



Existence of solutions of nonlinear differential equations with ψ -exponential or ψ -ordinary dichotomous linear part in a Banach space

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Abstract. In this article we consider nonlinear differential equations with ψ -exponential and ψ -ordinary dichotomous linear part in a Banach space. By the help of the fixed point principle of Banach sufficient conditions are found for the existence of ψ -bounded solutions of these equations on \mathbb{R} and \mathbb{R}_+ .

Keywords: ψ -dichotomy for ordinary differential equations, ψ -boundedness.

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1 Introduction

The problem of ψ -boundedness and ψ -stability of the solutions of differential equations in finite dimensional Euclidean spaces has been studied by many authors, as e.g. Akinyele [1], Constantin [6]. In these papers, the function ψ is a scalar continuous function (and increasing, differentiable and bounded in [1], nondecreasing and such that $\psi(t) \geq 1$ on \mathbb{R}_+ in [6]). In Diamandescu [8–15] and Boi [2–4] ψ is a nonnegative continuous diagonal matrix function.

Inspired by the famous monographs of Coppel [5], Daleckii and Krein [7] and Massera and Schaeffer [17], where the important notion of exponential and ordinary dichotomy is considered in detail, Diamandescu [8–12] and Boi [2–4] introduced and studied the ψ -dichotomy for linear differential equations in finite dimensional Euclidean space.

In our paper [16] we introduced the concept of ψ -dichotomy for arbitrary Banach spaces, where ψ is an arbitrary bounded invertible linear operator.

In this paper nonlinear perturbed differential equations with ψ -dichotomous linear part are considered in an arbitrary Banach space. We will show that some properties of these equations will be influenced by the corresponding ψ -dichotomous homogeneous linear equation. Sufficient conditions for the existence of ψ -bounded solutions of this equations on \mathbb{R} and \mathbb{R}_+ in case of ψ -exponential or ψ -ordinary dichotomy are found.

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2 Preliminaries

Let X be an arbitrary Banach space with norm $|\cdot|$ and identity I . Let $LB(X)$ be the space of all linear bounded operators acting in X with the norm $\|\cdot\|$. By J we shall denote \mathbb{R} or $\mathbb{R}_+ = [0, \infty)$.

We consider the nonlinear differential equation

$$\frac{dx}{dt} = A(t)x + F(t, x), \quad (2.1)$$

the corresponding linear homogenous equation

$$\frac{dx}{dt} = A(t)x \quad (2.2)$$

and the appropriate inhomogeneous equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad (2.3)$$

where $A(\cdot): J \rightarrow LB(X)$, $f(\cdot): J \rightarrow X$ are strongly measurable and Bochner integrable on the finite subintervals of J and $F(\cdot, \cdot): J \times X \rightarrow X$ is a continuous function with respect to t .

By a solution of equation (2.1) (or (2.2) or (2.3)) we will understand a continuous function $x(t)$ that is differentiable (in the sense that it is representable in the form $x(t) = \int_a^t y(\tau) d\tau$ of a Bochner integral of a strongly measurable function y) and satisfies (2.1) (or (2.2) or (2.3)) almost everywhere.

By $V(t)$ we will denote the Cauchy operator of (2.2).

Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ and $\psi(\cdot): J \rightarrow RL(X)$ be continuous for any $t \in J$ operator-function.

Definition 2.1 ([16]). A function $u(\cdot): J \rightarrow X$ is said to be ψ -bounded on J if $\psi(t)u(t)$ is bounded on J .

Let $C_\psi(X)$ denote the Banach space of all ψ -bounded and continuous functions with values in X with the norm

$$\|f\|_{C_\psi} = \sup_{t \in J} |\psi(t)f(t)|.$$

Definition 2.2 ([16]). The equation (2.2) is said to have a ψ -exponential dichotomy on J if there exist a pair of mutually complementary projections P_1 and $P_2 = I - P_1$ and positive constants N_1, N_2, ν_1, ν_2 such that

$$\|\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\| \leq N_1e^{-\nu_1(t-s)} \quad (s \leq t; s, t \in J) \quad (2.4)$$

$$\|\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\| \leq N_2e^{-\nu_2(s-t)} \quad (t \leq s; s, t \in J) \quad (2.5)$$

The equation (2.2) is said to have a ψ -ordinary dichotomy on J if (2.4) and (2.5) hold with $\nu_1 = \nu_2 = 0$.

Remark 2.3. For $\psi(t) = I$ for all $t \in J$ we obtain the notion of exponential and ordinary dichotomy in [5, 7, 17].

Let us introduce the principal Green function of (2.3) with the projections P_1 and P_2 from the definition for ψ -exponential dichotomy

$$G(t,s) = \begin{cases} V(t)P_1V^{-1}(s) & (t > s; t,s \in J) \\ -V(t)P_2V^{-1}(s) & (t < s; t,s \in J). \end{cases} \quad (2.6)$$

Clearly G is continuous except at $t = s$ where it has a jump discontinuity.

Definition 2.4. Let $r > 0$ be an arbitrary number. We say that the conditions (H) are fulfilled if there exist positive functions $m(t), k(t)$ such that

$$\text{H1. } |\psi(t)F(t,x)| \leq m(t) \quad (|\psi(t)x| \leq r, t \in J)$$

$$\text{H2. } |\psi(t)(F(t,x_1) - F(t,x_2))| \leq k(t)|\psi(t)(x_1 - x_2)| \quad (|\psi(t)x_1|, |\psi(t)x_2| \leq r, t \in J)$$

Definition 2.5. The nonnegative function $m(t)$ is said to be integrally bounded on J if the following inequality holds:

$$B(m(t)) = \sup_{t \in J} \int_t^{t+1} m(s) ds < \infty.$$

Definition 2.6. We say that the function $F(t,x)$ belongs to the class $ED_\psi(a_1, a_2, r)$ if the conditions (H) are fulfilled, the functions $m(t), k(t)$ are integrally bounded on J and $B(m(t)) \leq a_1, B(k(t)) \leq a_2$.

For each integrable on J function $m(t)$ we introduce the notation

$$L(m(t)) = \int_J m(s) ds.$$

Definition 2.7. We say that the function $F(t,x)$ belongs to the class $D_\psi(a_1, a_2, r)$ if the conditions (H) are fulfilled, the functions $m(t), k(t)$ are integrable on J and $L(m(t)) \leq a_1, L(k(t)) \leq a_2$.

3 Main results

Theorem 3.1. Let the following conditions be fulfilled:

1. The linear part of (2.1) has ψ -exponential dichotomy on \mathbb{R} with projections P_1 and P_2 .
2. The function $F(t,x)$ belongs to the class $ED_\psi(a_1, a_2, r)$.

Then for an arbitrary $r > 0$ for sufficient small values of a_1, a_2 the equation (2.1) has a unique solution $x(t)$, which is defined for $t \in \mathbb{R}$ and for which $|\psi(t)x(t)| \leq r$ ($t \in \mathbb{R}$).

Proof. Let $J = \mathbb{R}$. We consider in the space $C_\psi(X)$ the operator $Q: C_\psi(X) \rightarrow C_\psi(X)$ defined by the formula

$$Qx(t) = \int_J G(t,\tau)F(\tau,x(\tau)) d\tau \quad (3.1)$$

where G is defined by (2.6).

Let $x(t)$ be a solution of equation (2.1) that remains for $t \in J$ in the ball

$$S_{\psi,r} = \{x : \|x\|_{C_\psi} \leq r\}.$$

Then the function $F(t, x(t))$ is ψ -bounded on J and it follows (see [16, Theorem 3.6]) that such solution satisfies the integral equation

$$x(t) = Qx(t). \quad (3.2)$$

The converse is also true: a solution of the integral equation (3.2) which remains for $t \in J$ in the ball $S_{\psi, r}$ satisfies the differential equation (2.1) for $t \in J$.

Now we shall show that the ball $S_{\psi, r}$ is invariant with respect to Q and the operator Q is contracting.

First we shall prove that the operator Q maps the ball $S_{\psi, r}$ into itself. Indeed we have

$$|\psi(t)Qx(t)| \leq \left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right|.$$

We have

$$\begin{aligned} |\psi(t)Qx(t)| &\leq \left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right| \\ &\leq \int_J \|\psi(t)G(t, \tau)\psi^{-1}(\tau)\| |\psi(\tau)F(\tau, x(\tau))| d\tau \\ &= \int_{t \leq \tau} \|\psi(t)G(t, \tau)\psi^{-1}(\tau)\| |\psi(\tau)F(\tau, x(\tau))| d\tau \\ &\quad + \int_{t \geq \tau} \|\psi(t)G(t, \tau)\psi^{-1}(\tau)\| |\psi(\tau)F(\tau, x(\tau))| d\tau \\ &\leq N_2 \int_{t \leq \tau} e^{-\nu_2(\tau-t)} m(\tau) d\tau + N_1 \int_{t \geq \tau} e^{-\nu_1(t-\tau)} m(\tau) d\tau \\ &\leq N_2 \int_{s \geq 0} e^{-\nu_2 s} m(t+s) ds + N_1 \int_{s \leq 0} e^{\nu_1 s} m(t+s) ds \\ &\leq N_2 a_1 \sum_{k=0}^{\infty} e^{-\nu_2 k} + N_1 a_1 \sum_{k=0}^{\infty} e^{-\nu_1 k} = \frac{N_2 a_1}{1 - e^{-\nu_2}} + \frac{N_1 a_1}{1 - e^{-\nu_1}}. \end{aligned}$$

Hence by $a_1 \leq r \left(\frac{N_2}{1 - e^{-\nu_2}} + \frac{N_1}{1 - e^{-\nu_1}} \right)^{-1}$ we obtain

$$\left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right| \leq r.$$

Thus the operator Q maps the ball $S_{\psi, r}$ into itself.

Now we shall prove that the operator Q is a contraction in the ball $S_{\psi, r}$. Let $x_1, x_2 \in S_{\psi, r}$. We obtain

$$\begin{aligned} |\psi(t)Qx_1(t) - \psi(t)Qx_2(t)| &\leq \left| \psi(t) \int_J G(t, \tau) (F(\tau, x_1(\tau)) - F(\tau, x_2(\tau))) d\tau \right| \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| |\psi(\tau)(F(\tau, x_1(\tau)) - F(\tau, x_2(\tau)))| d\tau \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| k(\tau) |\psi(\tau)(x_1(\tau) - x_2(\tau))| d\tau \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| k(\tau) d\tau \sup_{\tau \in J} |\psi(\tau)(x_1(\tau) - x_2(\tau))| \\ &\leq \left(\frac{N_2 a_2}{1 - e^{-\nu_2}} + \frac{N_1 a_2}{1 - e^{-\nu_1}} \right) \sup_{\tau \in J} |\psi(\tau)(x_1(\tau) - x_2(\tau))|. \end{aligned}$$

Hence

$$\|Qx_1 - Qx_2\|_{C_\psi} \leq \left(\frac{N_2 a_2}{1 - e^{-v_2}} + \frac{N_1 a_2}{1 - e^{-v_1}} \right) \|x_1 - x_2\|_{C_\psi}.$$

Thus by $a_2 < \left(\frac{N_2}{1 - e^{-v_2}} + \frac{N_1}{1 - e^{-v_1}} \right)^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. \square

Corollary 3.2. *If the conditions of Theorem 3.1 are fulfilled and if, moreover, $F(t, 0) = 0$ ($t \in \mathbb{R}$) then $x = 0$ is a unique solution of (2.1) in $C_\psi(X)$.*

Proof. Let $F(t, 0) = 0$ ($t \in \mathbb{R}$). Then from H2 it follows

$$|\psi(t)F(t, x(t))| \leq k(t)|\psi(t)x(t)| \quad (t \in \mathbb{R}).$$

Thus every solution $x(t)$ except $x(t) \equiv 0$ ($t \in \mathbb{R}$) will leave any ball S_{ψ,r_1} ($r_1 < r$) by $t \rightarrow \infty$ or $t \rightarrow -\infty$. \square

Theorem 3.3. *Let the following conditions be fulfilled:*

1. *The linear part of (2.1) has ψ -ordinary dichotomy on \mathbb{R} with projections P_1 and P_2 .*
2. *The function $F(t, x)$ belongs to the class $D_\psi(a_1, a_2, r)$.*

Then for each $r > 0$ for sufficient small values of a_1, a_2 the equation (2.1) has a unique solution $x(t)$, which is defined for $t \in \mathbb{R}$ and for which $|\psi(t)x(t)| \leq r$ ($t \in \mathbb{R}$).

Proof. Let $J = \mathbb{R}$. In the proof of Theorem 3.1 it was mentioned that each solution $x(t)$ of equation (2.1) that remains for $t \in J$ in the ball $S_{\psi,r}$ satisfies the integral equation

$$x(t) = \int_J G(t, \tau) F(\tau, x(\tau)) d\tau$$

and vice versa.

We consider again in the space $C_\psi(X)$ the operator $Q: C_\psi(X) \rightarrow C_\psi(X)$ defined in (3.1).

For $|\psi(t)Qx(t)|$ we obtain the following estimate:

$$|\psi(t)Qx(t)| \leq \left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right|.$$

With $a_1 \leq r \max\{N_1, N_2\}$ we have

$$\begin{aligned} |\psi(t)Qx(t)| &\leq \left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right| \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| |\psi(\tau)F(\tau, x(\tau))| d\tau \\ &= \int_{t \leq \tau} \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| |\psi(\tau)F(\tau, x(\tau))| d\tau \\ &\quad + \int_{t \geq \tau} \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| |\psi(\tau)F(\tau, x(\tau))| d\tau \\ &\leq N_2 \int_{t \leq \tau} m(\tau) d\tau + N_1 \int_{t \geq \tau} m(\tau) d\tau \\ &\leq \max\{N_1, N_2\} \int_J m(\tau) d\tau \leq \max\{N_1, N_2\} a_1 \leq r. \end{aligned}$$

Thus the operator Q maps the ball $S_{\psi,r}$ into itself.

Now we shall prove that the operator Q is a contraction in the ball $S_{\psi,r}$. Let $x_1, x_2 \in S_{\psi,r}$. We obtain

$$\begin{aligned} |\psi(t)Qx_1(t) - \psi(t)Qx_2(t)| &\leq \left| \psi(t) \int_J G(t, \tau) (F(\tau, x_1(\tau)) - F(\tau, x_2(\tau))) d\tau \right| \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| |\psi(\tau)(F(\tau, x_1(\tau)) - F(\tau, x_2(\tau)))| d\tau \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| k(\tau) |\psi(\tau)(x_1(\tau) - x_2(\tau))| d\tau \\ &\leq \int_J \left\| \psi(t)G(t, \tau)\psi^{-1}(\tau) \right\| k(\tau) d\tau \sup_{\tau \in J} |\psi(\tau)(x_1(\tau) - x_2(\tau))| \\ &\leq (\max\{N_1, N_2\}a_2) \sup_{\tau \in J} |\psi(\tau)(x_1(\tau) - x_2(\tau))|. \end{aligned}$$

Hence

$$\|Qx_1 - Qx_2\|_{C_\psi} \leq (a_2 \max\{N_1, N_2\}) \|x_1 - x_2\|_{C_\psi}.$$

Thus by $a_2 < (\max\{N_1, N_2\})^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. \square

Theorem 3.4. *Let the following conditions be fulfilled:*

1. *The linear part of (2.1) has ψ -exponential dichotomy on \mathbb{R}_+ with projections P_1 and P_2 .*
2. *The function $F(t, x)$ belongs to the class $ED_\psi(a_1, a_2, r)$.*

Then for any $r > 0$ by sufficient small a_1, a_2 there exists $\rho < r$ such that the equation (2.1) has for each $\xi \in X_1 = P_1X$ with $|\psi(0)\xi| \leq \rho$ a unique solution $x(t)$ on \mathbb{R}_+ for which $P_1x(0) = \xi$ and $|\psi(t)x(t)| \leq r$ ($t \in \mathbb{R}_+$).

Proof. Let $J = \mathbb{R}_+$ and $x(t)$ be a solution of equation (2.1) that remains for $t \in J$ in the ball $S_{\psi,r} = \{x : \|x\|_{C_\psi} \leq r\}$. From the results obtained in [16, Theorem 3.6 and Remark 3.8] it follows that such $x(t)$ satisfies the integral equation

$$x(t) = V(t)\xi + \int_J G(t, \tau)F(\tau, x(\tau)) d\tau \quad (3.3)$$

where $\xi = P_1x(0)$. The converse is also true: a solution of the integral equation (3.3) satisfies the differential equation (2.1) for $t \in J$.

Let $\xi \in X_1$ and $|\psi(0)\xi| \leq \rho < r$. We consider in the space $C_\psi(X)$ the operator $Q: C_\psi(X) \rightarrow C_\psi(X)$ defined by the formula

$$Qx(t) = V(t)\xi + \int_J G(t, \tau)F(\tau, x(\tau)) d\tau \quad (3.4)$$

First we shall prove, that the operator Q maps the ball $S_{\psi,r}$ into itself. Indeed we have

$$|\psi(t)Qx(t)| \leq |\psi(t)V(t)\xi| + \left| \psi(t) \int_J G(t, \tau)F(\tau, x(\tau)) d\tau \right|.$$

For the first addend with $\rho \leq \frac{r}{2N_1}$ we obtain

$$|\psi(t)V(t)\xi| \leq N_1 e^{-\nu_1 t} |\psi(0)\xi| \leq N_1 e^{-\nu_1 t} \rho \leq \frac{r}{2}.$$

Using the same technique and notations as in the proof of Theorem 3.1 we obtain for the second addend the estimate

$$\left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right| \leq \frac{N_2 a_1}{1 - e^{-\nu_2}} + \frac{N_1 a_1}{1 - e^{-\nu_1}}.$$

Hence by $a_1 \leq \frac{r}{2} \left(\frac{N_2}{1 - e^{-\nu_2}} + \frac{N_1}{1 - e^{-\nu_1}} \right)^{-1}$ we obtain

$$\left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right| \leq \frac{r}{2}.$$

Thus the operator Q maps the ball $S_{\psi, r}$ into itself.

Now we shall prove that the operator Q is a contraction in the ball $S_{\psi, r}$. Let $x_1, x_2 \in S_{\psi, r}$. We obtain as in the proof of Theorem 3.1 the estimate

$$\|Qx_1 - Qx_2\|_{C_\psi} \leq \left(\frac{N_2 a_2}{1 - e^{-\nu_2}} + \frac{N_1 a_2}{1 - e^{-\nu_1}} \right) \|x_1 - x_2\|_{C_\psi}$$

By $a_2 < \left(\frac{N_2}{1 - e^{-\nu_2}} + \frac{N_1}{1 - e^{-\nu_1}} \right)^{-1}$ the operator Q is a contraction in the ball $S_{\psi, r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. \square

Theorem 3.5. *Let the following conditions be fulfilled:*

1. *The linear part of (2.1) has ψ -ordinary dichotomy on \mathbb{R}_+ with projections P_1 and P_2 .*
2. *The function $F(t, x)$ belongs to the class $D_\psi(a_1, a_2, r)$.*

Then for any $r > 0$ by sufficiently small a_1, a_2 there exists $\rho < r$ such that the equation (2.1) has for each $\xi \in X_1 = P_1 X$ with $|\psi(0)\xi| \leq \rho$ a unique solution $x(t)$ on \mathbb{R}_+ for which $P_1 x(0) = \xi$ and $|\psi(t)x(t)| \leq r$ ($t \in \mathbb{R}_+$).

Proof. Let $J = \mathbb{R}_+$, $\xi \in X_1$ and $|\psi(0)\xi| \leq \rho < r$. We consider again in the space $C_\psi(X)$ the operator $Q: C_\psi(X) \rightarrow C_\psi(X)$ defined by the formula (3.4).

First we shall prove, that the operator Q maps the ball $S_{\psi, r}$ into itself. We have

$$|\psi(t)Qx(t)| \leq |\psi(t)V(t)\xi| + \left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right|.$$

For the first addend with $\rho \leq \frac{r}{2N_1}$ we obtain

$$|\psi(t)V(t)\xi| \leq N_1 |\psi(0)\xi| \leq N_1 \rho \leq \frac{r}{2}.$$

For the second addend with $a_1 \leq \frac{r}{2 \max\{N_1, N_2\}}$ as in the proof of Theorem 3.3 we have

$$\left| \psi(t) \int_J G(t, \tau) F(\tau, x(\tau)) d\tau \right| \leq \max\{N_1, N_2\} a_1 \leq \frac{r}{2}.$$

Thus the operator Q maps the ball $S_{\psi, r}$ into itself.

Let $x_1, x_2 \in S_{\psi, r}$. As in the proof of Theorem 3.3 we obtain the estimate

$$\|Qx_1 - Qx_2\|_{C_\psi} \leq (a_2 \max\{N_1, N_2\}) \|x_1 - x_2\|_{C_\psi}.$$

Hence by $a_2 < (\max\{N_1, N_2\})^{-1}$ the operator Q is a contraction in the ball $S_{\psi, r}$.

From the fixed point principle of Banach it follows the existence of a unique fixed point of the operator Q .

In the proof of Theorem 3.4 it was already mentioned that every solution of the differential equation (2.1) which lies in the ball $S_{\psi,r}$ fulfil the equality

$$x(t) = Qx(t)$$

and vice versa. □

Corollary 3.6. *Let the conditions of Theorem 3.5 hold and let $x_1(t)$ and $x_2(t)$ be two solutions whose initial values fulfil $P_1x_1(0) = \xi$ and $P_1x_2(0) = \eta$. Let $N = \max\{N_1, N_2\}$.*

Then for $Na_2 < 1$ the following estimate holds

$$|\psi(t)(x_1(t) - x_2(t))| \leq \frac{N}{1 - Na_2} |\psi(0)(\xi - \eta)| \quad (t \in \mathbb{R}_+).$$

Proof. Applying the presentation (3.3) for the solutions x_1 and x_2 we obtain

$$x_1(t) - x_2(t) = V(t)(\xi - \eta) + \int_0^\infty G(t, \tau)(F(\tau, x_1(\tau)) - F(\tau, x_2(\tau))) d\tau.$$

From here and the conditions of Theorem 3.5 for $u(t) = \psi(t)(x_1(t) - x_2(t))$ we obtain

$$|u(t)| \leq N|\psi(0)(\xi - \eta)| + N \int_0^\infty k(\tau)u(\tau) d\tau.$$

Let us consider the equation

$$u(t) = \alpha + N \int_0^\infty k(\tau)u(\tau) d\tau, \tag{3.5}$$

where $\alpha = N|\psi(0)(\xi - \eta)|$. Let us introduce the functional $\Phi: C \rightarrow \mathbb{R}_+$, where C is the space of all bounded functions on \mathbb{R}_+ with values in \mathbb{R}_+ by the formula

$$(\Phi u)(t) = N \int_0^\infty k(\tau)u(\tau) d\tau.$$

For the norm of Φ we obtain the estimate

$$\|\Phi\| \leq N \int_0^\infty k(\tau) d\tau \leq Na_2.$$

For sufficiently small a_2 we have $\|\Phi\| \leq 1$.

Let I_C be the identity of the space C . Then the equation $(I_C - \Phi)u = \alpha$ has a bounded solution $u(t)$, i.e. there exists a constant $c = \sup_{t \in \mathbb{R}_+} |u(t)| < \infty$. We shall estimate the constant c from equation (3.5):

$$c \leq \alpha + Nc \int_0^\infty k(\tau) d\tau \leq \alpha + Nca_2,$$

i.e.

$$c \leq \frac{\alpha}{1 - Na_2}.$$

Finally we obtain

$$|\psi(t)(x_1(t) - x_2(t))| \leq \frac{N|\psi(0)(\xi - \eta)|}{1 - Na_2}.$$

□

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References

- [1] O. AKINYELE, On partial stability and boundedness of degree k , *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **8** 65(1978), 259–264. [MR0586610](#)
- [2] P. N. BOI, On the ψ -dichotomy for homogeneous linear differential equations, *Electron. J. Differ. Equ.* **2006**, No. 40, 1–12. [MR2213584](#)
- [3] P. N. BOI, Existence of ψ -bounded solutions for nonhomogeneous linear differential equations, *Electron. J. Differ. Equ.* **2007**, No. 52, 1–10. [MR2299606](#)
- [4] P. N. BOI, Behavior at infinity of ψ -evanescent solutions to linear differential equations, *Electron. J. Differ. Equ.* **2010**, No. 102, 1–9. [MR2680305](#)
- [5] W. A. COPPEL, *Dichotomies in stability theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, Berlin–New York, 1978. [MR0481196](#)
- [6] A. CONSTANTIN, Asymptotic properties of solutions of differential equations, *An. Univ. Timișoara Ser. Științ. Mat.* **30**(1992), No. 2–3, 183–225. [MR1330747](#)
- [7] J. L. DALECKII, M. G. KREIN, *Stability of solutions of differential equations in Banach space*, Translations of Mathematical Monographs, Vol. 43, American Mathematical Society, Providence, R.I., 1974. [MR0352639](#)
- [8] A. DIAMANDESCU, Note on the Ψ -boundedness of the solutions of a system of differential equations, *Acta Math. Univ. Comen., New Ser.* **73**(2004), No. 2, 223–233, ISSN 0862-9544. [MR2122210](#)
- [9] A. DIAMANDESCU, Existence of Ψ -bounded solutions for a system of differential equations, *Electron. J. Differ. Equ.* **2004**, No. 63, 1–6. [MR2057650](#)
- [10] A. DIAMANDESCU, A note on the existence of Ψ -bounded solutions for a system of differential equations on \mathbb{R} , *Electron. J. Differ. Equ.* **2008**, No. 128, 1–11. [MR2443151](#)
- [11] A. DIAMANDESCU, A note on existence of Ψ -bounded solutions for linear difference equations on \mathbb{Z} , *Kuwait J. Sci. Eng.* **36**(2009), No. 2A, 35–49. [MR2596982](#)
- [12] A. DIAMANDESCU, Existence of Ψ -bounded solutions for nonhomogeneous linear difference equations, *Appl. Math. E-Notes* **10**(2010), 94–102. [MR2658119](#)
- [13] A. DIAMANDESCU, Existence of Ψ -bounded solutions for nonhomogeneous Lyapunov matrix differential equations on \mathbb{R} , *Electron. J. Qual. Theory Differ. Equ.* **2010**, No. 42, 1–9. [MR2678384](#)
- [14] A. DIAMANDESCU, On Ψ -stability of nonlinear Lyapunov matrix differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2009**, No. 54, 1–18. [MR2546347](#)
- [15] A. DIAMANDESCU, Ψ -bounded solutions for a Lyapunov matrix differential equation, *Electron. J. Qual. Theory Differ. Equ.* **2009**, No. 17, 1–11. [MR2486660](#)

- [16] A. GEORGIEVA, H. KISKINOV, S. KOSTADINOV, A. ZAHARIEV, Psi-exponential dichotomy for linear differential equations in a Banach space, *Electron. J. Differ. Equ.* **2013**, No. 153, 1–13. [MR3084633](#)
- [17] J. L. MASSERA, J. J. SCHAEFFER, *Linear differential equations and function spaces*, Pure and Applied Mathematics, Vol. 21, Academic Press, New York–London, 1966. [MR0212324](#)