' \equiv 0$ and so $u \equiv 0$ on $[0, r_1]$. Therefore we also have $u(r_1) = 0$ and $u'(r_1) = 0$.

Now using (2.1) we obtain:

$$-r^{N-1}|u'|^{p-2}u' = \int_{r_1}^{r} t^{N-1} f(u)\, dt. \tag{2.15}$$

Since:

$$\frac{p-1}{p}|u'|^p + F(u) = E(r) < E(0) = 0 \text{ for } r > r_1, \tag{2.16}$$

it follows that $|u(r)| > 0$ for $r > r_1$. Combining this with the fact that $u(r_1) = 0$, we see that there exists an $\epsilon > 0$ such that $0 < |u(r)| < \beta$ for $r_1 < r < r_1 + \epsilon$. By (1.8) it follows that $|f(u)| > 0$ for $r_1 < r < r_1 + \epsilon$. Therefore, by (2.15) we see that $|u'| > 0$ for $r_1 < r < r_1 + \epsilon$. Using this fact and rewriting (2.16) we see that:

$$\frac{|u'|}{\sqrt[p]{|F(u)|}} < (\frac{p}{p-1})^{\frac{1}{p}} \text{ for } r_1 < r < r_1 + \epsilon. \tag{2.17}$$

Integrating (2.17) on $(r_1, r_1 + \epsilon)$, using (1.10), and that F is even gives:

$$\infty = \int_0^{|u(r_1+\epsilon)|} \frac{1}{\sqrt[p]{|F(t)|}}\, dt = \int_{r_1}^{r_1+\epsilon} \frac{|u'|}{\sqrt[p]{|F(u)|}} \leq (\frac{p}{p-1})^{\frac{1}{p}}\epsilon,$$

a contradiction. Thus we see that $r_1 = \infty$ and hence $u \equiv 0$.

Case 3: $f(d) \neq 0$.

We saw that the mapping T defined in lemma 2.1 is a contraction mapping. Therefore, T has a *unique* fixed point so that if u_1 and u_2 are solutions of (2.1)-(2.2) then there exists an $\epsilon > 0$ such that $u_1(r) \equiv u_2(r)$ on $[0, \epsilon)$. Let $[0, R)$ be the maximal half-open interval such that $u_1(r) \equiv u_2(r)$ on $[0, R)$. By continuity, $u_1(r) \equiv u_2(r)$ on $[0, R]$ and $u_1'(r) \equiv u_2'(r)$ on $[0, R]$.

As in the proof of lemma 2.3, if $u_1'(R) \neq 0$ then it follows from the standard existence-uniqueness theorem of ordinary differential equations that $u_1(r) \equiv u_2(r)$ on $[0, R + \epsilon)$ for some $\epsilon > 0$ contradicting the maximality of $[0, R)$.

If $u_1'(R) = 0$ and $f(u_1(R)) \neq 0$ then we can again apply the contraction mapping principle as in lemma 2.1 and show that $u_1(r) \equiv u_2(r)$ on $[0, R + \epsilon)$ for some $\epsilon > 0$ contradicting the maximality of $[0, R)$.

If $u_1'(R) = 0$ and $u_1(R) = \beta$ then as in Case 1 above we can show that $u_1(r) \equiv \beta$ for $r > R$ and $u_2(r) \equiv \beta$ for $r > R$. This contradicts the definition of R. A similar argument applies if $u_1'(R) = 0$ and $u_1(R) = -\beta$.

Finally, if $u_1'(R) = 0$ and $u_1(R) = 0$, then as in Case 2 above we can show that $u_1(r) \equiv 0$ for $r > R$ and $u_2(r) \equiv 0$ for $r > R$. This contradicts the definition of R.

Thus we see that in all cases we have $R = \infty$. This completes the proof. $\square$

Remark: Without assumptions (1.10) and (1.11), solutions of the initial value problem (2.1)-(2.2) are *not necessarily unique!* For example, let $f(u) = -|u|^{q-1}u$ where $1 \le q < p-1$. In addition to $u \equiv 0$,

$$u = C(p,q,N)r^{\frac{p}{p-1-q}}$$

where $C(p,q,N) = [\frac{(p-1-q)^p}{p^{p-1}[pq+N(p-1-q)]}]^{\frac{1}{p-1-q}}$ is also a solution of (2.1)-(2.2) with $u(0) = 0$ and $u'(0) = 0$. Note however that $\int_0 \frac{1}{\sqrt[p]{|F(t)|}}\,dt = \int_0 \frac{(q+1)^{\frac{1}{p}}}{t^{\frac{q+1}{p}}}\,dt < \infty$ since $1 \le q < p-1$. Similarly, if $f(u) = -|\delta - u|^{q-1}(\delta - u)$ and $1 \le q < p-1$ then $u \equiv \delta$ and

$$u = \delta - C(p,q,N)r^{\frac{p}{p-1-q}}$$

(with the same $C(p,q,N)$ as earlier) are both solutions of (2.1)-(2.2) but (1.11) is not satisfied.

Lemma 2.5. *Let u be a solution of (2.1)-(2.2) with $\gamma < d < \delta$ and suppose there exists an $r_1 > 0$ such that $u(r_1) = 0$. If (1.10) holds then $u'(r_1) \neq 0$.*

Proof.

This proof is from [1].

Suppose by way of contradiction that $u(r_1) = 0$ and $u'(r_1) = 0$. It follows that $E(r_1) = 0$. (In fact, it follows from lemma 2.4 that $u \equiv 0$ on $[r_1, \infty)$). Now let $r_0 = \inf\{r \le r_1 \,|\, E(r) = 0\}$. Since E is continuous, decreasing, and $E(0) = F(d) > 0$ we see that $r_0 > 0$ and that $E(r) > 0$ for $0 \le r < r_0$.
If $r_0 < r_1$ then $E(r) \equiv 0$ on (r_0, r_1) and thus $-\frac{N-1}{r}|u'|^p = E'(r) \equiv 0$ on (r_0, r_1). Therefore $u \equiv 0$ on (r_0, r_1) and thus $u(r_0) = u'(r_0) = 0$.

Integrating (2.12) on (r, r_0) and using that $E(r_0) = 0$ gives:

$$\frac{p-1}{p}|u'|^p + F(u) = \int_r^{r_0} \frac{N-1}{r}|u'|^p\,dt. \tag{2.18}$$

Letting $w = \int_r^{r_0} \frac{N-1}{r}|u'|^p\,dt$, we see that $w' = -\frac{N-1}{r}|u'|^p$. Thus (2.18) becomes:

$$w' + \frac{\alpha}{r}w = \frac{\alpha}{r}F(u) \text{ where } \alpha = \frac{p(N-1)}{p-1}. \tag{2.19}$$

By (1.9) it follows that there is an ϵ with $0 < \epsilon < \frac{1}{2}r_0$ such that $F(u(r)) \le 0$ on $(r_0 - \epsilon, r_0)$. and so solving the first order linear equation (2.19) gives:

$$w = \frac{\alpha}{r^\alpha}\int_r^{r_0} t^{\alpha-1}|F(u)|\,dt \text{ for } r_0 - \epsilon < r < r_0.$$

Rewriting (2.18) we obtain:

$$|u'|^p = \frac{p}{p-1}[|F(u)| + \frac{\alpha}{r^\alpha}\int_r^{r_0} t^{\alpha-1}|F(u(t))|\,dt] \text{ for } r_0 - \epsilon < r < r_0. \tag{2.20}$$

In addition, since $E(r) > 0$ for $r < r_0$, we see that:

$$|u'| > (\frac{p}{p-1})^{\frac{1}{p}}\sqrt[p]{|F(u)|} \ge 0 \text{ for } r_0 - \epsilon < r < r_0.$$

Thus u is monotone on $(r_0-\epsilon, r_0)$.
Since $F' = f < 0$ on $(0,\beta)$ (by (1.8)) we see that:

$$|F(u(t))| < |F(u(r))| \text{ for } r_0-\epsilon < r < t < r_0. \tag{2.21}$$

Substituting (2.21) into (2.20) gives:

$$|u'|^p \le (\frac{p}{p-1})\frac{r_0^\alpha}{r^\alpha}|F(u)| \le (\frac{p}{p-1})(\frac{r_0}{r_0-\epsilon})^\alpha |F(u)| \le 2^\alpha(\frac{p}{p-1})|F(u)| \text{ for } r_0-\epsilon < r < r_0.$$

Finally, dividing by $|F(u)|$, taking roots, integrating on (r, r_0), and using (1.10) we obtain:

$$\infty = \int_0^{|u(r)|} \frac{1}{\sqrt[p]{|F(t)|}}\,dt \le 2^{\frac{N-1}{p-1}}(\frac{p}{p-1})^{\frac{1}{p}}(r_0-r)$$

a contradiction. Thus $u'(r_1)\neq 0$ and this completes the proof. □

Lemma 2.6. *Let u be a solution of (2.1)-(2.2) where $\gamma < d < \delta$. Then $u' < 0$ on a maximal nonempty open interval $(0, M_{d,1})$, where either:*

(a) $M_{d,1} = \infty$, $\lim_{r\to\infty} u'(r) = 0$, $\lim_{r\to\infty} u(r) = L$ where $|L| < d$ and $f(L) = 0$,

or

(b) $M_{d,1}$ is finite, $u'(M_{d,1}) = 0$, and $f(u(M_{d,1})) \le 0$.

In either case, it follows that there exists a unique (finite) number $\tau_d \in (0, M_{d,1})$ such that $u(\tau_d) = \gamma$ and $u' < 0$ on $(0, \tau_d]$.

Proof.

From (2.8) we have:

$$\lim_{r\to 0^+}\frac{|u'(r)|^{p-2}u'(r)}{r} = -\frac{f(d)}{N}.$$

For $\gamma < d < \delta$ the right hand side of the above equation is negative by (1.8). Hence for small values of $r > 0$ we see that $u(r,d)$ is decreasing.

If u is not everywhere decreasing, then there is a first critical point, $r = M_{d,1} > 0$, with $u'(M_{d,1}) = 0$ and $u' < 0$ on $(0, M_{d,1})$. From (2.1) we have:

$$r^{N-1}|u'(r)|^{p-2}u'(r) = \int_r^{M_{d,1}} t^{N-1} f(u(t))\,dt.$$

If $f(u(M_{d,1})) > 0$ then the above equation implies $u' > 0$ for $r < M_{d,1}$ and r sufficiently close to $M_{d,1}$ which contradicts that $u' < 0$ on $(0, M_{d,1})$. Therefore $f(u(M_{d,1})) \le 0$ and so $u(M_{d,1}) \le \beta < \gamma$. Thus, there exists $\tau_d \in (0, M_{d,1})$ with the stated properties.

On the other hand, suppose that $u(r)$ is decreasing for all $r > 0$. We showed in lemma 2.2 that $|u(r)| < d < \delta$ for $r > 0$. Thus $\lim_{r\to\infty} u(r) = L$ with $|L| \le d < \delta$.
Dividing (2.1) by r^N and taking limits as $r\to\infty$ we see that:

$$\lim_{r\to\infty}\frac{|u'|^{p-2}u'}{r} = -\frac{f(L)}{N}. \tag{2.22}$$

We know from (2.10) that u' is bounded for all $r \geq 0$ and so the limit of the left hand side of (2.22) is 0. Thus $f(L) = 0$ and since $|L| \leq d < \delta$ we see that $L = -\beta, 0$, or β. Thus there exists a (finite) τ_d with the stated properties.

Finally, the fact that $\lim_{r\to\infty} u'(r) = 0$ can be seen as follows. In lemma 2.2 we saw that the energy $E(r) = \frac{p-1}{p}|u'(r)|^p + F(u(r))$ is decreasing and bounded below by $F(\beta)$, therefore $\lim_{r\to\infty} E(r)$ exists. Since $\lim_{r\to\infty} u(r) = L$, we see that $\lim_{r\to\infty} F(u(r)) = F(L)$. Also, since $\frac{p-1}{p}|u'(r)|^p = E(r) - F(u(r))$ and both $E(r)$ and $F(u(r))$ have a limit as $r \to \infty$, it follows that $|u'|$ has a limit as $r \to \infty$. This limit must be zero since u is bounded. This completes the proof. □

Lemma 2.7. *Suppose $\gamma < d^* < \delta$. Then $\lim_{d\to d^*} u(r, d) = u(r, d^*)$ uniformly on compact subsets of $\mathbb{R}$ and $\lim_{d\to d^*} u'(r, d) = u'(r, d^*)$ uniformly on compact subsets of $\mathbb{R}$. Further, if (1.11) holds then $\lim_{d\to\delta^-} u(r, d) = \delta$ uniformly on compact subsets of $\mathbb{R}$.*

Proof.

If not, then there exists an $\epsilon_0 > 0$, a compact set K, and sequences $r_j \in K$, d_j with $\gamma < d_j < \delta$ and $\lim_{j\to\infty} d_j = d^*$ such that

$$|u(r_j, d_j) - u(r_j, d^*)| \geq \epsilon_0 > 0 \text{ for every } j. \tag{2.23}$$

However, by lemma 2.2 we know that $|u(r, d_j)| < \delta$ and $|u'(r, d_j)| \leq (\frac{p}{p-1})^{\frac{1}{p}}[F(\delta) + |F(\beta)|]^{\frac{1}{p}}$ for all j so that by the Arzela-Ascoli theorem there is a subsequence of the d_j (still denote d_j) such that $u(r, d_j)$ converges uniformly on K to a function $u(r)$ and $|u(r)| \leq \delta$. From (2.3) we see that $u'(r, d_j)$ converges uniformly on K a function $v(r)$ where $-v = \frac{1}{r^{\frac{N-1}{p-1}}}\Phi_{p'}[\int_0^r t^{N-1} f(u(t))\, dt]$.

Taking limits in the equation $u(r, d_j) = d_j + \int_0^r u'(s, d_j)\, ds$, we see that $u(r) = d + \int_0^r v(s)\, ds$. Hence $u'(r) = v(r)$, that is $-u' = \frac{1}{r^{\frac{N-1}{p-1}}}\Phi_{p'}[\int_0^r t^{N-1} f(u(t))\, dt]$, and thus u is a solution of (2.1)-(2.2) with $d = d^*$. So by lemma 2.4, $u(r) = u(r, d^*)$. Therefore, given $\epsilon = \epsilon_0 > 0$ and the compact set K we see that for all $r \in K$ we have:

$$|u(r, d_j) - u(r, d^*)| < \epsilon_0$$

which contradicts (2.23). This completes the proof of the first part of the theorem.

An identical argument shows that $\lim_{d\to\delta^-} u(r, d) = u(r)$ uniformly on compact sets where $|u(r)| \leq \delta$ and u solves (2.1)-(2.2) with $d = \delta$. To complete the proof we need to show $u(r) \equiv \delta$. Let $r_1 = \sup\{\, r \geq 0 \mid E(r) = E(0) = F(\delta)\}$. Since E is decreasing we see that if $r_1 = \infty$ then E is constant and hence $u \equiv \delta$ and we are done.

Therefore we suppose $r_1 < \infty$.

By the definition of r_1 we have:

$$\frac{p-1}{p}|u'|^p + F(u) = E(r) < E(0) = F(\delta) \text{ for } r > r_1. \tag{2.24}$$

Thus, it follows that $u(r) < \delta$ for $r > r_1$. Also by (1.8) it follows that $f(u) > 0$ for $r_1 < r < r_1 + \epsilon$ for some $\epsilon > 0$. Therefore, by (2.15) we see that $u' < 0$ for $r_1 < r < r_1 + \epsilon$. Using this fact and rewriting (2.24) we see that:

$$\frac{-u'}{\sqrt[p]{F(\delta) - F(u)}} < (\frac{p}{p-1})^{\frac{1}{p}} \ \text{ for } r_1 < r < r_1 + \epsilon. \tag{2.25}$$

Integrating (2.25) on $(r_1, r_1 + \epsilon)$ and using (1.11) gives:

$$\infty = \int_{u(r_1+\epsilon)}^{\delta} \frac{1}{\sqrt[p]{F(\delta) - F(t)}}\, dt = \int_{r_1}^{r_1+\epsilon} \frac{-u'}{\sqrt[p]{F(\delta) - F(u)}} \leq (\frac{p}{p-1})^{\frac{1}{p}}\epsilon,$$

a contradiction. Hence $r_1 = \infty$ and $u \equiv \delta$. □

3. Energy Estimates

From lemma 2.6 we saw for $\gamma < d < \delta$ that $u(r,d)$ is decreasing on $[0,\tau_d]$. Therefore $u^{-1}(y,d)$ exists for $\gamma \le y \le d$.

Lemma 3.1. *For $\gamma \le y < d < \delta$ we have:*

$$\lim_{d\to\delta^-} u^{-1}(y,d) = \infty.$$

Note: In particular this implies that $\tau_d \to \infty$ as $d \to \delta^-$ since $u^{-1}(\gamma,d) = \tau_d$.

Proof.

We fix y_0 with $\gamma \le y_0 < d$ and suppose by way of contradiction that there exists d_k with $d_k < \delta$ and $d_k \to \delta$, $u^{-1}(y_0,d_k) = b_k$, and that the b_k are bounded.
Then there is a subsequence of the b_k (still denote b_k) such that $b_k \to b_0$ for some real number b. By lemma 2.2 we have that $|u(r,d_k)|$ and $|u'(r,d_k)|$ are uniformly bounded on say $[0,b+1]$. Thus by lemma 2.7, $\lim_{k\to\infty} u(r,d_k) = \delta$ *uniformly* on $[0,b+1]$. On the other hand, $y_0 = \lim_{k\to\infty} u(b_k,d_k) = \delta$ - a contradiction since $y_0 < d < \delta$. □

Lemma 3.2.

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{d-y}{[F(d)-F(y)]^{\frac{1}{p}}} \le \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_y^d \frac{dt}{[F(d)-F(t)]^{\frac{1}{p}}} \le u^{-1}(y,d) \text{ for } \gamma < y < d.$$

Proof.

Rewriting (2.13) gives:

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{|u'(r,d)|}{[F(d)-F(u(r,d))]^{\frac{1}{p}}} \le 1. \tag{3.1}$$

Since $u'(r,d) < 0$ on $(0,\tau_d)$, integrating (3.1) on $(0,r)$ where $0 < r \le \tau_d$ we obtain:

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{u(r,d)}^d \frac{dt}{[F(d)-F(t)]^{\frac{1}{p}}} \le r.$$

Denoting $y = u(r,d)$ and using the fact that $F' = f > 0$ on (γ,δ) we obtain:

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{d-y}{[F(d)-F(y)]^{\frac{1}{p}}} \le \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_y^d \frac{dt}{[F(d)-F(t)]^{\frac{1}{p}}} \le u^{-1}(y,d). \tag{3.2}$$

This completes the proof. □

Lemma 3.3.

$$\lim_{d\to\delta^-} [E(0) - E(\tau_d)] = 0.$$

Integrating (2.12) on $(0,\tau_d)$ gives:

$$E(0) - E(\tau_d) = \int_0^{\tau_d} \frac{N-1}{t} |u'(t,d)|^p \, dt.$$

Using (2.13) we obtain:

$$E(0) - E(\tau_d) \le \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} (N-1) \int_0^{\tau_d} \frac{1}{t} [F(d) - F(u(t,d))]^{\frac{p-1}{p}} |u'(t,d)| \, dt.$$

Now changing variables with $y = u(t,d)$ we obtain:

$$E(0) - E(\tau_d) \le (\frac{p}{p-1})^{\frac{p-1}{p}}(N-1)\int_\gamma^d \frac{[F(d)-F(y)]^{\frac{p-1}{p}}}{u^{-1}(y,d)}\,dy. \tag{3.3}$$

Since $[F(d)-F(y)]^{\frac{p-1}{p}} \le F(\delta)^{\frac{p-1}{p}}$ for $\gamma \le y \le d$ we see by lemma 3.1 that:

$$\lim_{d\to\delta^-} \frac{[F(d)-F(y)]^{\frac{p-1}{p}}}{u^{-1}(y,d)} = 0 \text{ for } \gamma \le y < d. \tag{3.4}$$

Also, by (3.2) and the mean value theorem we see that:

$$\int_\gamma^d \frac{[F(d)-F(y)]^{\frac{p-1}{p}}}{u^{-1}(y,d)}\,dy \le (\frac{p}{p-1})^{\frac{1}{p}}\int_\gamma^d \frac{F(d)-F(y)}{d-y}\,dy \le (\frac{p}{p-1})^{\frac{1}{p}}(\delta-\gamma)\max_{[\gamma,\delta]} f.$$

Therefore by (3.4) and the dominated convergence theorem it follows that:

$$\lim_{d\to\delta^-}\int_\gamma^d \frac{[F(d)-F(y)]^{\frac{p-1}{p}}}{u^{-1}(y,d)}\,dy = 0.$$

Therefore by (3.3):

$$\lim_{d\to\delta^-}[E(0)-E(\tau_d)] = 0.$$

This completes the proof. □

Lemma 3.4. *Suppose u is monotonic on (τ_d, t). Then*

$$E(\tau_d) - E(t) \le \frac{C}{\tau_d}$$

where $C = 2\delta(N-1)(\frac{p}{p-1})^{\frac{p-1}{p}}[F(\delta)+|F(\beta)|]^{\frac{p-1}{p}}$. (Note that C is independent of d).

Proof.

Integrating (2.12) on (τ_d, t), estimating, and using (2.13) gives:

$$E(\tau_d)-E(t) = \int_{\tau_d}^t \frac{N-1}{s}|u'|^p\,ds \le \frac{N-1}{\tau_d}\int_{\tau_d}^t |u'|^{p-1}|u'|\,ds$$

$$\le \frac{N-1}{\tau_d}(\frac{p}{p-1})^{\frac{p-1}{p}}\int_{\tau_d}^t [F(\delta)-F(u)]^{\frac{p-1}{p}}|u'|\,ds = \frac{N-1}{\tau_d}(\frac{p}{p-1})^{\frac{p-1}{p}}\int_{u(t)}^\gamma [F(\delta)-F(t)]^{\frac{p-1}{p}}\,dt$$

$$\le \frac{2\delta(N-1)}{\tau_d}(\frac{p}{p-1})^{\frac{p-1}{p}}[F(\delta)+|F(\beta)|]^{\frac{p-1}{p}} = \frac{C}{\tau_d}$$

where $C = 2\delta(N-1)(\frac{p}{p-1})^{\frac{p-1}{p}}[F(\delta)+|F(\beta)|]^{\frac{p-1}{p}}$.

This completes the proof. □

Lemma 3.5. *Suppose $\gamma < d^* < \delta$. Let $u(r,d^*)$ be a solution of (2.1)-(2.2) with k zeros and suppose $\lim_{r\to\infty} u(r,d^*) = 0$. Then for d sufficiently close to d^*, $u(r,d)$ has at most $k+1$ zeros.*

Proof.

From (2.12) we know that $E'(r,d^*) \le 0$ and since E is bounded from below by $F(\beta)$, we see that $\lim_{r\to\infty} E(r,d^*)$ exists. Also by assumption $\lim_{r\to\infty} u(r,d^*) = 0$ and since F is continuous we have $\lim_{r\to\infty} F(u(r,d^*)) = 0$. Since $\frac{p-1}{p}|u'(r,d^*)|^p = E(r,d^*) - F(u(r,d^*))$ and the limits of both terms on the right hand side of this equation exist as $r \to \infty$ we see that $\lim_{r\to\infty} |u'(r,d^*)|$ exists and since by assumption $\lim_{r\to\infty} u(r,d^*) = 0$ (so that $u(r,d^*)$ is bounded) we therefore must have:

$$\lim_{r\to\infty} u'(r,d^*) = 0. \tag{3.5}$$

Combining (3.5) with the assumption that $\lim_{r\to\infty} u(r,d^*) = 0$, we see by (2.11) that:

$$\lim_{r\to\infty} E(r,d^*) = 0. \tag{3.6}$$

Combining (3.6) with the fact that $E'(r,d^*) \le 0$, we see that $E(r,d^*) \ge 0$ for all $r \ge 0$.

Claim.

$$E(r,d^*) > 0 \text{ for all } r \ge 0. \tag{3.7}$$

Proof of claim. First note that $E(0,d^*) = F(d^*) > 0$. Now suppose $E(r_0,d^*) = 0$ for some $r_0 > 0$. Then from (3.6) and the fact that E is decreasing it then follows that $E \equiv 0$ on $[r_0,\infty)$. Thus, $-\frac{N-1}{r}|u'|^{p-1} = E' \equiv 0$ on $[r_0,\infty)$. Therefore $u(r,d^*) \equiv u(r_0,d^*)$ for $r \ge r_0$ and since $\lim_{r\to\infty} u(r,d^*) = 0$ we see that $u(r,d^*) \equiv 0$ for $r \ge r_0$. This implies $u'(r_0,d^*) = 0$. However, by lemma 2.5 $u'(r_0,d^*) \ne 0$ - a contradiction. This completes the proof of the claim.

By assumption $u(r,d^*)$ has k zeros. Let us denote the kth zero of $u(r,d^*)$ as y^*. Henceforth we assume without loss of generality that $u(r,d^*) > 0$ for $r > y^*$. By (3.7) we see that $\frac{p-1}{p}|u'(y^*,d^*)|^p = E(y^*,d^*) > 0$. Also since $\lim_{r\to\infty} u(r,d^*) = 0$ it follows that there exists an $M^* > y^*$ such that $u'(M^*,d^*) = 0$. Again by (3.7) we see that $F(u(M^*,d^*)) = E(M^*,d^*) > 0$ which implies $u(M^*,d^*) > \gamma$. Now by (2.1) we obtain:

$$-r^{N-1}|u'(r,d^*)|^{p-1}u'(r,d^*) = \int_{M^*}^{r} s^{N-1} f(u(s,d^*))\,ds.$$

By (1.8) we have $f(u(M^*,d^*)) > 0$, so from the above equation we see that $u(r,d^*)$ is decreasing for $r > M^*$ as long as $u(r,d^*)$ remains greater than β. In particular, since $\lim_{r\to\infty} u(r,d^*) = 0$, we see that there exists s^*, t^* with $M^* < s^* < t^*$ such that $u(s^*) = \frac{u(M^*)+\gamma}{2}$ and $u(t^*) = \gamma$.

Now let d_n be any sequence such that $\lim_{n\to\infty} d_n = d^*$. Then by lemmas 2.4 and 2.7, for some subsequence of d_n (still denoted d_n) we see that $u(r,d_n)$ converges uniformly on compact sets to $u(r,d^*)$ and that $u'(r,d_n)$ converges uniformly on compact sets to $u'(r,d^*)$.

In particular we see that $u(r,d_n)$ converges uniformly to $u(r,d^*)$ on $[0,t^*+1]$. Since $\gamma < d < \delta$, we see by lemma 2.5 that if $u(r_0,d^*) = 0$ and $r_0 > 0$ then $u'(r_0,d^*) \ne 0$ and so by lemma 2.7 for sufficiently large n we see that $u(r,d_n)$ has exactly k zeros on $[0,t^*+1]$. Further for sufficiently large n there exists a $t_n \in [s^*,t^*+1]$ such that $u(t_n,d_n) = \gamma$ and $\lim_{n\to\infty} t_n = t^*$.

We now assume by way of contradiction that $u(r,d_n)$ has at least $(k+2)$ interior zeros. We denote z_n as the $(k+1)$st zero of $u(r,d_n)$ and w_n as the $(k+2)$nd zero of $u(r,d_n)$. Since $u(r,d_n)$ converges uniformly to $u(r,d^*)$ on $[0,t^*+1]$, we see that for large n we have $z_n > t^*+1$ and in fact:

$$\lim_{d\to\infty} z_n = \infty, \tag{3.8}$$

for if some subsequence of z_n (still denoted z_n) were uniformly bounded by some $B < \infty$ then a further subsequence (still denoted z_n) would converge to some z^* with $y^* < t^*+1 \le z^* \le B$. Since $u(r, d_n)$ converges uniformly to $u(r, d^*)$ on $[0, z^*+1]$, we would then have that $u(z^*, d^*) = 0$ and since $z^* \ge t^*+1 > y^*$, z^* would then be a $(k+1)$st zero of $u(r, d^*)$. However by assumption $u(r, d^*)$ has only k zeros - a contradiction. Thus (3.8) holds.

By assumption $\gamma < d^* < \delta$ so that for sufficiently large n we have that $\gamma < d_n < \delta$ so by lemma 2.5 we have that $u'(w_n, d_n) \ne 0$. Thus $\frac{p-1}{p}|u'(z_n)|^p = E(z_n) \ge E(w_n) = \frac{p-1}{p}|u'(w_n)|^p > 0$ so we see that there exists m_n with $z_n < m_n < w_n$, $u'(r, d_n) < 0$ on $[z_n, m_n)$, and $u'(m_n, d_n) = 0$. Also $|u(m_n, d_n)| > \gamma$ since $F(u(m_n)) = E(m_n) \ge E(w_n) > 0$. Hence there exists a_n, b_n, c_n with $z_n < a_n < b_n < c_n < m_n$ such that $u(a_n) = -\beta$, $u(b_n) = -\frac{\beta+\gamma}{2} \equiv \tau$, and $u(c_n) = -\gamma$.
Now as in the proof of lemma 2.7 with $\alpha = \frac{p(N-1)}{p-1}$ we have $(r^\alpha E)' = \alpha r^{\alpha-1}F(u)$. Integrating this on $[t_n, c_n]$, using the fact that $F(u) \le 0$ on $[t_n, c_n]$, and that $F(u(r, d_n)) \le F(\tau) < 0$ on $[a_n, b_n]$ we obtain:

$$0 \le \frac{p-1}{p} c_n |u'(c_n)|^p = c_n^\alpha E(c_n) = t_n^\alpha E(t_n) + \int_{t_n}^{c_n} \alpha r^{\alpha-1} F(u)\,dr \le t_n^\alpha E(t_n) + \int_{a_n}^{b_n} \alpha r^{\alpha-1} F(u)\,dr$$

$$\le t_n^\alpha E(t_n) + F(\tau)[b_n^\alpha - a_n^\alpha] \le t_n^\alpha E(t_n) + F(\tau) b_n^{\alpha-1}[b_n - a_n]. \tag{3.9}$$

From lemma 2.2 we know that $|u'| \le (\frac{p}{p-1})^{\frac{1}{p}}[F(\delta) + |F(\beta)|]^{\frac{1}{p}}$. Integrating this on $[a_n, b_n]$ gives:

$$b_n - a_n \ge c > 0 \tag{3.10}$$

where $c = (\frac{\gamma-\beta}{2})(\frac{p}{p-1})^{\frac{-1}{p}}[F(\delta) + |F(\beta)|]^{\frac{-1}{p}}$. Substituting (3.10) into (3.9) and using the fact that $F(\tau) < 0$ we see that we obtain:

$$0 \le t_n^\alpha E(t_n) + cF(\tau) b_n^{\alpha-1}. \tag{3.11}$$

In addition, since $b_n \ge z_n$ we see from (3.8) that:

$$\lim_{n\to\infty} b_n = \infty. \tag{3.12}$$

Finally, by lemma 2.7 we know that $u(r, d_n)$ converges uniformly to $u(r, d^*)$ on $[0, t^*+1]$ and $u'(r, d_n)$ converges uniformly to $u'(r, d^*)$ on $[0, t^*+1]$ and $t_n \to t^*$. Therefore, we see that:

$$\lim_{n\to\infty} t_n^\alpha E(t_n, d_n) = (t^*)^\alpha E(t^*, d^*) \tag{3.13}$$

Substituting (3.12)-(3.13) into (3.11) and recalling that $F(\tau) < 0$, and $\alpha = \frac{p(N-1)}{p-1} > 1$ (since $N \ge 2$), we see that the right hand side of (3.11) goes to $-\infty$ as $n \to \infty$ which contradicts the inequality in (3.11).
This completes the proof. □

4. Proof of the Main Theorem

Proof.

For $k \in \mathbb{N} \cup \{0\}$, define

$$A_k = \{d \in (\beta, \delta) | u(r, d) \text{ has exactly } k \text{ zeros on } [0, \infty)\}.$$

Observe first that $(\beta, \gamma) \subset A_0$ because for any $d \in (\beta, \gamma)$ we have $E(0, d) = F(d) < 0$ so that by (2.12) $E(r, d) < 0$ for all $r > 0$. Thus $u(r, d) > 0$ for if $u(r_0, d) = 0$ then $E(r_0, d) = \frac{p-1}{p}|u'(r_0, d)|^p \ge 0$ - a contradiction. Thus we see that A_0 is nonempty.

We now assume that $d > \gamma$ and we apply lemma 3.4 at $t = M_{d,1}$ where $M_{d,1}$ is defined in lemma 2.6 and we combine this with lemma 3.3 to obtain:

$$\lim_{d \to \delta^-} F(u(M_{d,1})) = F(\delta) > 0.$$

Thus

$$|u(M_{d,1})| > \gamma \text{ for } d \text{ sufficiently close to } \delta. \tag{4.1}$$

This implies that $M_{d,1} < \infty$ for if $M_{d,1} = \infty$, then from lemma 2.6 we see that $u(M_{d,1}) = \lim_{r\to\infty} u(r)$, $|u(M_{d,1})| < d < \delta$, and $f(u(M_{d,1})) = 0$ which implies $|u(M_{d,1})| \leq \beta$ - contradicting (4.1). Thus $M_{d,1} < \infty$ and by lemma 2.6 we see that $f(u(M_{d,1})) \leq 0$ so by (1.8) we have $u(M_{d,1}) \leq \beta$. Combining this with (4.1) we see that we must have $u(M_{d,1}) < -\gamma < 0$. Therefore for $d < \delta$ and d sufficiently close to δ, we see that $u(r,d)$ must have a first zero, $z_{d,1}$.

Thus we see that A_0 is bounded above by a quantity that is strictly less than δ. We now define:

$$d_0 = \sup A_0$$

and we note that $d_0 < \delta$.

Lemma 4.1.

$$u(r, d_0) > 0 \text{ for } r \geq 0.$$

Proof.

Suppose there exists a smallest value of r, r_0, such that $u(r_0, d_0) = 0$. By Lemma 2.5, $u'(r_0, d_0) \neq 0$ thus $u(r, d_0)$ becomes negative for r slightly larger than r_0. By lemma 2.7 it follows that if $d < d_0$ is sufficiently close to d_0 then $u(r,d)$ must also have a zero close to r_0. However by the definition of d_0 if $d < d_0$ then $u(r,d) > 0$ - a contradiction. This completes the proof. □

Lemma 4.2

$$u'(r, d_0) < 0 \text{ for } r > 0.$$

Proof

We will show that $M_{d_0,1} = \infty$ where $M_{d_0,1}$ is defined in lemma 2.6. If $M_{d_0,1} < \infty$ then by lemma 2.7 for d slightly larger than d_0 we also have $M_{d,1} < \infty$. Also, since $u(r, d_0) > 0$ then $u(M_{d_0,1}, d_0) > 0$ and again by lemma 2.7 we also have $u(M_{d,1}, d) > 0$ for d sufficiently close to d_0. By lemma 2.6 it follows that $f(u(M_{d,1}, d)) \leq 0$ so that $0 \leq u(M_{d,1}, d) \leq \beta$ thus $E(M_{d,1}, d) < 0$. Since E is decreasing we see that $E(r,d) < 0$ for $r \geq M_{d,1}$.
For d slightly larger than d_0, $u(r,d)$ must have a first zero, $z_{d,1}$, (by definition of d_0) and $z_{d,1} > M_{d,1}$ since $u(r,d) > 0$ on $[0, M_{d,1}]$. Thus, we have $0 \leq E(z_1, d) \leq E(M_{d,1}, d) < 0$ - a contradiction. This completes the proof. □

From lemmas 2.6, 4.1, and 4.2 we see that $\lim_{r\to\infty} u(r, d_0) = L$ where $f(L) = 0$ where $L < d_0 < \delta$ and since $u(r, d_0) > 0$ we have that $L = 0$ or $L = \beta$. We also see that $\lim_{r\to\infty} E(r, d_0) = F(L)$.

Lemma 4.3.

$$\lim_{d \to d_0^+} z_{d,1} = \infty$$

Proof.

If $z_{d,1} \leq C$ for $d > d_0$ then as in the proof of (3.8) there would be a subsequence d_n with $d_n \to d_0$ and $z_{d_n,1} \to z$. By lemma 2.7 it then would follow that $u(z, d_0) = 0$ which contradicts that $u(r, d_0) > 0$. This completes the proof. □

Lemma 4.4. $L = 0$

Proof.

We know that $L = 0$ or $L = \beta$ so suppose $L = \beta$. Then $\lim_{r\to\infty} E(r, d_0) = F(L) = F(\beta) < 0$ so there exists an r_0 such that $E(r_0, d_0) < 0$. Thus for $d > d_0$ and d sufficiently close to d_0 we have by lemma 2.7 $E(r_0, d) < 0$. Since $E(z_{d,1}, d) \geq 0$ we see that $z_{d,1} < r_0$ which contradicts lemma 4.3. Thus $\lim_{r\to\infty} u(r, d_0) = 0$ and this completes the proof. □

By definition of d_0, if $d > d_0$ then $u(r, d)$ has *at least* one zero. By lemma 3.4, if d is close to d_0 then $u(r, d)$ has *at most* one zero. Therefore for $d > d_0$ and d sufficiently close to d_0, $u(r, d)$ has exactly one zero. Thus the set A_1 is nonempty and $d_0 < \sup A_1$.

As we saw in the first part of the proof of the main theorem, $M_{d,1} < \infty$ and $u(M_{d,1}) < -\gamma$ for d sufficiently close to δ. By a similar argument as in lemma 2.6, it can be shown that there exists an $M_{d,2}$ with $M_{d,1} < M_{d,2} \leq \infty$ such that $u'(r, d) > 0$ on $(M_{d,1}, M_{d,2})$. Also, by lemma 3.4 we see that

$$0 \leq E(0) - E(M_{d,2}) = [E(0) - E(\tau_d)] + [E(\tau_d) - E(M_{d,1})] + [E(M_{d,1}) - E(M_{d,2}]$$

$$\leq [E(0) - E(\tau_d)] + \frac{C}{\tau_d} + \frac{C}{M_{d,1}} \text{ where } C \text{ is independent of d.}$$

By lemmas 3.1, 3.3 and the fact that $\tau_d < M_{d,1}$ we see:

$$\lim_{d\to\delta^-} F(u(M_{d,2})) = E(0) = F(\delta) > 0.$$

As at the beginning of the proof of the main theorem we may also show that $M_{d,2} < \infty$ and $u(M_{d,2}) > \gamma$ for d sufficiently close to δ. Therefore, there exists $z_{d,2}$ such that $M_{d,1} < z_{d,2} < M_{d,2}$ and $u(z_{d,2}, d) = 0$. Therefore A_1 is bounded above by a quantity strictly less than δ.

Let:

$$d_1 = \sup A_1$$

and note that $d_0 < d_1 < \delta$.

In a similar way in which we proved that $u(r, d_0) > 0$ and $\lim_{r\to\infty} u(r, d_0) = 0$ we can show that $u(r, d_1)$ has exactly one zero and that $\lim_{r\to\infty} u(r, d_1) = 0$.

In a similar way we may show by induction that A_k is nonempty and bounded above by a quantity strictly less than δ. Let

$$d_k = \sup A_k.$$

It can be shown that $u(r, d_k)$ has exactly k zeros and that $\lim_{r\to\infty} u(r, d_k) = 0$.

This completes the proof of the main theorem. □

References

1. B. Franchi, E. Lanconelli, and J. Serrin, *Existence and Uniqueness of Nonnegative Solutions of Quasilinear Equations in* $\mathbb{R}^N$, Advances in Mathematics **118** (1996), 177–243.
2. J. Iaia, H. Warchall, and F.B. Weissler, *Localized Solutions of Sublinear Elliptic Equations: Loitering at the Hilltop*, Rocky Mountain Journal of Mathematics **27(4)** (1997), 1131–1157.

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